APPLIED ELASTICITY

BY

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PREFACE

In writing this book I have tried to see the subject from the point of view of the engineer rather from that of the mathematician. It is for that reason that the title *Applied Elasticity* has been chosen. Although there is not much in the book that could not be covered by the usual title *Theory of Elasticity*, yet this theory has been developed only so far as it seemed likely to lead to the solution of practical problems. Moreover, in the course of the work, only such problems as were deemed to have a practical interest have been chosen to illustrate the theory.

One very important departure from the strict mathematical theory is to be found in the use of approximate methods of solution based on the principle of minimum energy. The application of this method requires nothing more difficult than some simple integration, and the probable errors in the results attained are generally much smaller than the probable errors due to ignorance of the values of the elastic constants. The method is ideal for dealing with problems on stability, since, with very little effort, it usually gives buckling loads to within one per cent. The process can also be applied, with unexpected success, to the task of finding the periods of normal oscillations of elastic bodies. Mathematically these oscillation problems are identical with the stability problems before mentioned.

Some of the results of this book are here published, as far as I know, for the first time. Among these are Arts. 279 to 283 on the deflexion of a thin plate under normal pressure when the stretching of the middle surface is taken into account; Arts. 307, 308, 309, giving the approximate method of finding the buckling loads of deep beams; and Arts. 330, 331, 338, on the vibrations of a disk of variable thickness. I wish also to mention the successful application of the energy method to the problem of the buckling of thin tubes in Art. 324, in spite of the difficulty, to which the late Lord Rayleigh called attention, in getting an accurate expression for the energy in a bent plate.

The equations have been numbered consecutively through the chapters, and the number before the dot denotes the number of the chapter. Thus (14.53) is the number used to indicate equation 53 in chapter 14.

A few pieces of analysis have been put into three Appendices, named A, B, and C. The 10th equation in Appendix A is numbered (A. 10).

I am under great obligation to Mr J. D. Cockcroft, M. Sc. Tech., for his most conscientious reading, either in the proofs or in manuscript, of all the book except the Appendices. He has worked through nearly every piece of analysis and arithmetic given in the book, and through his labours innumerable errors, big and small, have been eliminated.

So much tedious arithmetic has been involved in the production of the book that it will not surprise me to learn that many errors still remain in the printed pages. I shall, however, be content if there are no serious errors of principle.

JOHN PRESCOTT.

June, 1924.

CONTENTS

APPLIED ELASTICITY

CHAPTER I

ANALYSIS OF STRESS

1. Definition of Stress.

When body is in equilibrium under the action of forces applied at different points of its bounding surface it is quite clear that the effect of the forces must be somehow transmitted through the body. Even when the body is not in equilibrium, or when the forces act at points not on the boundary of the body, it is still true in general that actions are transmitted through the body. Then it follows that, across any small plane area in the body, forces are exerted by the matter on one side of the area on the matter on the other side. Thus, if F is this force across a small area A, we can resolve F into two components, N perpendicular to the area, and S in the plane of the area. We will suppose, for clearness, that the force F shown in fig. 1 is the action of the matter to the right of the area on the matter to the left. By Newton's third law the action of the matter to the left of the area on the matter to the right is the force F reversed. The normal component force N is called a *tension* if its direction is away from the matter on which it acts. Thus in fig. 1, the matter on which F acts being supposed to lie to the left of the area, the force N is shown as a tension. If the force N is towards the matter on which it acts it is called a *thrust*. Physically a tension is an action inside a body resisting the separation of the particles of the body, and a thrust is an action resisting the crushing together of the particles. A thrust may be regarded as a negative tension.

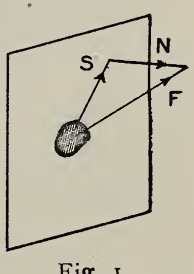

Fig. 1

The component force S in the plane of the area is called a *shearing force,* or a *shear force,* and it can have any direction in the area depending on the direction of F in space.

The mean normal force per unit area of A is $\frac{N}{A}$, and this is the *mean normal stress* on the area. If N is a tension this normal stress is called a *tensional stress,* whereas if N is a thrust the stress is a *compressive stress* or a *negative tensional stress.*

The mean tangential force per unit area on A is $\frac{S}{A}$, and this is called the *mean shear stress* on the area A.

If the area A is infinitely small and concentrated round a point C the mean normal stress and the mean shear stress are then called the normal stress and the shear stress at the point C for an area having the direction (or orientation) of the area A. It will be shown later what is the relation between the stresses at a point C for areas in different directions through that point.

If the structure of matter is molecular there is probably no such thing as stress at a point, for the very idea of stress at a point involves the idea of continuity of matter, which is quite opposed to the molecular theory. Nevertheless, molecular distances are so small compared with any lengths we usually measure that the area can be made very small without its dimensions being allowed to approach the smallness of intermolecular distances, and we could then take the stresses on such a small area as the stresses at a point, the point being the centre of the area. In effect, for the purposes of the the theory of elasticity, our results will be quite good enough if our points at which stresses are taken are very small areas and not Euclidean points.

2. Component Stresses parallel to coordinate axes.

Let C be a point in an elastic solid situated at the point (x, y, z) relative to three rectangular coordinate axes OX, OY, OZ, fixed in the body. Let three very small areas be taken at C perpendicular respectively to the three coordinate axes. Let the tensional stresses in the material in the directions of OX, OY, OZ, be denoted by P_1, P_2, P_3, compressive stresses being indicated by negative values of the P's. The shearing stress on the area perpendicular to any one axis can be resolved into two components parallel to the other two axes. Thus there are two component shear stresses on each of the three areas, six component shear stresses in all on the three planes through the point C. It will shortly be proved that these six shear stresses fall into three pairs, each pair having equal members, so that there are only three different component shear stresses at any point. Let us consider only one of these areas, namely the area perpendicular to the axis OZ, and suppose the outward normal to the matter, on which the stresses shown in fig. 2 act, is parallel to the positive direction along OZ; that is, if OZ points vertically upwards, the matter on which the stresses act lies below the area in the figure. We will denote the component stresses on the area perpendicular to OZ by S_{zx} and S_{zy}, the first being parallel to

OX and the second parallel to OY. There are in all six of these component shear stresses at C, the other four being on the other two planes through C perpendicular to the other two axes One of these other stresses is S_{xz}, which is the component shear stress acting on the plane perpendicular to OX in the direction of OZ. We shall show that $S_{xy}=S_{yx}$; $S_{yz}=S_{zy}$; $S_{zx}=S_{xz}$.

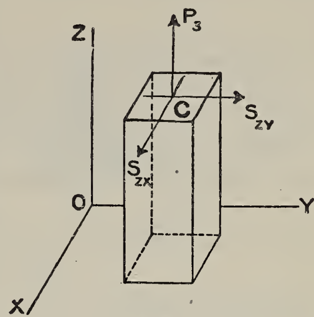

Fig. 2

It should first of all be observed that, since action and reaction are equal an opposite, the stresses on the plane in fig. 2 are all reversed if we are considering the action on the matter above the area. Moreover the positive directions of the component shear stresses are determined by the fact that the positive tension and the positive shear stresses on any one of the three faces are all the same as the positive directions along the three co-ordinate axes, or all contrary to these directions, and there is no trouble in determining the direction of a positive tension. Thus in fig. 2, P_3, S_{zx}, S_{zy} are all in the same directions as OZ, $\overrightarrow{OX}$, $\overrightarrow{OY}$.

Let us consider the forces on a small rectangular block with edges parallel to the coordinate axes, the centre of the block being at (x, y, z). Since the dimensions of the block are small, and will be ultimately infinitely small, we may regard the stresses at the centre of each face as the mean stresses over the area, and the resultant force on each face may be regarded as acting at the centre of the face. The errors due to these assumptions are quantities of smaller orders than the quantities retained in our equations, and therefore in the limit these errors vanish.

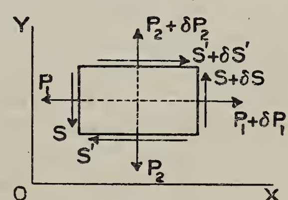

Fig. 3

In fig. 3 only the plan of the block on the xy plane is shown. The stresses perpendicular to the xy plane, and the shear stresses on the planes parallel to the xy plane, are not shown because they have no moment about that line through the point (x, y, z) which is parallel to the z-axis.

Since the area of the face on which S acts is $\delta y \delta z$ the force due to this stress is $S\delta y\delta z$. By taking moments about the line through C perpendicular to the xy plane we get

$$\tfrac{1}{2}\delta x\{S+(S+\delta S)\}\delta y\delta z - \tfrac{1}{2}\delta y\{S'+(S'+\delta S')\}\delta x\delta z = 0.$$

Therefore, dividing by $\delta x \delta y \delta z$ and then making the block infinitely small, which makes $\delta S \rightarrow 0$ and $\delta S' \rightarrow 0$, we get

$$S - S' = 0$$

or

$$S = S'.$$

But S, being the shear stress in the y direction on a plane perpendicular to the axis of x, is the value of the stress S_{xy} at the middle of the face on which it acts, which point ultimately coincides with C. Likewise when the block is infinitely small the stress S' is the stress S_{yx} at the point C. Therefore our result says that

$$S_{xy} = S_{yx};$$

and in the same way we can prove that

$$S_{yz} = S_{zy} \text{ and } S_{xz} = S_{zx}.$$

We may now put

$$S_1 = S_{yz} = S_{zy},$$
$$S_2 = S_{zx} = S_{xz},$$
$$S_3 = S_{xy} = S_{yx}.$$

Thus S_1 is the value of any of the shear stresses which tends to turn the block about an axis through its centre parallel to the axis of x. We thus see the connection between the suffix 1 and the x-axis. The suffixes 2 and 3 have the same connection with the axes of y and z respectively.

3. Principal Axes of Stress.

We have now shown that the stress system at any point on planes parallel to the coordinate planes is completely specified by means of six stresses, namely, P_1, P_2, P_3, S_1, S_2 and S_3. By considering the equilibrium of a small portion of the elastic body bounded by one oblique plane and three planes parallel to the coordinate planes we can find the state of stress on this oblique plane in terms of the six stresses mentioned above. It follows then that the stress at a point across any plane through that point can be completely determined in terms of these six stresses and the known angles which determine the position of the plane. It will be shown later (Art 6) that there is one set of three mutually perpendicular planes through each point of the body on which the shear stresses are all zero, so that, if the coordinate axes are taken perpendicular to these three planes, the stresses S_1, S_2, S_3, are each zero. These particular coordinate axes are called the principal axes of stress for that point, and the three planes perpendicular to these axes are the principal planes at the point.

4. Principal Axes when $S_1 = S_2 = 0$.

As a first step towards the general problem of finding the principal axes of stress for a given point where the six stress-components are known let us first consider the case where S_1 and S_2 are both zero. We shall show that in this case the principal axes can be reached by merely rotating the axes about the axis of z.

In fig. 4 the stresses on opposite faces of a block of dimensions δx, δy, δz, are shown as equal stresses, whereas actually they should differ by small quantities of the order δx or δy. These small differences, if we took account of them, would not alter the result of the following reasoning.

Let DK be a plane parallel to the z-axis, and let us consider the equilibrium of the wedge shaped portion DAK under the stresses P_1, P_2, P_3, S_3 and the stresses P and S on the oblique plane DK.

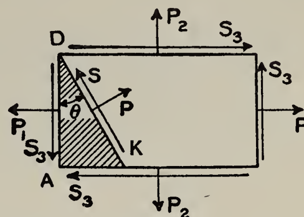

Fig. 4

The area of the face represented by AK in the figure is $\delta z\, \delta y \tan\theta$, and the area of DK is $\delta z\, \delta y \sec\theta$. Then the normal forces on these areas are $P_2 \delta z\, \delta y \tan\theta$ and $P\delta z\, \delta y \sec\theta$, and the tangential stresses are $S_3 \delta z\, \delta y \tan\theta$ and $S\delta z\, \delta y \sec\theta$. Then resolving all the forces on the wedge in the direction of S we get, for equilibrium,

$$S\delta z\, \delta y \sec\theta = (P_2\delta z\, \delta y \tan\theta + S_3\delta z\, \delta y)\cos\theta$$
$$- (P_1\delta z\, \delta y + S_3\delta z\, \delta y \tan\theta)\sin\theta.$$

Therefore

$$S = (P_2 \sin\theta + S_3\cos\theta)\cos\theta - (P_1\cos\theta + S_3\sin\theta)\sin\theta$$
$$= S_3\cos 2\theta + \tfrac{1}{2}(P_2 - P_1)\sin 2\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.1)$$

The angle θ which makes S zero is given by

$$\tfrac{1}{2}(P_1 - P_2)\sin 2\theta = S_3 \cos 2\theta,$$

or

$$\tan 2\theta = \frac{2S_3}{P_1 - P_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.2)$$

Now as θ varies from 0 to 90° the angle 2θ varies from 0 to 180° and therefore $\tan 2\theta$ varies firstly from 0 to ∞ and then from $-\infty$ to 0, thus passing once through every positive and negative number. It follows then that, whatever values S_3, P_1 and P_2 have, there is one value of θ between 0 and 90° satisfying equation (1.2). Let this angle be α. If therefore the axes of x and y be turned about OZ through the angle $(90^\circ - \alpha)$ in the direction from OX towards OY then the shear stress S_3 for this new set of axes is zero. Also S_1 and S_2 are still zero since our original assumption that S_1 and S_2 were zero meant that the resultant tangential stress on the planes perpendicular to OZ was zero, and these planes have not been altered. The axes in the new position are the principal axes for the stresses at the point C.

5. Components, parallel to the coordinate axes. of the resultant stress on any oblique plane.

The six stresses on three mutually perpendicular planes through O being given it is required to find the component forces per unit area parallel to the three axes on any oblique plane through O.

Let ABC be an oblique plane parallel to the one through O, which point is taken as origin of coordinates for convenience. We assume that the stresses over all the faces of the block OABC, which is supposed to be small, are uniform, and then it follows that the resultant force on any face passes through the centre of gravity of that face. Thus for example, the point of intersection of F_1, F_2, F_3, is the centre of gravity of the face ABC. When the block OABC is reduced to infinitesimal dimensions the stresses over the faces become uniform in any case, and the stresses on the oblique plane ABC become the stresses on an oblique plane through O.

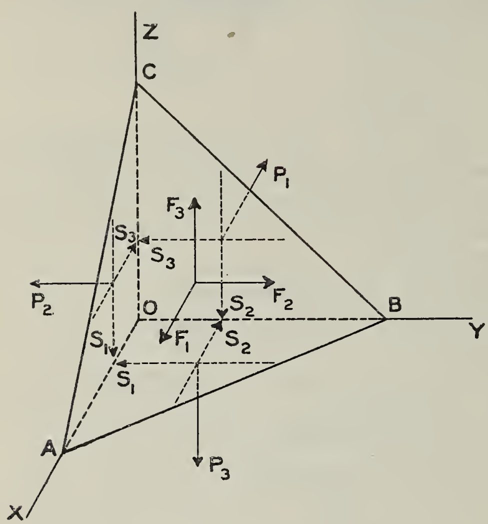

Fig. 5

Let l, m, n, denote the direction-cosines of the outward normal to the face ABC; that is, the cosines of the angles between a positive tension on the face ABC and the axes OX, OY, OZ; and let a denote the area of the triangle ABC. Then the projected areas OBC, OCA, OAB are la, ma, na. Let F_1, F_2, F_3, denote the components of the resultant stress on the oblique plane ABC.

Now, assuming the block to be in equilibrium, and resolving the forces on the block parallel to OX, we get

$$F_1 a = P_1 la + S_3 ma + S_2 na.$$

Therefore
$$F_1 = lP_1 + mS_3 + nS_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.3)$$

Likewise, by resolving parallel to the other axes,

$$F_2 = lS_3 + mP_2 + nS_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.4)$$
$$F_3 = lS_2 + mS_1 + nP_3 \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.5)$$

By reducing the block OABC to infinitesimal dimensions equations (1.3) (1.4) (1.5) give relations between the stresses at a point, namely the point O in figure 5.

6. Principal axes for any given system of stresses.

It has been shown in the last article that the components, parallel to the axes, of the force per unit area on an oblique plane through any point at which the six independent stresses are given, are

$$F_1 = lP_1 + mS_3 + nS_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.6)$$
$$F_2 = lS_3 + mP_2 + nS_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.7)$$
$$F_3 = lS_2 + mS_1 + nP_3 \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.8)$$

Now if there is no shear stress on the oblique area ABC then the resultant of F_1, F_2, and F_3, is a purely tensional stress, that is, a stress in the direction l, m, n. Suppose P is this tensional stress. Then

$$lP = F_1, \ mP = F_2, \ nP = F_3;$$

that is,

$$\frac{F_1}{l} = \frac{F_2}{m} = \frac{F_3}{n}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.9)$$

or

$$P_1 + \frac{m}{l} S_3 + \frac{n}{l} S_2 = P_2 + \frac{l}{m} S_3 + \frac{n}{m} S_1$$
$$= P_3 + \frac{l}{n} S_2 + \frac{m}{n} S_1.$$

We may write these two conditions thus

$$P_1 - P_2 + \left(\frac{m}{l} - \frac{l}{m}\right) S_3 + \frac{n}{l} S_2 - \frac{n}{m} S_1 = 0$$

and

$$P_2 - P_3 + \left(\frac{n}{m} - \frac{m}{n}\right) S_1 + \frac{l}{m} S_3 - \frac{l}{n} S_2 = 0.$$

Let $\alpha = \frac{l}{m}$ and $\beta = \frac{n}{m}$. Then these last equations become

$$P_1 - P_2 + \left(\frac{1}{\alpha} - \alpha\right) S_3 + \frac{\beta}{\alpha} S_2 - \beta S_1 = 0$$

and

$$P_2 - P_3 + \left(\beta - \frac{1}{\beta}\right) S_1 + \alpha S_3 - \frac{\alpha}{\beta} S_2 = 0$$

Clearing these of fractions we get

$$(\alpha^2 - 1)S_3 - \alpha(P_1 - P_2) - \beta(S_2 - \alpha S_1) = 0 \quad . \quad . \quad . \quad (1.10)$$
$$(\beta^2 - 1)S_1 - \beta(P_3 - P_2) - \alpha(S_2 - \beta S_3) = 0 \quad . \quad . \quad . \quad (1.11)$$

The value of β from equation (1.10) is

$$\beta = \frac{(\alpha^2 - 1)S_3 - \alpha(P_1 - P_2)}{S_2 - \alpha S_1}. \quad . \quad . \quad . \quad . \quad (1.12)$$

On substituting this in equation (1.11) and multiplying up by $(S_2 - \alpha S_1)^2$ we get

$$\left[\left\{(\alpha^2 - 1)S_3 - \alpha(P_1 - P_2)\right\}^2 - (S_2 - \alpha S_1)^2\right] S_1$$
$$+ (P_2 - P_3)(S_2 - \alpha S_1)\left\{(\alpha^2 - 1)S_3 - \alpha(P_1 - P_2)\right\}$$
$$+ \alpha(S_2 - \alpha S_1)\left[\left\{(\alpha^2 - 1)S_3 - \alpha(P_1 - P_2)\right\}S_3 - (S_2 - \alpha S_1)S_2\right] = 0 \quad (1.13)$$

The coefficient of α^4 in this equation is

$$S_3^2 S_1 - S_1 S_3^2 = 0$$

so that equation (1.13) is a cubic equation in α, since the coefficient of α^3 does not vanish identically. Now a cubic equation has at least one real root, and therefore there is certainly one possible real value of the ratio $\frac{l}{m}$, and corresponding to this value of α there is one real value of β given by (1.12). Then the general relation between direction-cosines, namely,

$$l^2 + m^2 + n^2 = 1$$

gives

$$m^2(\alpha^2 + 1 + \beta^2) = 1,$$

which determines m. Thus one set of direction-cosines is αm, m, βm. It follows then that there is certainly one plane on which there is no shear stress.

As we have now proved that there is one plane through the given point on which there is no shear stress let the axes be turned so that the new axis OZ′ is perpendicular to this plane. Then, stresses on planes parallel to the new axes being denoted by dashed letters, our new conditions are that $S'_1 = 0$ and $S'_2 = 0$. Then the stresses relative to the new axes are exactly similar to those in Art 4, where it was proved that, by rotating the axes about OZ (in the present case about OZ′) a position can always be found where the stress S_3, referred to the last position of the axes, is also zero. This proves that, whatever the state of stress at a point, there is always a set of mutually perpendicular axes for which the stresses S_1, S_2, S_3, are all zero at that point; that is, there is certainly one set of principal axes for each point of a stressed body. Moreover, there is only one set, for the three roots for α given by (1.13) would only give the three principal axes in a different order. The stresses P_1, P_2, P_3, on the principal planes through a point are called the principal stresses at that point.

7. Stress on an oblique plane in terms of principal stresses. If the coordinate axes are principal axes for a point C then, since $S_1 = S_2 = S_3 = 0$, equations (1.6), (1.7), (1.8), give

$$\left.\begin{aligned} F_1 &= lP_1 \\ F_2 &= mP_2 \\ F_3 &= nP_3 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.14)$$

If P_1, P_2, P_3, are all equal then

$$\frac{F_1}{l} = \frac{F_2}{m} = \frac{F_3}{n} = P_1, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.15)$$

which are the conditions that the resultant stress on the oblique plane should be a purely normal stress equal to P_1. This shows that, if the three principal stresses are all equal at any point, then the stress across any plane at that point is exactly the same as it is across each principal plane, from which it follows that any set of rectangular planes through the point are principal planes. If P is negative then there is a pure thrust across every plane, this latter being the condition in the atmosphere or in a liquid, and we shall refer to it as a state of hydrostatic thrust.

In the general case, where P_1, P_2, P_3, are unequal we get

$$\frac{F_1^2}{P_1^2}+\frac{F_2^2}{P_2^2}+\frac{F_3^2}{P_3^2}=l^2+m^2+n^2=1 \quad . \quad . \quad . \quad . \quad (1.16)$$

If we regard F_1, F_2, F_3, as the coordinates of a point, relative to an origin at the point where the stresses act, then the vector drawn from the origin to the point F_1, F_2, F_3, is the resultant stress on the oblique plane, and equation (1.16) shows that the end of this vector lies on the ellipsoid whose equation is

$$\frac{x^2}{P_1^2}+\frac{y^2}{P_2^2}+\frac{z^2}{P_3^2}=1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.17)$$

This shows that, of the three principal stresses at a point, one is the maximum and another the minimum stress across any plane through that point.

The ellipsoid (1.17) or (1.16) is called the ellipsoid of stress for the point whose principal stresses are P_1, P_2, P_3.

The tensional stress across the oblique plane is the sum of the components of F_1, F_2, F_3, along the normal to the plane. Thus, denoting this tension by P, we get

$$\begin{aligned} P &= lF_1+mF_2+nF_3 \\ &= l^2P_1+m^2P_2+n^2P_3 \quad . \quad . \quad . \quad . \quad . \quad (1.18) \end{aligned}$$

The resultant stress is

$$R=\sqrt{F_1^2+F_2^2+F_3^2}=\sqrt{l^2P_1^2+m^2P_2^2+n^2P_3^2} \quad . \quad (1.19)$$

If $P_2=P_3=P_1$ then, since $l^2+m^2+n^2=1$,

$$P=R=P_1,$$

which shows in another way that the resultant stress across any oblique plane when the three principal stresses are equal is a normal stress of the same magnitude as each principal stress.

8. Shear stress on a plane inclined to two of the principal stresses.

Since we are now dealing with principal stresses we must put $S_3=0$ in equation (1.1), and then we find, for the shear stress on a plane inclined to the principal stresses P_1 and P_2 but parallel to P_3,

$$S=\tfrac{1}{2}(P_2-P_1)\sin 2\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.20)$$

If we vary θ in this it is clear that S has its maximum value when $\sin 2\theta=1$, that is, when $\theta=45°$. Moreover S depends only on the difference of the tensional stresses, so that the addition of the same amount to both stresses would not alter S.

Again, if $P_1=-P_2$, that is, if the principal stresses are equal in magnitude but one is a tension and the other a compression, then

$$S=P_2\sin 2\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.21)$$

and the maximum shear stress is equal to P_2. It is easy to verify this directly. Figs 6 a and 6 b show the related stresses

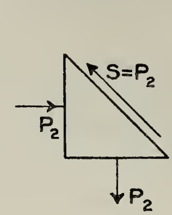

Fig. 6a

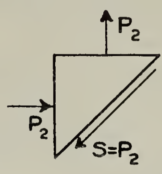

Fig. 6b

9. Stresses on planes which are rotated relatively to OZ. Given the stresses P_1, P_2, S_3, on planes perpendicular to axes OX, OY, it is required to find the stresses on planes perpendicular to axes OX′ OY′, which are in the plane XOY but inclined at an angle θ to the original axes.

Let P'_1, P'_2, S'_3, denote the stresses on the faces perpendicular to the new axes. Since the direction of P'_1 in fig. 7 is parallel to OX′ it follows that the direction of S'_3 is parallel to OY′ on the same face, according to the rule given in Art 2.

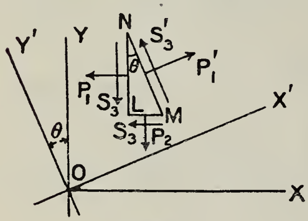

Fig. 7

The triangle LMN (fig. 7) represents a small prismatic block of length δz perpendicular to the plane XOY.

Let A denote the area of the face MN. Then the areas of the faces LN and LM are A cos θ and A sin θ. Thus the force due to P_1 on the face LN has as magnitude $P_1 A \cos\theta$. Similarly the force due to S_3 on the same face has a magnitude $S_3 A \cos\theta$. The forces on the face LM are $P_2 A \sin\theta$ and $S_3 A \sin\theta$. Now resolving all the forces on the block in the direction of OX′, we get, for equilibrium,

$$\begin{aligned} P'_1 A &= (P_1 A \cos\theta)\cos\theta + (P_2 A \sin\theta)\sin\theta \\ &\quad + (S_3 A \cos\theta)\sin\theta + (S_3 A \sin\theta)\cos\theta \\ &= A\{P_1 \cos^2\theta + P_2 \sin^2\theta + 2S_3 \sin\theta\cos\theta\}, \end{aligned}$$

whence
$$P'_1 = P_1 \cos^2\theta + P_2 \sin^2\theta + 2S_3 \sin\theta\cos\theta \quad . \quad . \quad . \quad (1.22)$$

Since OY′ makes an angle $\left(\theta + \frac{\pi}{2}\right)$ with OX we get the tensional stress P'_2 by putting $\left(\theta + \frac{\pi}{2}\right)$ for θ in the expression for P'_1; that is, by putting cos θ for sin θ and -sin θ for cos θ. Thus

$$P'_2 = P_1 \sin^2\theta + P_2 \cos^2\theta - 2S_3 \cos\theta\sin\theta \quad . \quad . \quad . \quad (1.23)$$

Again by resolving the forces on the block in fig. 7 in the direction OY′ the condition for equilibrium is

$$S'_3A = (P_2A \sin\theta)\cos\theta - (P_1A\cos\theta)\sin\theta + (S_3A\cos\theta)\cos\theta - (S_3A\sin\theta)\sin\theta,$$

whence

$$S'_3 = (P_2 - P_1)\sin\theta\cos\theta + S_3(\cos^2\theta - \sin^2\theta) \quad . \quad . \quad . \; (1.24)$$

10. Rotation of the axes in three dimensions.

Suppose the stresses on faces perpendicular to three given rectangular axes OX, OY, OZ, are known. It is required to find the stresses on faces perpendicular to axes OX′, OY′, OZ′, whose direction-cosines relative to the first axes are (l_1, m_1, n_1), (l_2, m_2, n_2), (l_3, m_3, n_3).

Consider the equilibrium of a small tetrahedron which has its faces perpendicular respectively to OX, OY, OZ, OX′. Figure 5 may be taken to represent this tetrahedron. Then OX′ is perpendicular to the face ABC, whose area is a.

Let dashed letters represent the stresses relative to the axes OX′, OY′, OZ′. Then using the results in (1.3), (1.4), (1.5), with l_1, m_1, n_1, for l, m, n, in the expressions for the F's we get, since P'_1 is the sum of the components of F_1, F_2, F_3, in the direction (l_1, m_1, n_1),

$$\begin{aligned} P'_1 &= l_1F_1 + m_1F_2 + n_1F_3 \\ &= l_1^2P_1 + m_1^2P_2 + n_1^2P_3 \\ &\quad + 2m_1n_1S_1 + 2n_1l_1S_2 + 2l_1m_1S_3 \quad . \quad . \; (1.25) \end{aligned}$$

We can write down from symmetry the other two tensional stresses. Thus for P'_2,

$$\begin{aligned} P'_2 &= l_2^2P_1 + m_2^2P_2 + n^2P_3 \\ &\quad + 2m_2n_2S_1 + 2n_2l_2S_2 + 2l_2m_2S_3 \quad . \quad . \; (1.26) \end{aligned}$$

Again, since the sum of the components of F_1, F_2, F_3, in the direction OY′ is S'_3, we get

$$\begin{aligned} S'_3 &= l_2F_1 + m_2F_2 + n_2F_3 \\ &= l_1l_2P_1 + m_1m_2P_2 + n_1n_2P_3 + (m_2n_1 + m_1n_2)S_1 \\ &\quad + (n_2l_1 + n_1l_2)S_2 + (l_2m_1 + l_1m_2)S_3 \quad . \quad . \quad . \quad . \; (1.27) \end{aligned}$$

We should get the expressions for S'_2 by replacing l_2, m_2, n_2, by l_3, m_3, n_3, in the last result; and we should get S'_1 by replacing l_1, m_1, n_1, by l_3, m_3, n_3.

11. Validity of the stress-relations at a point in all cases.

To simplify the reasoning it has been assumed so far that the body under stresses was in equilibrium and was acted on by no body forces such as gravity or magnetic or electrical forces. But the stress relations are not altered if we do take account of such forces, or if we assume that the body is being accelerated. In dealing with the motion or equilibrium of an element of dimensions δx, δy, δz, any of these body forces, or the inertia due to acceleration, introduces a term into the equations proportional to the product $\delta x\delta y\delta z$, whereas the terms due to the stresses are proportional to areas such as $\delta x\,\delta y$, and when δx, δy, and δz, are made infinitely small the terms of the third order vanish in comparison with terms of the second order; that is, the effect of body forces or accelerations vanishes in comparison with the effect of the stresses, so that the final equations contain only stresses and give the same relations as if the body forces or accelerations were not taken into account.

It is necessary to remark that the stress-relations that remain true when body forces or accelerations are taken into account are those relations not involving differences of stresses on parallel faces, for these differences are, of course, of smaller order than the stresses themselves.

CHAPTER II

RELATIONS BETWEEN STRESS AND STRAIN

12. Elasticity.

Although we often speak of rigid bodies there are no absolutely rigid bodies, for every body alters its shape or size under the action of stress. If, for instance, the three principal stresses at every point of a body are all equal tensional stresses, in which case the stress across any plane in the body is also a tensional stress of the same intensity, then every element of the body slightly increases its volume without altering its shape—a spherical element becomes a sphere of larger size, and a cubical element becomes a cube of larger size. Again a rod under the action of a pair of balancing pulls at its ends has is lenght slightly increased by these pulls.

A body whose shape or size is altered by stress is said to be strained. To every kind of *stress* there is a corresponding *strain.* If the stresses are not too great (and the limit of greatness can only by determined by experiment for any particular material), the strained body will recover its original shape and size when the stress is removed. This property which a body possesses of recovering from strain is called *elasticity.* The elasticity is *perfect* if the body recovers completely. Some bodies, such as steel, recover completely after very large stresses, while others, such as cast iron and lead, do not completely recover from comparatively small stresses. The strain that remains when the stress is removed is called *permanent set.* If forces are applied to any rigid body and these forces are gradually increased the body will be perfectly elastic (that is, would recover its original size and shape it the forces ceased to act) until some definite magnitude of these forces is reached, and any further increase of the forces would produce permanent set.

13. Isotropic Bodies.

A body which has the same elastic properties in every direction is called an *isotropic* body. A substance such as wood with a fibral structure has certainly not the same properties in every direction. For instance, if forces are applied to a cube of wood with one pair of faces perpendicular to the fibres, the same tension is not likely to produce the same extension when the tension is along the fibres as when the tension is perpendicular to the fibres.

14. Hooke's Law.

If a pair of balancing pulls are applied at the ends of a rod or string the increase of length of the rod or string is proportional to the pull. This was expressed shortly by Hooke thus:— "the force is proportional to the extension." This is Hooke's Law in its simplest form. The generalised Hooke's law may be expressed in the following way:— the relative displacements of the particles of a body by any given set of forces will be increased in the ratio $n:1$ if the forces are all increased in that same ratio.

15. Young's modulus of elasticity.

Suppose a rectangular block of dimensions $a \times b \times c$ is under the action of a uniform tensional stress over a pair of opposite faces and no other stresses on the six faces, as shown in fig. 8, then it is known by experiment that the length a increases and the lengths b and c decrease. Let the new lengths of the edges of the block be $a+u$, $b-v$, $c-w$. Then the three ratios $\frac{u}{a}$, $\frac{v}{b}$, $\frac{w}{c}$ are called the strains

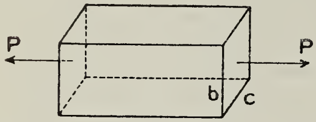

Fig. 8

of the block in the three directions, the first being an extensional strain and the two latter compressive strains. It is clearly convenient to treat a compressive strain as a negative extensional strain. Then, regarding extensional strains as positive, the three strains are $\frac{u}{a}$, $-\frac{v}{b}$, $-\frac{w}{c}$. Let these be denoted by α, β, γ. Then Hooke's law gives

$$P = E\alpha, \qquad (2.1)$$

or

$$\alpha = \frac{P}{E}, \qquad (2.2)$$

E being a constant which is called Young's modulus of elasticity.

16. Poisson's Ratio.

The other experimental fact is that

$$\beta = \gamma = -\sigma\alpha = -\sigma\frac{P}{E} \qquad (2.3)$$

σ being a constant called *Poisson's Ratio* for the material. The statement in words is that "a tension in one direction only causes, in any perpendicular direction, a negative extensional strain bearing a constant ratio to the extensional strain in the direction of the tension".

17. Shear strain.

Suppose a block, originally rectangular, is subjected to equal shear stresses S on opposite faces AB, CD, and the equal shear stresses, which we have shown (Art 2) must accompany these, on the faces

BC, DA, and no other stresses on the six faces of the block. In this case two opposite faces of the block are distorted into parallelograms, the pair of angles at A and C in fig. 9 being decreased. The radian measure of the change of angle θ at one of the corners is called the shear strain, and Hooke's law for this case is expressed by the equation

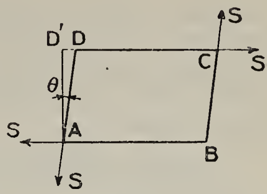

Fig. 9

$$S = n\theta, \quad \text{(2.4)}$$

n being a constant called the *modulus of rigidity*, or sometimes the *shear modulus*, of the material. In practice θ is so small that we may take either $\sin\theta$ or $\tan\theta$ for θ. Then we could express Hooke's law in either of the forms

$$S = n \sin\theta = n\frac{D'D}{AD} \quad \text{(2.5)}$$

or

$$S = n \tan\theta = n\frac{D'D}{AD'}, \quad \text{(2.6)}$$

18. The Bulk Modulus.

Suppose a uniform tensional stress, P, and no shear stress acts on every face of a rectangular block. It has been proved that the only action across every plane in the block is an equal tensional stress. Under such a system of stresses the extensional strain is the same in every direction, and there is no change of shape, but necessarily an increase in volume. Let V be the original volume of the block, and v the increase in volume under the stresses. Then Hooke's law for this case is expressed by the equation

$$P = k\frac{v}{V}, \quad \text{(2.7)}$$

k being a constant called the *bulk modulus* of the material. The ratio $\frac{v}{V}$ may be called the volume-strain or bulk-strain.

19. Strain due to simultaneous stresses.

It is an experimental fact, which must be regarded as fundamental to the subject of elasticity, that, when several stresses exist simultaneously in any element of a body, each stress produces its own strain just as if no other stress were present.

20. Relations between the elastic constants for an isotropic body.

We have defined four elastic constants, namely, E, σ, n, and k. For an isotropic body only two of these are independent, and therefore any two of them can be expressed in terms of the other two. There are many ways of obtaining these relations, but it is probably best to get them by considering simple cases from different points of view, and using the fact, stated in the last article, that simultaneous stresses produce their own strains independently. We shall now consider two special cases.

Consider a cube (fig. 10) each face of which is subjected to a uniform tensional stress P. Let c be the natural length of each edge and a the strain of each edge, so that the new length is $c(1+a)$, and therefore the new volume is $c^3(1+a)^3 = c^3(1+3a)$ neglecting powers of a beyond the first. Then the bulk-strain is $3a$, and consequently

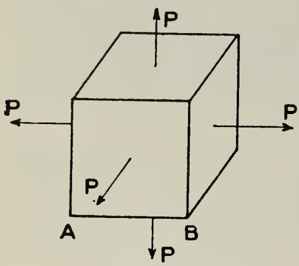

Fig. 10

$$P = 3ka \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.8)$$

If P acted on only one pair of opposite faces the extensional strain perpendicular to those faces would be $\frac{P}{E}$. This is the strain of AB due to the stresses parallel to AB. Again the strain of AB due to the stresses on one of the other pairs of opposite faces is $-\sigma\frac{P}{E}$ and the strain of AB due to the stresses on the third pair of faces is $-\sigma\frac{P}{E}$ again. Hence the total extensional strain of AB is $\frac{P}{E} - 2\sigma\frac{P}{E}$. Therefore

$$\frac{P}{E}(1-2\sigma) = a \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.9)$$

Equating the values of $\frac{P}{a}$ from (2.8) and (2.9) we get

$$3k = \frac{E}{1-2\sigma} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.10)$$

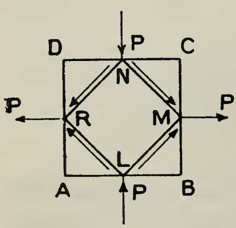

Fig. 11

Consider next the same cubical block (Fig. 11) with a uniform tensional stress P over one pair of opposite faces and an equal uniform compressive stress over another pair of opposite faces and no other stresses. We know by Article 8 that there is a uniform shear stress of intensity P over the faces LM, MN, NR, and RL. The figure was a square in the unstrained state and becomes a parallelogram with unequal diagonals in the strained state. The extensional strain of AB is $\frac{P}{E}$ due to the stresses parallel to AB, and $\sigma\frac{P}{E}$ due to the stresses perpendicular to AB. Thus the total strain of AB (or CD) is an extensional strain of magnitude $\frac{P}{E}(1+\sigma)$. There is also an equal compressive strain in AD. If α be written for $\frac{P}{E}(1+\sigma)$ the new lengths of AB and AD are $c(1+\alpha)$ and $c(1-\alpha)$ respectively. Let θ be the diminution of the angle LRN, so that the actual value of the angle is $(\frac{1}{2}\pi-\theta)$. Now by trigonometry, applied to the triangle LNR,

$$\cos(\tfrac{1}{2}\pi-\theta)=\frac{LR^2+RN^2-LN^2}{2\cdot LR\cdot RN}=\frac{2LR^2-LN^2}{2\cdot LR^2}$$

Now

$$LR^2=AL^2+AR^2,$$
$$LN^2=4\cdot AR^2.$$

Therefore

$$\begin{aligned}2\cdot LR^2-LN^2&=2\{AL^2-AR^2\}\\&=2\{AL-AR\}\{AL+AR\}\\&=2\times c\alpha\times c=2c^2\alpha;\end{aligned}$$

and, neglecting α^2, $LR^2=AL^2+AR^2=\frac{1}{2}c^2$.

Consequently

$$\begin{aligned}\sin\theta&=\cos(\tfrac{1}{2}\pi-\theta)\\&=\frac{2c^2\alpha}{c^2}=2\alpha\end{aligned}$$

This is the shear strain of the parallelogram LMNR, and the shear stress is P. Therefore

$$P=n\sin\theta=2n\alpha$$

or

$$\alpha=\frac{P}{2n}.$$

But

$$\alpha=\frac{P}{E}(1+\sigma),$$

whence

$$2n=\frac{E}{1+\sigma} \qquad (2.11)$$

Equations (2.10) and (2.11) express k and n in terms of E and σ, thus showing that only two of the four constants are independent for an isotropic body.

21. Strain in terms of displacements in two dimensions. We shall suppose that a naturally plane body (or sheet) is strained in such a way that all the particles remain in one plane after the strain. It is necessary to refer all the displacements to a pair of axes fixed relatively to some particles of the body. Let the origin O be situated at one of the particles of the body, and if that particle moves O is supposed to move with it; and let the axis OX always pass through one other given particle of the body. The axis OY is always perpendicular to OX and in the plane of the particles.

Let the particle which was at (x, y) before strain be situated at $(x+u, y+v)$ after strain. Both u and v are functions of x and y since each will vary with both x and y. We have to investigate the change of size and shape of an element, which, in the natural state, is a rectangle with sides δx and δy.

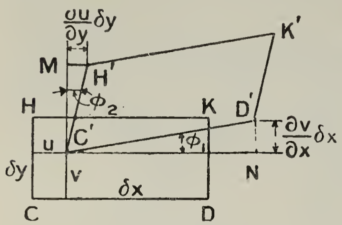

Fig. 12

The rectangle CDKH (fig. 12) is displaced relatively to the axes into the position C′D′K′H′. The displacement of C has components u and v. Now suppose

$$u = f(x, y).$$

Then, denoting the displacement of D parallel to OX by $(u+\delta u)$, we get, since the coordinates of D are $(x+\delta x, y)$,

$$u + \delta u = f(x+\delta x, y).$$

Therefore

$$\begin{aligned}\delta u &= f(x+\delta x, y) - f(x, y)\\ &= \frac{\partial f(x,y)}{\partial x}\delta x \text{ neglecting } (\delta x)^2\\ &= \frac{\partial u}{\partial x}\delta x.\end{aligned}$$

But δu is the displacement of D′ relative to C′ and is therefore the excess of C′N over CD. Now the length of CD is δx, and the horizontal projection of C′D′ is greater than CD by $\frac{\partial u}{\partial x}\delta x$. Hence the extensional strain in the direction of OX is

$$\frac{\text{C}'\text{N} - \text{CD}}{\text{CD}} = \frac{\frac{\partial u}{\partial x}\delta x}{\delta x} = \frac{\partial u}{\partial x}.$$

Likewise the extensional strain in the direction of OY is $\frac{\partial v}{\partial y}$.

Again let

$$v = \text{F}(x, y)$$

and let $(v+\delta v)$ be the value of v for D′. Then

$$v + \delta v = \text{F}(x+\delta x, y),$$

and therefore

$$\begin{aligned}\delta v &= \text{F}(x+\delta x, y) - \text{F}(x, y)\\ &= \frac{\partial \text{F}(x,y)}{\partial x}\delta x\\ &= \frac{\partial v}{\partial x}\delta x.\end{aligned}$$

This δv is the vertical displacement of D′ relative to C′, and is shown as ND′ in fig. 12. Similarly MH′ is $\frac{\partial u}{\partial y}\delta y$.

The shear strain for the lines C′D′ and C′H′ is, by definition, the whole change of angle at C′; that is, the shear strain is $(\varphi_1 + \varphi_2)$. But

$$\varphi_1 = \frac{\text{ND}'}{\text{C}'\text{N}} \text{ approximately}$$
$$= \frac{\partial v}{\partial x},$$

and
$$\varphi_2 = \frac{\partial u}{\partial y}.$$

Therefore the shear strain is

$$\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}.$$

22. Strains in terms of displacements in three dimensions. In the last article we showed that, when a plane body is strained into another plane body, the component extensional strains at a point (x, y) are

$$\frac{\partial u}{\partial x} \text{ parallel to OX}$$

and
$$\frac{\partial v}{\partial y} \text{ parallel to OY.}$$

Also the shear strain for lines parallel to the coordinate axes is

$$\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}.$$

Let us now go into three dimensions and choose our origin at one of the particles, and OX through another particle as before. Also let the plane XOY always pass through a third particle which is at safe distance from OX in the unstrained state. (This distance will be safe if it is greater than the largest value of v.) The axes OY and OZ are always perpendicular to OX and to one another. This completely determines the axes in any state of strain.

Let a particle originally at (x, y, z) move to $(x + u, y + v, z + w)$. The displacements parallel to the $x\ y$ plane are exactly the same as if w were zero, and therefore the extensional strains parallel to OX and OY, as well as the shear strain of the faces perpendicular to the axis OZ, are just the same as in the last article.

It follows then that the three extensional strains are

$$\frac{\partial u}{\partial x}, \frac{\partial v}{\partial y} \text{ and } \frac{\partial w}{\partial z}.$$

Likewise the three component shear strains are

$$\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right), \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) \text{ and } \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right).$$

We shall denote the extensional strains by α, β, γ, respectively, and the shear strains by a, b, c, respectively. That is

$$\alpha = \frac{\partial u}{\partial x}, \ \beta = \frac{\partial v}{\partial y}, \ \gamma = \frac{\partial w}{\partial z}, \quad . \quad . \quad . \quad . \quad . \quad . \ (2.12)$$

$$a = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}, \ b = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}, \ c = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \quad . \quad . \ (2.13)$$

23. Stress-strain relations.

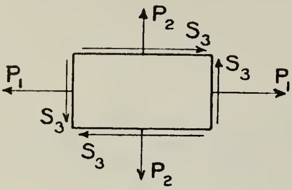

Fig. 13

Fig. 13 represents one view of a small rectangular block under tensional stresses P_1, P_2, P_3, and shear stresses S_1, S_2, S_3. The shear stresses S_1, and S_2, which cause no strains in the plane of the figure, are not shown; and P_3 is not shown since it is perpendicular to the plane of the figure. The strains also are not shown in this figure.

The extensional strain in the direction of P_1 is $\frac{P_1}{E}$ due to P_1, $-\sigma\frac{P_2}{E}$ due to P_2, and $-\sigma\frac{P_3}{E}$ due to P_3. Hence

$$\alpha = \frac{1}{E}\left\{P_1 - \sigma(P_2 + P_3)\right\}; \quad . \quad . \quad . \quad . \quad . \ (2.14)$$

also

$$\beta = \frac{1}{E}\left\{P_2 - \sigma(P_3 + P_1)\right\}, \quad . \quad . \quad . \quad . \quad . \ (2.15)$$

and

$$\gamma = \frac{1}{E}\left\{P_3 - \sigma(P_1 + P_2)\right\}. \quad . \quad . \quad . \quad . \quad . \ (2.16)$$

Again the relations between shear stress and shear strain give

$$S_1 = na = n\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \quad . \quad . \quad . \quad . \quad . \quad . \ (2.17)$$

$$S_2 = nb = n\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) \quad . \quad . \quad . \quad . \quad . \quad . \ (2.18)$$

$$S_3 = nc = n\left(\frac{\partial u}{dy} + \frac{\partial v}{\partial x}\right) \quad . \quad . \quad . \quad . \quad . \quad . \ (2.19)$$

24. Tensional stresses in terms of strains.

Equations (2.14), (2.15), (2.16), give strains in terms of stresses. The equations can be solved for the stresses P_1, P_2, P_3, in terms of α, β, γ. Adding the three equations we get

$$\alpha + \beta + \gamma = \frac{1 - 2\sigma}{E}(P_1 + P_2 + P_3),$$

whence

$$P_1 + P_2 + P_3 = \frac{E}{1 - 2\sigma}(\alpha + \beta + \gamma)$$
$$= 3k(\alpha + \beta + \gamma) \text{ by equation (2.10).}$$

Now the strained volume of a small block of natural dimensions δx, δy, δz, is

$$\delta x(1 + \alpha) \times \delta y(1 + \beta) \times \delta z(1 + \gamma) = \delta x \delta y \delta z(1 + \alpha + \beta + \gamma)$$

neglecting products of α, β and γ. Then it follows that the ratio of the increase of volume to the original volume is $(\alpha + \beta + \gamma)$. Call this $\triangle$. Then

$$P_1 + P_2 + P_3 = 3k\triangle \quad \ldots \quad (2.20)$$

Therefore

$$P_2 + P_3 = 3k\triangle - P_1,$$

and substituting this in equation (2.14) that equation gives

$$E\alpha = P_1 - \sigma(3k\triangle - P_1)$$
$$= P_1(1 + \sigma) - 3\sigma k\triangle,$$

whence

$$P_1 = \frac{3\sigma k}{1 + \sigma}\triangle + \frac{E}{1 + \sigma}\alpha$$
$$= (m - n)\triangle + 2n\alpha$$

where m and n are constants, n being the modulus of rigidity.

In terms of E and σ

$$m - n = \frac{\sigma E}{(1 + \sigma)(1 - 2\sigma)},$$

and thus

$$m = \frac{2\sigma n}{1 - 2\sigma} + n = \frac{n}{1 - 2\sigma} \quad \ldots \quad (2.21)$$

Consequently the three equations for the tensional stresses in terms of strains are

$$\left.\begin{aligned} P_1 &= (m - n)\triangle + 2n\alpha \\ P_2 &= (m - n)\triangle + 2n\beta \\ P_3 &= (m - n)\triangle + 2n\gamma \end{aligned}\right\} \quad \ldots \quad (2.22)$$

where

$$\triangle = \alpha + \beta + \gamma$$
$$= \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \quad \ldots \quad (2.23)$$

25. Relations between stresses and external forces.

We shall deal with a rectangular block of dimensions $\delta x \times \delta y \times \delta z$ having its centre at x, y, z, and take the means stresses on the faces to be the stresses at the middle of the face concerned. The middle points of the two faces perpendicular to the axis of x have coordinates

$(x \pm \frac{1}{2}\delta x, y, z)$, and the middle points of the other faces are at $(x, y \pm \frac{1}{2}\delta y, z)$ and $(x, y, z \pm \frac{1}{2}\delta z)$.

Let the body force per unit mass at (x, y, z) have components X, Y, Z, and let ρ be the density of the body. The mass of the block being $\rho\,\delta x\,\delta y\,\delta z$, the body force on the block has components $\rho X\,\delta x\,\delta y\,\delta z$, $\rho Y\,\delta x\,\delta y\,\delta z$, $\rho Z\,\delta x\,\delta y\,\delta z$. Suppose also that the element has component accelerations f_1, f_2, f_3.

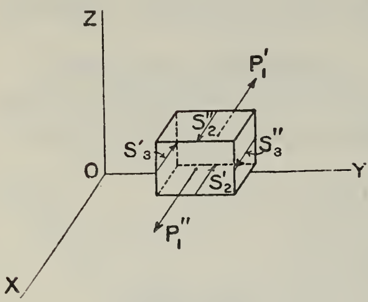

Fig. 14

Fig. 14 shows a perspective view of the block, and only those stresses are indicated that act parallel to the x-axis. A similar figure for the y direction would show P'_2, P''_2, and also the shear stresses S'_1, S''_1, S'_3, S''_3, all acting parallel to OY.

Now let P_1, P_2, P_3, S_1, S_2, S_3 denote the values of the stresses at (x, y, z). Then since P''_1 is supposed to act at $(x + \frac{1}{2}\delta x, y, z)$ we find, neglecting powers of δx beyond the first,

$$P''_1 = P_1 + \frac{\partial P_1}{\partial x} \times (\tfrac{1}{2}\delta x).$$

Also, since P'_1 acts at $(x - \frac{1}{2}\delta x, y, z)$,

$$P'_1 = P_1 + \frac{\partial P_1}{\partial x}(-\tfrac{1}{2}\delta x).$$

Hence
$$P''_1 - P'_1 = \frac{\partial P_1}{\partial x}\delta x.$$

Again, since S'_3 and S''_3 are supposed to act at $(x, y - \frac{1}{2}\delta y, z)$ and $(x, y + \frac{1}{2}\delta y, z)$, it follows in the same way that

$$S''_3 - S'_3 = \frac{\partial S_3}{\partial y}\delta y.$$

Also
$$S''_2 - S'_2 = \frac{\partial S_2}{\partial z}\delta z.$$

The force on the block in the direction OX due to P'_1 and P''_1 is

$$(P''_1 - P'_1)\ \delta y\,\delta z = \frac{\partial P_1}{\partial x}\delta x\,\delta y\,\delta z.$$

Similarly the forces in the direction OX due to the stresses S'_2, S''_2 and S'_3, S''_3 are

$$(S''_2 - S'_2)\,\delta x\,\delta y = \frac{\partial S_2}{\partial z}\,\delta x\,\delta y\,\delta z,$$

and
$$(S''_3 - S'_3)\,\delta x\,\delta z = \frac{\partial S_3}{\partial y}\,\delta x\,\delta y\,\delta z.$$

Then the total component action of the stresses on the faces of the block in the direction OX is

$$\left(\frac{\partial P_1}{\partial x} + \frac{\partial S_3}{\partial y} + \frac{\partial S_2}{\partial z}\right)\delta x\,\delta y\,\delta z.$$

Since the component of the body force on this element is $\varrho X\,\delta x\,\delta y\,\delta z$ and the component acceleration is f_1 the equation of motion of the element is

$$\left\{\frac{\partial P_1}{\partial x} + \frac{\partial S_3}{\partial y} + \frac{\partial S_2}{\partial z} + \varrho X\right\}\delta x\,\delta y\,\delta z = (\varrho\,\delta x\,\delta y\,\delta z)f_1,$$

whence

$$\frac{\partial P_1}{\partial x} + \frac{\partial S_3}{\partial y} + \frac{\partial S_2}{\partial z} + \varrho X = \varrho f_1 \quad \text{. . . . . (2.24)}$$

The two corresponding equations, obtained by resolving parallel to the other axes, are

$$\frac{\partial P_2}{\partial y} + \frac{\partial S_1}{\partial z} + \frac{\partial S_3}{\partial x} + \varrho Y = \varrho f_2 \quad \text{. . . . . (2.25)}$$

$$\frac{\partial P_3}{\partial z} + \frac{\partial S_2}{\partial x} + \frac{\partial S_1}{\partial y} + \varrho Z = \varrho f_3 \quad \text{. . . . . (2.26)}$$

If the body is in equilibrium the accelerations f_1, f_2, f_3, are zero. If, however, the body is not in equilibrium but nevertheless the three coordinate axes—which, it must be recalled, are determined by means of three particles in the body—are themselves at rest, then the displacements u, v, w, are true displacements in space, and therefore

$$f_1 = \frac{\partial^2 u}{\partial t^2},\ f_2 = \frac{\partial^2 v}{\partial t^2},\ f_3 = \frac{\partial^2 w}{\partial t^2} \quad \text{. . . . . (2.27)}$$

t denoting the time measured from any particular instant.

These last forms of the accelerations are applicable when we are dealing with small oscillations of an elastic body.

The equations (2.24), (2.25), (2.26), will be called the equations of equilibrium or of motion according as the f's are all zero or not all zero.

26. Equations of motion in terms of displacements.

Since
$$P_1 = (m - n)\,\triangle + 2n\frac{\partial u}{\partial x},$$

$$S_3 = n\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right),$$

$$S_2 = n\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right),$$

and
$$\triangle = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z},$$

then equation (2.23) becomes

$$(m-n)\frac{\partial \triangle}{\partial x} + 2n\frac{\partial^2 u}{\partial x^2} + n\left\{\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 w}{\partial z \partial x} + \frac{\partial^2 u}{\partial z^2}\right\} + \varrho X = \varrho f_1,$$

or
$$m\frac{\partial \triangle}{\partial x} + n\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + \varrho X = \varrho f_1.$$

This equation is written

$$m\frac{\partial \triangle}{\partial x} + n\nabla^2 u + \varrho X = \varrho f_1 \quad . \quad . \quad . \quad . \quad . \quad (2.28)$$

The two corresponding equations for the y and z directions are

$$m\frac{\partial \triangle}{\partial y} + n\nabla^2 v + \varrho Y = \varrho f_2 \quad . \quad . \quad . \quad . \quad . \quad (2.29)$$

$$m\frac{\partial \triangle}{\partial z} + n\nabla^2 w + \varrho Z = \varrho f_3 \quad . \quad . \quad . \quad . \quad . \quad (2.30)$$

27. Relations between the six stresses.

Since the six stresses can be expressed in terms of the three displacements u, v, w, and their rates of change in space, it follows that there must be three independent relations between the six stresses. There would be no great difficulty in deducing such relations, but they would not be of much use when they were deduced. When the three equations of motion are expressed in terms of stresses these equations contain six unknown stresses, whereas in terms of displacements there are only three unknown displacements. It is for this reason that the equations in terms of displacements are more useful than those in terms of stresses, although the three additional equations due to the three relations between the stresses would give us the requisite number of equations from which to find the six unknowns. Nevertheless, three equations and three unknowns are much preferable to six equations and six unknowns. When, however, we are dealing with a problem where the coordinate axes are known to be the principal axes for every point of the body, then our stress equations contain only three unknown stresses, and these equations may in such a case be preferable to the equations in terms of displacements, particularly as these equations give stresses directly, which are the things we are usually aiming to find. Moreover, we may make the six stresses satisfy any three arbitrary equations and then find the six stresses from these three equations and the other three relations that we know always exist between the stresses. In this way we get the solution to some problem in elasticity. The three permanent relations between the stresses can be got by eliminating u, v, w, from equations (2.17) (2.18) (2.19), and (2.22).

28. Solution of problems in elasticity.

The object of a problem in elasticity is usually to find the stresses in a body, and in some cases to find the strains due to given body forces and given conditions at the boundary of the body.

The stresses and strains are completely determined by means of equations (2.28), (2.29), and (2.30), if, at the same time, the conditions at the boundary of the body are given. If the theory of differential equations had been carried far enough it should be possible to write down, from equations (2.28), (2.29) and (2.30) alone, values of u, v, w, corresponding to given values of X, Y, Z, and f_1, f_2, f_3. These expressions would contain arbitrary functions of x, y, z, the form of which functions would be determined by the known conditions at the boundary of the body. Unfortunately, however, pure mathematics has not reached the stage of solving these equations in general terms, and therefore we have to be content very often with the reverse process of finding any solutions of the equations and then finding out the problems of which they are the solutions. That is, the real problem is; given the forces X, Y, Z, and the accelerations f_1, f_2, f_3, calculate the displacements and consequently the stresses. The easier problem is; assume some displacements and calculate the forces. Luckily many of the most important problems have comparatively easy solutions.

29. The equation for the volume strain $\triangle$.

If we are dealing with a problem of equilibrium, so that f_1, f_2, f_3, are all zero, we can get a differential equation for $\triangle$ alone. Assume that the accelerations are zero and differentiate (2.28), (2.29), (2.30), with respect to x, y, z, respectively, and add the corresponding sides of the resulting equations. Then since

$$\begin{aligned}\frac{\partial}{\partial x}(\nabla^2 u) &= \frac{\partial}{\partial x}\left\{\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right\} \\ &= \left\{\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right\}\frac{\partial u}{\partial x} \\ &= \nabla^2\frac{\partial u}{\partial x}\end{aligned}$$

the final equation reduces to

$$(m+n)\nabla^2\triangle+\frac{\partial(\varrho X)}{\partial x}+\frac{\partial(\varrho Y)}{\partial y}+\frac{\partial(\varrho Z)}{\partial z}=0 \quad . \quad . \quad (2.31)$$

and when X, Y, Z, are given as functions of the position x, y, z, this equation contains only the one unknown $\triangle$.

When $\triangle$ has been found from equation (2.31) then u, v, w, can be found in turn from (2.28), (2.29), (2.30).

30. Stresses due to a number of different forces acting simultaneously.

When several forces act on a body each particular force or system of forces produces its own stresses and strains exactly as if none of the other forces acted. This means that the total stress at any point is merely the sum of the stresses due to each force acting separately. For this purpose an acceleration must be treated as if it were one of the acting forces because it is clear from equations (2.23), (2.24), (2.25) that, if no forces act on any portion of an elastic body, there will be stresses in that portion if the accelerations f_1, f_2, f_3, are not all zero. Moreover, forces of given magnitude at the boundary of a body must be treated in the same way as forces X, Y, Z, acting inside the body. But forces arising at the boundary, due to fixing that boundary, need not be treated as forces producing stresses since they are themselves merely stresses caused by other forces and proportional to those other forces.

CHAPTER III

SOME PARTICULAR SOLUTIONS OF THE EQUATIONS OF EQUILIBRIUM

31. Recapitulation of equations.

As it is a great convenience to have all our important equations gathered together those we have already proved are collected below.

Equations of motion

$$\left.\begin{aligned} m\frac{\partial\triangle}{\partial x}+n\nabla^2 u+\varrho X&=\varrho f_1,\\ m\frac{\partial\triangle}{\partial y}+n\nabla^2 v+\varrho Y&=\varrho f_2,\\ m\frac{\partial\triangle}{\partial z}+n\nabla^2 w+\varrho Z&=\varrho f_3,\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (3.1)$$

$$\text{where } \triangle=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z},$$

$$\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}.$$

The equations of equilibrium, obtained by putting zero for each of the component accelerations f_1, f_2, f_3, in equations (3.1), are

$$\left.\begin{aligned} m\frac{\partial\triangle}{\partial x}+n\nabla^2 u+\varrho X&=0\\ m\frac{\partial\triangle}{\partial y}+n\nabla^2 v+\varrho Y&=0\\ m\frac{\partial\triangle}{\partial z}+n\nabla^2 w+\varrho Z&=0\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (3.2)$$

The relations between the stresses and strains, the latter being expressed in terms of the space variations of the displacements, are

$$\left.\begin{aligned} P_1&=(m-n)\triangle+2n\frac{\partial u}{\partial x}\\ P_2&=(m-n)\triangle+2n\frac{\partial v}{\partial y}\\ P_3&=(m-n)\triangle+2n\frac{\partial w}{\partial z}\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (3.3)$$

$$\left.\begin{aligned} S_1 &= n\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) \\ S_2 &= n\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \\ S_3 &= n\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (3.4)$$

Equations (3.3) express the tensional stresses in terms of the extensional strains. The same relations among these six quantities can be shown by equations giving the strains in terms of the stresses, as is done in equations (2.14), (2.15), (2.16). These equations are

$$\left.\begin{aligned} E\frac{\partial u}{\partial x} &= P_1 - \sigma(P_2 + P_3) \\ E\frac{\partial v}{\partial y} &= P_2 - \sigma(P_3 + P_1) \\ E\frac{\partial w}{\partial z} &= P_3 - \sigma(P_1 + P_2) \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \quad (3.5)$$

Adding these last three equations we get

$$E\triangle = (1 - 2\sigma)(P_1 + P_2 + P_3),$$

or

$$P_1 + P_2 + P_3 = \frac{E}{1 - 2\sigma}\triangle = 3k\triangle, \quad . \; . \; . \quad (3.6)$$

k being the bulk-modulus.

Relations between the elastic constants.

$$\left.\begin{aligned} 3k &= \frac{E}{1 - 2\sigma} \\ 2n &= \frac{E}{1 + \sigma} \\ m &= \frac{n}{1 - 2\sigma} \\ \sigma &= \frac{m - n}{2m} \\ E &= \frac{n(3m - n)}{m} \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \quad (3.7)$$

We shall now turn to some of the simplest solutions of the equations of equilibrium and find the stresses corresponding to them, as well as the body forces X, Y, Z.

32. Homogeneous strain.

Assume that the body is in equilibrium and that the displacements are

$$u = ax, \; v = by, \; w = cz, \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (3.8)$$

where a, b, c, are constants.

Then $$\frac{\partial u}{\partial x}=a,\ \frac{\partial v}{\partial y}=b,\ \frac{\partial w}{\partial z}=c,\ \triangle=a+b+c, \quad . \quad . \quad . \quad (3.9)$$

$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial z}=\frac{\partial v}{\partial x}=\frac{\partial v}{\partial z}=\frac{\partial w}{\partial y}=\frac{\partial w}{\partial x}=0 \quad . \quad . \quad . \quad (3.10)$$

Therefore

$$P_1=(m-n)(a+b+c)+2na$$
$$=(m+n)a+(m-n)(b+c) \quad . \quad . \quad . \quad . \quad . \quad (3.11)$$
$$P_2=(m+n)b+(m-n)(c+a) \quad . \quad . \quad . \quad . \quad . \quad (3.12)$$
$$P_3=(m+n)c+(m-n)(a+b) \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.13)$$
$$S_1=S_2=S_3=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.14)$$

Thus P_1, P_2, P_3, are principal stresses and are constant everywhere.

Equations (3.2) give

$$X=0,\ Y=0,\ Z=0,$$

so that the body forces are zero. Then the only forces on the body are forces at the boundary, and if the body is bounded by faces perpendicular to the axes there must be pulls across these faces of amounts P_1, P_2, P_3, per unit area.

33. Two tensions zero.

The simplest case of homogeneous strain, from a physical point of view, is the one in which $P_2=0$ and $P_3=0$. These conditions require that

$$(m+n)b+(m-n)(c+a)=0$$
$$(m+n)c+(m-n)(a+b)=0$$

whence, by subtraction,

$$2n(b-c)=0$$

that is, $$b=c$$

and therefore $$b=c=-\frac{m-n}{2m}a=-\frac{1-\frac{n}{m}}{2}a$$

$$=-\frac{1-(1-2\sigma)}{2}a$$

$$=-\sigma a \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.15)$$

Thus the three displacements are

$$u=ax;\ v=-\sigma ay;\ w=-\sigma az; \quad . \quad . \quad . \quad . \quad (3.16)$$

and the strains

$$\alpha=a;\ \beta=-\sigma a;\ \gamma=-\sigma a; \quad . \quad . \quad . \quad . \quad (3.17)$$

which only show that the mathematics agrees with our original assumptions as expressed by equation (2.3).

34. The three tensions equal.

Another simple case occours when

$$P_1=P_2=P_3, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.18)$$

and therefore

$$a = b = c, \quad \ldots \ldots \quad (3.19)$$

whence

$$P_1 = (3m - n)\,a = \left(\frac{3}{1 - 2\sigma} - 1\right) na = \frac{2(1+\sigma)}{1 - 2\sigma} na$$

$$= \frac{3k}{n} na = 3ka \quad \ldots \ldots \quad (3.20)$$

This is the case where the stress is a pure tension of the same intensity across every plane in the body. There must, of course, be a tension $3ka$ across unit area all over the boundary of the solid. If the P's are negative, and therefore a negative, the stress is a hydrostatic thrust.

35. The stress zero in one direction and the strain zero in a perpendicular direction.

Another useful case is the one where

$$P_3 = 0 \text{ and } b = 0 \quad \ldots \ldots \quad (3.21)$$

Equations (3.11), (3.12), (3.13), now give

$$P_1 = (m + n)a + (m - n)c$$
$$P_2 = (m - n)(c + a)$$
$$0 = (m + n)c + (m - n)a$$

The last of these gives a relation between a and c, by means of which we can express P_1 and P_2 in terms of a alone. Thus

$$P_1 = (m + n)\,a - (m - n)\frac{m - n}{m + n}\,a$$

$$= \frac{4mn}{m + n}\,a$$

$$= \frac{4n}{1 + \dfrac{n}{m}}\,a$$

$$= \frac{4n}{2 - 2\sigma}\,a$$

$$= \frac{E}{1 - \sigma^2}\,a; \quad \ldots \ldots \quad (3.22)$$

and

$$P_2 = (m - n)\left(1 - \frac{m - n}{m + n}\right) a$$

$$= \frac{2n(m - n)}{m + n}\,a$$

$$= \frac{E}{1 + \sigma}\;\frac{1 - (1 - 2\sigma)}{1 + (1 - 2\sigma)}\,a$$

$$= \frac{\sigma E}{1 - \sigma^2}\,a = \sigma P_1 \quad \ldots \ldots \quad (3.23)$$

Also
$$c = -\frac{m-n}{m+n}a$$
$$= -\frac{1-(1-2\sigma)}{1+(1-2\sigma)}a$$
$$= -\frac{\sigma}{1-\sigma}a \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3\cdot24)$$

This last case would be useful if we were dealing with the stretching of thin sheets by a pull in one direction, the length in a perpendicular direction being kept constant by some means. The stresses P_1 and P_2 are parallel to the faces of the sheet.

36. A prism (or rod) hanging vertically under its own weight.

The problem before us is to find a solution of the equations of equilibrium which satisfy the following conditions, if possible:—

$$X = \text{constant},\ P_2 = 0,\ P_3 = 0:$$
$$S_1 = S_2 = S_3 = 0;$$
$$\left.\begin{array}{l} u = 0 \\ v = 0 \\ w = 0 \end{array}\right\} \text{ at the origin;}$$
$$P_1 = 0 \text{ where } x = l,$$

l being the length of the rod.

It is easy to verify that the following displacements make the shear stresses S_1, S_2, S_3, all zero.

$$\left.\begin{array}{l} u = ax^2 + bx + c(y^2 + z^2) \\ v = -2cxy + hy \\ w = -2cxz + hz \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad (3\cdot25)$$

Also the displacements are zero at the origin.

Now
$$\frac{\partial u}{\partial x} = 2ax + b$$
$$\frac{\partial v}{\partial y} = -2cx + h$$
$$\frac{\partial w}{\partial z} = -2cx + h$$
$$\triangle = 2(a-2c)x + b + 2h$$
$$\nabla^2 u = 2(a+2c)$$
$$\frac{\partial \triangle}{\partial x} = 2(a-2c)$$

Therefore
$$-\rho X = m\frac{\partial \triangle}{\partial x} + n\nabla^2 u$$
$$= 2a(m+n) - 4c(m-n) \quad . \quad . \quad . \quad (3\cdot26)$$

Also
$$-\varrho Y = m\frac{\partial\Delta}{\partial y} + n\nabla^2 v$$
$$= 0, \quad \text{(3.27)}$$
and
$$-\varrho Z = 0. \quad \text{(3.28)}$$
The stresses are
$$P_1 = (m-n)\Delta + 2n\frac{\partial u}{\partial x}$$
$$= 2x\{(m+n)a - 2(m-n)c\} + (m+n)b + 2(m-n)h \quad \text{(3.29)}$$
$$P_2 = (m-n)\Delta + 2n\frac{\partial v}{\partial y}$$
$$= 2x\{(m-n)a - 2mc\} + (m-n)b + 2mh. \quad \text{(3.30)}$$
$$P_3 = P_2 \quad \text{(3.31)}$$
To make P_2 and P_3 each zero at all points it is necessary that
$$2mc = (m-n)a, \quad \text{or } c = \sigma a \quad \text{(3.32)}$$
and
$$2mh = -(m-n)b, \quad \text{or } h = -\sigma b \quad \text{(3.33)}$$
Using these to express c and h in terms of a and b we find that
$$P_1 = \frac{2n(3m-n)}{m}ax + \frac{n(3m-n)}{m}b \quad \text{(3.34)}$$
The condition that $P_1 = 0$ when $x = l$ now gives
$$b = -2al \quad \text{(3.35)}$$
Then
$$P_1 = -\frac{2n(3m-n)}{m}a(l-x) = -2Ea(l-x) \quad \text{(3.36)}$$
and
$$\varrho X = -\frac{2n(3m-n)}{m}a = -2Ea \quad \text{(3.37)}$$

If we now write ϱg for ϱX, which, in the present problem is the weight of unit volume of the solid, then
$$P_1 = \varrho g(l-x),$$
which is the weight of a column of the material of unit area extending from the point (x, y, z) to the lower free end of the solid where $x = l$.

The displacements in terms of the one constant a are
$$u = ax^2 - 2alx + \sigma a(y^2 + z^2)$$
$$= a\{x^2 - 2lx + \sigma(y^2 + z^2)\} \quad \text{(3.38)}$$
$$v = 2\sigma ay(l-x) \quad \text{(3.39)}$$
$$w = 2\sigma az(l-x) \quad \text{(3.40)}$$

The stress P_1 is constant over the top surface of the solid where $x = 0$, and its value could, of course, have been found by simply considering the weight supported by the stress. The vertical displacement at this upper surface is, however,
$$u = \sigma a(y^2 + z^2) = -\frac{\sigma\varrho g}{2E}(y^2 + z^2) \quad \text{(3.41)}$$

If we write r for the distance of a particle of the top surface from the x-axis then the displacement of the particle is

$$u_0 = \sigma a r^2 = -\frac{\sigma \varrho g}{2E} r^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.42)$$

Regarding u_0 as the x-coordinate this is the equation to the surface into which the originally plane top surface is bent. It is the surface of revolution obtained by rotating the parabola

$$x = -\frac{\sigma \varrho g}{2E} y^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.43)$$

about the x-axis.

The radius of curvature of this parabola at the origin is

$$\frac{1}{2\sigma a} = \frac{E}{\sigma \varrho g} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.44)$$

which is very large for most materials. Unless the area of the top surface of the solid is very great the parabola may be regarded as an arc of a circle with the above radius of curvature as its radius.

37. The same problem from another point of view.

The same problem will now be treated in a much simpler way. It has been solved rigorously for the sake of showing how the equations of equilibrium can be used, and at the same time of showing what sort of error there is in the usual simple method of treating such a problem.

Let P_1 denote the *mean* tensional stress across a horizontal plane at distance x from the top end, and let u denote the mean displacement of that plane of particles in the downward direction. Then, since the total tension across the plane must balance the weight below the plane, we get, denoting the area of the section by A,

$$P_1 A = \varrho g A(l - x) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.45)$$

or

$$P_1 = \varrho g(l - x) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.46)$$

Also, assuming that the same relation between mean stress and mean strain exists as between actual stress and actual strain, we get

$$E\frac{du}{dx} = P_1 = \varrho g(l - x), \quad . \quad . \quad . \quad . \quad . \quad (3.47)$$

whence

$$u = \frac{\varrho g}{E}(lx - \tfrac{1}{2}x^2), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.48)$$

no constant being needed in the integration because $u = 0$ when $x = 0$.

In this case u is a function of x only since u is the mean displacement for all values of y and z and one value of x. The present method cannot, therefore, show any variation of u with y or z such as we got by the previous method. It will be observed, however, that the value of u given by the present solution is the same as that along the axis of x (where $y = z = 0$) given by the last solution. Moreover, at points where x is large compared with y or z, the present value of u is practically the same as the more exact value of u. Then it follows that the present solution is sufficiently exact for a thin rod.

The exact solution shows that, to make P_1 constant, the top surface cannot remain plane. If the solution were altered so as to make the top surface plane after the strain then P_1 would not be constant, and, in addition, shear stresses would be introduced on the planes parallel to the coordinate planes. These shear stresses would have to be zero at the vertical faces of the solid, and, in the case of a thin rod, they would be small compared with P_1 everywhere except near the lower end where P_1 itself would be small.

38. Straight rod bent into a circular arc.

Another very interesting problem is given by the displacements

$$\left.\begin{aligned} u &= axz \\ v &= -\sigma a y z \\ w &= -\frac{a}{2}(x^2+\sigma z^2-\sigma y^2) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3{\cdot}49)$$

Here

$$\frac{\partial u}{\partial x}=az,\ \frac{\partial v}{\partial y}=-\sigma a z,\ \frac{\partial w}{\partial z}=-\sigma a z, \quad . \quad . \quad . \quad (3{\cdot}50)$$

$$\triangle = a\ (1-2\sigma)z \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3{\cdot}51)$$

$$\nabla^2 u=0,\ \nabla^2 v=0,\ \nabla^2 w=-a \quad . \quad . \quad . \quad (3{\cdot}52)$$

Therefore

$$-\rho X = m\frac{\partial\triangle}{\partial x}+n\nabla^2 u=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (3{\cdot}53)$$

$$-\rho Y = m\frac{\partial\triangle}{\partial y}+n\nabla^2 v=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (3{\cdot}54)$$

$$\begin{aligned} -\rho Z &= m\frac{\partial\triangle}{\partial z}+n\nabla^2 w \\ &= a\left\{(1-2\sigma)m-n\right\} \\ &= 0 \qquad \text{by equations } (3{\cdot}7) \end{aligned} \quad . \quad . \quad . \quad (3{\cdot}55)$$

Thus the three body forces are zero.

Again

$$\begin{aligned} P_1 &= (m-n)\triangle+2n\frac{\partial u}{\partial x} \\ &= \left\{(1-2\sigma)(m-n)+2n\right\}az \\ &= \left\{2\sigma n+2n\right\}az \\ &= 2naz(1+\sigma)=Eaz \end{aligned} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3{\cdot}56)$$

$$\begin{aligned} P_2 &= \left\{(1-2\sigma)(m-n)-2\sigma n\right\}az \\ &= 0 \end{aligned} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3{\cdot}57)$$

and

$$\begin{aligned} P_3 &= \left\{(1-2\sigma)(m-n)-2\sigma n\right\}az \\ &= 0 \end{aligned} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3{\cdot}58)$$

Also

$$\left.\begin{aligned} S_1 &= n\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) \\ &= na(\sigma y - \sigma y) = 0 \\ S_2 &= n\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \\ &= na(x - x) = 0 \\ S_3 &= n\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) = 0 \end{aligned}\right\} \quad \ldots \ldots \quad (3{\cdot}59)$$

Thus the only stress that is not zero is P_1, and this is

$$P_1 = Eaz \quad \ldots \ldots \quad (3{\cdot}60)$$

Now let us suppose that our equations apply to a rod of uniform section, and we will suppose that, before the strain, the x-axis was the line passing through the centres of inertia of the normal sections of the rod. The line in any cross section where the stress P_1 is zero is called the *neutral axis* of the section Thus the y-axis is the neutral axis of the section in fig 15. Let dA denote the element of area of the cross section at distance z from the axis of y, the strip PQ in fig. 15. Then the total tension across this section is

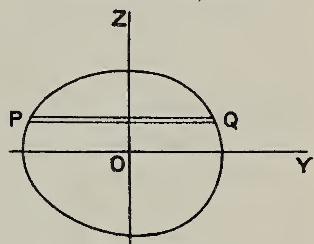

Fig. 15

$$T = \int P_1\, dA = \int Eaz\,dA = Ea\int z\,dA \quad \ldots \ldots \quad (3{\cdot}61)$$

But, by the equation for the position of the centre of inertia of an area,

$$\int z\,dA = \bar{z}A,$$

and in our case $\bar{z} = 0$ by our hypothesis that the x-axis passes through the centre of inertia. Consequently

$$T = 0.$$

Taking moments about OY of the tensions on the elements of area, we get, as the total moment,

$$\begin{aligned} M &= \int z P_1\, dA \\ &= Ea\int z^2 dA \\ &= EIa \quad \ldots \ldots \quad (3{\cdot}62) \end{aligned}$$

where I denotes the moment of inertia of the area about OY.

If we take strips of area parallel to the z-axis and denote one of these strips by dA the total moment of the tensions about OZ is

$$\begin{aligned} M' &= -\int y P_1 dA \\ &= -Ea\int yz\,dA \\ &= -EaI_{yz} \quad \ldots \ldots \quad (3{\cdot}63) \end{aligned}$$

where I_{yz} denotes the *product of inertia* of the area relative to the axes OY, OZ.

If OY, OZ, are principal axes of inertia at O of the cross section then I_{yz} is zero, and therefore $M' = 0$. If either OY or OZ is an axis of symmetry of the cross section then these two axes are principal axes of inertia, for it is clear in this case that I_{yz} is zero, because the axis of symmetry divides the area into two portions in one of which the integral $\int yz dA$ is positive, and in the other of which it has an equal negative value.

We see then that the assumed displacements correspond to a tensional stress across normal sections of the rod, which stress is proportional to the distance of the element of the section from the xy plane. These stresses are positive, that is, tensional, when z is positive, and negative, that is, compressive, when z is negative.

Also if OY, OZ, are principal axes of inertia of the section these stresses are equivalent to a couple of magnitude EIa, and this couple is constant along the rod. This state of the rod could, then, be produced by applying a pair of balancing couples at the ends of the rod, the stress system across any section on the portion of the rod on one side of that section being the action necessary to balance the couple at the end of that portion.

Let us consider again the case where OY is the neutral axis but not a principal axis of inertia of the section. We have already found, in equations (3.62) and (3.63), one pair of component couples acting across the section. The axes of these couples are the axes of OY and OZ, and the resultant couple across the section can be obtained by adding vectors of magnitude M and M′ drawn along OY and OZ respectively. The method consists merely in representing a couple by a vector perpendicular to its plane, and the right-handed screw system is understood here.

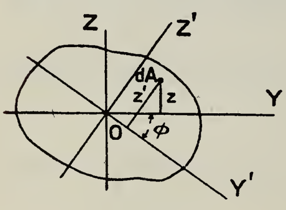

Fig. 16

The resultant couple across the section can, however, be represented by a more useful pair of components than the pair in equations (3.62) and (3.63). These more useful components have as axes the principal axes of inertia of the section. Let OY′, OZ′ (fig. 16) be the principal axes of inertia of the section. Then, if y', z', are the coordinates of an element of area dA referred to the new axes, the moment about OY′ of the stresses across the section is

$$M_{y'} = \int z' P_1 dA$$
$$= Ea \int z' z dA$$

But

$$z = z' \cos \varphi - y' \sin \varphi.$$

Therefore

$$M_{y'} = Ea \int (z'^2 \cos \varphi - z' y' \sin \varphi) dA$$
$$= Ea \cos \varphi \int z'^2 dA$$

since $$\int z'y' \, dA = 0$$

because the new axes are principal axes. If we write $I_{y'}$, for the moment of inertia of the section about OY′ then

$$M_{y'} = EI_{y'} \, a \cos \varphi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.64)$$

On comparing this with the result in (3.62) we see that the component couple about a principal axis, which is inclined at the angle φ with the neutral axis, has the same magnitude as if the principal axis itself were the neutral axis and as if at the same time a were replaced by $a \cos \varphi$. It is shown in the next article that a represents the curvature of the bent rod. Then if OY′ were the neutral axis $M_{y'}$ would be the couple corresponding to a curvature $a \cos \varphi$ in a plane perpendicular to OY′. It can also be shown that $a \cos \varphi$ is the curvature of the curve we should get by projecting the actual curve of the rod on the plane perpendicular to OY′.

The component couple about OZ′ is clearly

$$M_{z'} = EI_{z'} \, a \sin \varphi.$$

If we are given the component couples $M_{y'}$ and $M_{z'}$ the magnitude of the resultant couple is $\sqrt{M_{y'}^2 + M_{z'}^2}$, and its axis makes an angle $\tan^{-1} \dfrac{M_{z'}}{M_{y'}}$ with OY′. The resultant curvature is

$$a = \frac{1}{E} \sqrt{\left(\frac{M_{y'}}{I_{y'}}\right)^2 + \left(\frac{M_{z'}}{I_{z'}}\right)^2}$$

and the neutral axis, which is perpendicular to the actual curve of the rod, makes an angle

$$\varphi = \tan^{-1} \frac{M_{z'} \, I_{y'}}{M_{y'} \, I_{z'}}$$

with OY′. We see again, what equation (3.63) has already shown, that the axis of the resultant couple does not coincide with the neutral axis except when the neutral axis coincides with one of the principal axes of the section.

39. The form of the strained rod.

The displacements of points originally on the x-axis are obtained by putting $y = 0$, $z = 0$ in the expressions for the displacements. These displacements are

$$\left.\begin{array}{l} u_0 = 0, \;\; v_0 = 0, \\ w = -\frac{1}{2} a x^2 \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.65)$$

Since w is a displacement in the z-direction we may write z for w to get the equation to the curve into which the old x-axis is strained. This curve is

$$z = -\tfrac{1}{2} a x^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.66)$$

The radius of curvature of this parabola is

$$R = \frac{\left\{1+\left(\frac{dz}{dx}\right)^2\right\}^{\frac{3}{2}}}{-\frac{d^2z}{dx^2}}$$

$$= \frac{(1+a^2x^2)^{\frac{3}{2}}}{a}$$

$$= \frac{\left(1+4\frac{z^2}{x^2}\right)^{\frac{3}{2}}}{a}$$

$$= \frac{1}{a} \text{ approximately} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.67)$$

This last approximation is obtained by neglecting z^2 in comparison with x^2, z being the displacement w in this equation. This is in agreement with all our previous work since we have always assumed that the displacements were so small that they could be neglected in comparison with the dimensions of the body we were dealing with. All the more may we neglect the square of a displacement in comparison with the square of a dimension of the body. Then it follows that, to the degree of approximation that is usual in elasticity, the radius of curvature is constant along the rod.

In terms of the radius of curvature the couple M about OY is

$$M = \frac{EI}{R} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.68)$$

The shape of the cross section is also altered. It is easiest to study the section containing the origin. At this section $x = 0$ and therefore

$$\left.\begin{aligned} u &= 0, \\ v &= -\sigma a y z \\ &= -\frac{\sigma}{R} y z, \\ w &= \frac{\sigma}{2R}(y^2 - z^2). \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.69)$$

The first of these equations shows that the originally plane section remains plane.

Suppose the unstrained form of the cross section was a rectangle with sides parallel to the axes of y and z. Let the equations of these sides before strain be

$$y = \pm b \text{ and } z = \pm c.$$

The line $\quad y = b$

becomes

$$y = b + v$$

$$= b - \frac{\sigma b}{R} z; \quad \ldots\ldots (3.70)$$

and the line $y = -b$

becomes

$$y = -b + \frac{\sigma b}{R} z \quad \ldots\ldots (3.71)$$

Thus these two edges remain straight lines but each is turned through the small angle $\frac{\sigma b}{R}$ and they are turned in contrary directions.

The other two edges are changed into the two curves

$$z = \pm c + \frac{\sigma}{2R}(y^2 - c^2) \quad \ldots\ldots (3.72)$$

These are equal parabolas and the two curves differ from the curve

$$z = \frac{\sigma}{2R} y^2 \quad \ldots\ldots (3.73)$$

only in being bodily displaced parallel to the axis of z. Comparing (3.73) with the curve of the central line of the rod, namely

$$z = -\frac{x^2}{2R} \quad \ldots\ldots (3.74)$$

which we have shown has a radius R, we see that the two strained edges have a radius $\frac{R}{\sigma}$, and the vector representing the radius of curvature of either of the edges is drawn in the direction contrary to

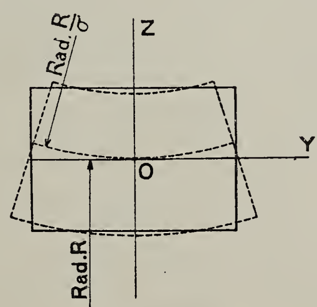

Fig. 17

that in which the radius of the central line is drawn. The strained section is shown in fig. 17.

We have shown that the cross-section containing the origin remains plane after strain, and it can easily be shown that all other cross-

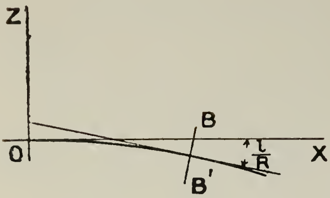

Fig. 18

sections remain plane and perpendicular to the line of centres of the cross sections. Thus the cross-section B′B (fig. 18) whose equation before strain is

$$x = l$$

is displaced by the strain to

$$x = l + u$$
$$= l + \frac{l}{R} z \quad . \quad (3\cdot75)$$

This is a plane parallel to the plane

$$x = \frac{l}{R} z$$

which latter contains the y-axis and is inclined to the z-axis at the small angle $\frac{l}{R}$ Thus the original plane is rotated about a line parallel to the y-axis through this small angle $\frac{l}{R}$. But, the equation to the line of centres being

$$z = -\frac{x^2}{2R},$$

the slope of this curve at the point $x = l$ is

$$\left(\frac{dz}{dx}\right)_{x=l} = -\frac{l}{R} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3\cdot76)$$

which is equal to the angle through which the cross-section is turned. It is clear that the curve and the cross-section have been turned in the same direction and therefore that each is perpendicular to the other.

40. Case of failure of the preceding solution.

It might be imagined that everything in the preceding solution is quite rigorous and that therefore it is the correct solution to the problem of a beam bent by a pair of opposing couples at its ends. But there is at least one simple case in which the solution fails. Suppose, for instance, that the cross-section is a rectangle and that the breadth $2b$ is several times as great as the depth $2c$. The rod would then be a strip like a steel rule, and for such a strip the radius R might easily be little greater than $2b$, or even, for a very wide strip, less than $2b$. Now the displacement of a point in the z-direction is

$$w = \frac{\sigma}{2R}(y^2 - c^2),$$

and this, for points on the edges where $y = b$ is

$$w = \frac{\sigma}{2R}(b^2 - c^2) = \frac{\sigma b^2}{2R} \text{ approximately} \quad . \quad . \quad . \quad (3\cdot77)$$

For a thin strip this might easily be greater than $2c$, in which case a point would be displaced right across the width of the strip. But there are many points in our theory which are justified only on the assumption that displacements are small compared with the dimensions of the body. Then our present solution will only hold provided $\frac{\sigma b^2}{2R}$ is small compared with $2c$; that is, b^2 must be small compared with Rc. We shall return to this point in Chapter 14 when we deal with the bending of thin plates.

CHAPTER IV.

THE EMPIRICAL BASIS OF ELASTICITY.

41. Hooke's law.

We have already assumed some of the empirical properties of materials in Chapter II. We shall state briefly the laws deduced from experiments on elastic bodies.

Under the action of forces an elastic body is deformed or strained. If the forces do not exceed a limit which, for any particular material, and for forces applied in a given way, is determined only by experiment, then the body recovers its original size and shape when these forces are removed. If the forces exceed the experimental limit mentioned then the body only partially recovers its size and shape when the forces are removed. The body is said to be perfectly elastic within the limits in which it completely recovers from the strains. When the body does not completely recover, the forces, and therefore the stresses in the body, are said to have exceeded the elastic limit, and the strain that remains after the forces cease to act is called a *permanent set*.

Hooke's law has already been stated in Chapter II. If can be expressed shortly in the following form:— Within the elastic limits the stress producing any strain is proportional to that strain. A more generalised law, which includes Hooke's Law, and of which we have already made use in Chapter II, can be expressed thus:— *Each force acting on an elastic body produces its own strains independently of the other forces.* To put it in another way:— *The total strains due to several forces is the sum of the strains due to each force separately.* It is easy to see that Hooke's Law is involved in this last law, for, if a particular force produces certain strains, then a force of n times the magnitude will produce strains n times as great, since each of the n forces produces the same strains.

Another assumption in the theory of elasticity, which agrees very well with experiment, is that, within the elastic limits the ratio of stress to strain is the same for positive and negative stresses.

The limits within which Hooke's Law holds and those within which no permanent set is produced are not precisely the same for all bodies, but they are nearly the same for steels and wrought iron. The mathematical theory of elasticity really only applies up to the limits within which Hooke's Law is true.

42. The yield point.

In finding the tensile strength of any given material, small bars, usually of circular cross-section, whose longitudinal sections are as shown in figs. 19(*a*) and 19(*b*) are gripped at the ends, and put in

Fig. 19a

Fig. 19b

tension till they break, the tension and extension being observed continuously till fracture occurs. The graph showing the relation between stress and strain for such a test-piece of steel is shown in fig. 20.

The portion OA is a straight line, but AB is slightly curved, the lower side being the concave side. The point B, at which the strain begins to increase greatly for very little increase of stress, is called the *yield point*. The point A determines the elastic limit for the purposes of the mathematical theory, but there would probably be very little error for most materials in using the theory up to the point B. The maximum stress is reached at D, and is considerably greater than the stress at the elastic limit.

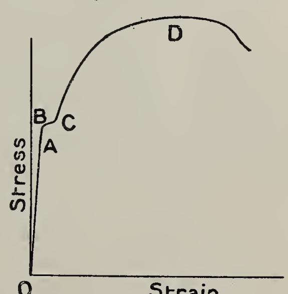

Fig. 20

The stress at the elastic limit for cast iron is so low compared with the maximum stress that the material will stand before fracture that it can hardly be called an elastic material at all. Nevertheless, it is in a different class from such a ductile material as lead, because it does recover from much of its strain when the elastic limit has been far exceeded, whereas lead hardly recovers at all. Its behaviour will be best understood if we conceive that, in the figure for cast iron corresponding to fig. 20, the length OA is a small proportion of OB. Also the elastic limit is not so well defined as for steels and wrought iron.

43 Ultimate stress.

The ultimate stress of a test piece is defined to be the quotient obtained on dividing the maximum pull on the piece during the process of breaking by the *original* area of the thin part of the section, where fracture ultimately takes place. This does not coincide with the maximum stress in fig. 20 because in that figure the stress is calculated on the actual area of the section at any instant, and as the bar lengthens the area of the section decreases to a measurable extent.

44 Factor of safety.

In actual engineering practice it is usual to calculate, as nearly as possible, by the theory of elasticity, the greatest stress to which any piece of material in a structure will be subjected, which stress is called the working stress, and the fraction

$$\frac{\text{ultimate stress}}{\text{working stress}}$$

is called the factor of safety of that particular piece. The ultimate stress used in the calculation is, of course, the ultimate stress of a similar piece of material which has been tested. It is common, when dealing with material whose stress depends very much on what may be regarded as the unknown accidents of manufacture, to make test pieces at the same time and under the same conditions as the actual material that is to be used in a structure. This gives some guarantee that the ultimate stress assumed in calculating the factor of safety is very nearly correct for the piece to which the calculation applies.

45. Viscosity in solid bodies.

Any elastic body may be set vibrating in different sorts of ways, and these vibrations always die out after a short time. Since the vibrations usually take place in air the whole of the damping effect might be attributed to the action of the air on the body; that is, it is possible that the whole of the energy in the vibrating body is dissipated into the air by means of air waves, such as those that produce sound. But it appears to be well established by experiment that the air alone is not sufficient to account for the whole of the damping effect; that, in fact, there is a sort of internal frictional resistance to the relative motion of the particles of the body, this resistance depending on the relative velocities, and increasing as these relative velocities increase. The property of an elastic body which causes such resistance is exactly similar to viscosity in fluids.

46. Elastic fatigue.

Although the strength and other elastic properties of a body are usually unaltered by stresses within the elastic limit there are circumstances in which this is not true. When a body is subjected for some time to rapidly alternating forces the properties of the material may be very much changed. Experiments show that such a body may be broken by a stress far below that at which it would have broken before the application of the alternating force. The mere repeated application and removal of a load, if the change takes place rapidly, is known to weaken an elastic body.

47. Theories of elastic failure.

Something in an elastic body has certainly given way when permanent set is brought about, and it seems reasonable to regard the point at which permanent set begins as the beginning of elastic failure. The theory of elasticity does not help us to make calculations beyond the elastic limit because this theory is all based on the assumption that Hooke's law is true, and there is no adequate mathematical theory beyond that point. Experiment has not settled at what particular state of stress or strain rupture or permanent set occurs. Three theories have been advanced but no decisive experiments seem to have determined which is right. The first theory states that failure occurs when the greatest principal stress reaches a certain definite limit for a given material; the second that the greatest principal strain is the deciding factor; and the third, that failure occurs where the greatest difference between the maximum and minimum principal stresses at a point reaches a fixed amount. The last theory amounts nearly to the same thing as taking the shearing stress as the deciding factor.

It is easy to think of rational objections to all the above theories. The last surely cannot be entirely right because it would mean that a body could not be ruptured by an infinite hydrostatic thrust or the corresponding tension. It seems probable that the factor that determines rupture is different in different cases. Possibly the first theory is right in many cases, but if the same piece of material had been subjected to a different set of stresses the shearing force might have been the deciding factor. In fact, instead of saying that any one of the three theories is right it is probably much nearer the mark to say that each contains a part of the truth, and that failure really occurs when some more complicated combination of the stresses and strains reaches a certain limit, and in certain simple cases the condition may reduce to one of the three given. To give a geometrical illustration, the point (x, y) certainly lies outside the circle $x^2 + y^2 = a^2$ when $x > a$, or when $y > a$; but there are many points outside the circle for which neither x nor y is greater than a. It may possibly be that all the three conditions for non-failure must be true simultaneously; that is, failure may occur when any one of the three following conditions is satisfied:—

(1) if the greatest principal stress exceeds a certain given magnitude;
(2) if the greatest principal strain exceeds a certain given magnitude;
(3) if the greatest difference between the maximum and minimum principal stresses at any point exceeds a given magnitude.

We have, however, enough experimental data on which to build a mathematical theory within the elastic limits, and it rests with the experimentalists to settle the doubtful points in their own domain.

CHAPTER V.

THE BENDING OF THIN RODS BY TRANSVERSE FORCES.

48. Rod bent into a circulur arc.

In Chapter III the stresses were obtained in a uniform rod in which the line of centres of inertia of the cross-sections was bent into the form a circular arc. The results are valid provided the breadth of the cross-section is not much greater than the depth. A thin rod, in this chapter, will be taken to mean a rod whose length is much greater than its breadth or depth, and of which the breadth and depth are not greatly unequal. It is not the absolute dimensions but the ratios of these dimensions that matter.

Taking the axes in the same positions as in Art 38, in which, it must be remembered, the x-axis passed through the centres of inertia of the cross-sections before strain, we shall assume that the axes of y and z are principal axes of inertia of the cross-section which contains the origin. Then the stresses are equivalent to a couple given by

$$M = \frac{EI}{R},$$

I being the moment of inertia of the section about the y-axis, and R the radius of the circle into which the line of centres of inertia is bent. The plane of this latter circle is the zx plane. The line of particles which lay on the y-axis before strain are distributed on a circle of radius $\frac{R}{\sigma}$ after strain, but it will be convenient in future to assume that this line remains straight after the strain. This will not make any difference to our results as long as the maximum displacement in the z-direction due to this curvature is small compared with the dimension of the rod parallel to the z-direction, and this will always be true for the sort of rod we are dealing with in this chapter.

49. Rod under transverse forces.

Suppose a rod AB is in equilibrium under the action of a number of forces perpendicular to its length such as P_1, P_2, ... Q_1, Q_2 etc in fig. 21. Since the portion CB is in equilibrium the action of AC on CB across the section at C must be such as to balance the forces P_3 and Q_3. This action must consist of a force F parallel to Q_3 such that

$$F = P_3 - Q_3, \qquad (5.1)$$

and a couple M which will balance the couple formed by F and the resultant of P_3 and $-Q_3$. By taking moments about C for the portion CB (fig. 22) we find that the moment of this couple is

$$M = CH \times P_3 - CK \times Q_3 \qquad (5.2)$$

The force F is the total shearing force across the section at C, and M is called the bending moment at C. We see that F is equal to the algebraic sum of all the forces on one side of the section and M is the sum of the moments of the same forces about a line through C perpendicular to the plane containing the forces. For definiteness we

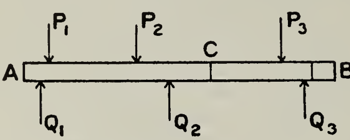

Fig. 21

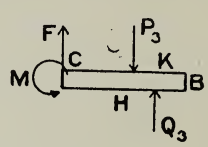

Fig. 22

shall refer to forces such as P_3, which are drawn downwards in the figure, as downward forces, though as far as the theory is concerned they might be horizontal forces.

We have shown the shearing force at C on the portion CB as an upward force, but it may easily be a force in the contrary direction. We shall, however, make it our rule that the shearing force F on a horizontal rod is positive when it has the direction shown in fig. 22, and negative when it acts in the contrary direction. This is quite in accordance with regarding a thrust as a negative tension. Since a shearing force is a reaction between two parts of a body it has contrary directions on the two portions. The rule is that the shearing force on any section is positive when the part of the rod to the right of the section exerts a downward force on the part to the left; or what amounts to the same thing, when the part to the left of the section exerts an upward force on the part to the right.

A similar rule for the sign of the bending moment is needed. The rule we shall use is that the bending moment is positive when the part to the left of a section exerts a counter-clockwise couple on the part to the right.

In order to fix the rules of signs for F and M it is worth while to observe, and remember, that both F and M are positive for a rod built into a wall at the left hand end and supporting a load at the other as in fig. 23.

Measuring x from the fixed end O in this case, and neglecting the weight of the rod itself, the shearing force and bending moment for this beam are given by

$$F = W \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.3)$$

$$M = W(l - x) \quad . \quad . \quad . \quad . \quad (5.4)$$

l being the total length of the beam.

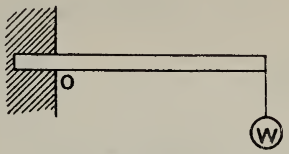

Fig. 23

It is clear that the shearing force in a beam changes, in passing an isolated load, by an amount equal to the load. The shearing force and bending moment for a beam under several isolated loads are shown by diagrams for one case in fig. 24.

It should be observed that, in passing a load, although there is a sudden change in the shearing force of amount equal to that load, there is no sudden change in the bending moment. Thus the bending moment has the same value immediately on opposite sides of a load. It is the slope of the bending moment diagram that makes a sudden change.

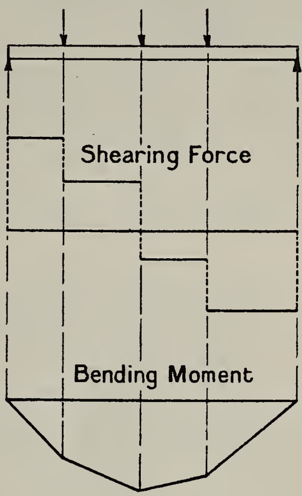

Fig. 24

50. Beams under distributed loads.

Let the load per unit length on a beam be w at distance x from some origin taken on the line of centres of the beam. Then wdx is the load on dx, and w may be constant or a function of x. The shearing stress and bending moment are F and M at x, and $(F + dF)$ and $(M + dM)$ at $(x + dx)$.

The forces on this element are shown in fig. 25. Suppose that wdx acts at a distance fdx from P. Then f is a fraction which is approximately one half. Resolving the forces perpendicular to the beam

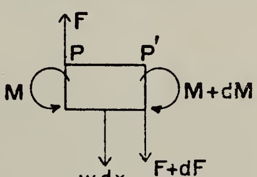

Fig. 25

$$(F + dF) + wdx = F;$$

or

$$dF = -wdx$$

whence

$$\frac{dF}{dx} = -w$$

Again, taking moments about P′, we get, since each of the couples M and $(M + dM)$ has the same moment about all points in its plane,

$$(M + dM) + F dx = M - w dx \times (1 - f) dx;$$

or
$$dM = -F dx - w(1 - f)(dx)^2,$$

whence
$$\frac{dM}{dx} = -F - w(1 - f)\,dx$$
$$= -F$$

since, in the limit, $dx = 0$.

We have now obtained the equations for beams under distributed loads, namely,

$$\frac{dF}{dx} = -w \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.5)$$

and
$$\frac{dM}{dx} = -F, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.6)$$

from which
$$\frac{d^2M}{dx^2} = -\frac{dF}{dx}$$
$$= w \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.7)$$

If there are concentrated loads on a beam as well as distributed loads then equations (5.5), (5.6), (5.7), still remain true between any pair of loads, but, by considering the equilibrium of an element on which a finite load acts, it is easy to show that the values of the shearing force in the immediate neighbourhood of this finite load, and on opposite sides of it, differ by the amount of the load. That is, there is the same discontinuity in the shearing force diagram as in fig. 24, but there is no discontinuity in the bending moment.

51. Relation between the bending moment and deflexion. If M is the bending moment in a beam and R the radius it was proved in chapter III that, when M is constant and when the principal axes of inertia of the section of the beam are respectively in the plane of the couple M and perpendicular to this plane,

$$M = \frac{EI}{R} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.8)$$

Now M is not constant in beams under transverse loads. A shearing force F exists, as equation (5.6) shows, only when M is variable. But when we are dealing with a beam whose length is much greater than its breadth or depth there is no appreciable error in using the formula (5.8) for M just as if M were constant. We shall assume provisionally that (5.8) is correct and show later that is justified.

There is a line of fibres crossing any section of the beam that are neither stretched nor shortened. These are along the y-axis in fig. 15. This line is called the neutral axis of the section. The tensional stress at distance z from this axis is, by equation (3.60),

$$P_1 = Eaz = \frac{Ez}{R} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.9)$$

The relations expressed by (5.8) and (5.9) are usually written,

$$\frac{f}{z} = \frac{M}{I} = \frac{E}{R} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.10)$$

f being now written for the more cumbersome symbol P_1.

In dealing with beams we shall use use η for the downward deflection of a point on the line of centres of inertia of the cross-section of the beam. It will be seen than η is the displacement, parallel to the negative direction along the z-axis, of a point on the line of centres of the beam, and therefore that it is what we have previously denoted by $-w$ for such a point. It is more convenient, however, to have the displacement reckoned positive when it is downward. If it were not that we already have a y-axis in the plane of one section of the beam it would have been convenient to write y for η.

In equation (5.10) R is now the radius of curvature of the $x\eta$ curve. The abscissa x is measured from any convenient origin, but

Fig. 26

will always be reckoned positive towards the right in our figures. For this curve

$$\pm \frac{1}{R} = \frac{\dfrac{d^2\eta}{dx^2}}{\left\{1 + \left(\dfrac{d\eta}{dx}\right)^2\right\}^{\frac{3}{2}}} \quad . \quad . \quad . \quad . \quad . \quad (5.11)$$

and since, in beams, $\left(\dfrac{d\eta}{dx}\right)^2$ is always small compared with unity, we may take

$$\frac{1}{R} = \pm \frac{d^2\eta}{dx^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.12)$$

Then the equation for the bending moment is

$$M = EI\frac{d^2\eta}{dx^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.13)$$

the positive sign being chosen for R since our convention that M is reckoned positive when the curve is concave on the lower side requires that M should be positive when $\dfrac{d^2\eta}{dx^2}$ is positive. In fig. 26 $\dfrac{d\eta}{dx}$ starts by being positive at O and becomes zero at D, from which it follows that $\dfrac{d\eta}{dx}$ is decreasing as x increases, and therefore that $\dfrac{d^2\eta}{dx^2}$ is negative, which agrees with our convention for the sign of M.

Now for a uniform beam, for which I is constant, equations (5.7) and (5.13) give

$$EI\frac{d^4\eta}{dx^4} = w \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.14)$$

Also equations (5.6) and (5.13) give

$$EI\frac{d^3\eta}{dx^3} = -F \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.15)$$

52. Solution of beam problems.

When w is given as a function of x (or as a constant) equation (5.14) can be integrated four times in succession and thus η can be found, and consequently F and M can be found. The complete integral involves, however, four constants, one of which appears at each integration. The problem is not solved until these constants have been determined. It is the conditions of the beam at its ends that determine these constants, and there are two conditions for each end, four conditions in all from which the four constants can be determined. We shall state what these conditions are.

(1) For a free end, with no load on that end,

$$\left.\begin{aligned} F = 0 \text{ i.e. } \frac{d^3\eta}{dx^3} &= 0, \\ \text{and} \qquad M = 0 \text{ i.e. } \frac{d^2\eta}{dx^2} &= 0, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (5.16)$$

If there is a load W on the free end then $F = W$ at that end.

(2) For a supported end, that is, an end resting on a support but not gripped,

$$\left.\begin{aligned} M = 0 \text{ i.e. } \frac{d^2\eta}{dx^2} &= 0, \\ \text{and} \qquad \eta \text{ is known} & \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (5.17)$$

(3) For a clamped end, where both the position of the beam and its direction are fixed, as when it is built into a rigid wall,

$$\left.\begin{aligned} \frac{d\eta}{dx} &\text{ is known, and is usually zero,} \\ \text{and} \qquad \eta &\text{ is known} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (5.18)$$

Thus we have in every case enough information to determine η completely, and therefore to determine completely F, M, and the stress f.

53. Distribution of shear stress in a beam.

We have found that there is a shearing force F across a section of a beam, but this cannot be distributed as a uniform stress across the section for this would require an equal shear stress over planes perpendicular to the z-axis, and in particular, over the top and bottom surfaces of the beam. No such boundary forces are applied and therefore the shear stresses near the highest and lowest parts of a cross-section must be zero.

Let us consider a beam of uniform rectangular section. The following argument is good for such a beam.

Let $2b$ be the breadth of the beam and $2c$ the depth. Consider the forces on the bundle of fibres of length dx, width $2b$ across a section, and height dz. These form a thin rectangular plate as shown in fig. 27, where dx is shown much greater than dz.

The tensional stresses at the two ends are f and $f+df$, where

$$df = \frac{df}{dx}dx.$$

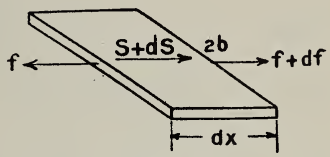

Fig. 27

Let the shear stresses on the upper and lower faces of the plate be S and S + dS, where

$$dS = \frac{dS}{dz}dz.$$

Then, resolving in the direction of dx, we get

$$df \times 2bdz + dS \times 2bdx = 0.$$

Dividing by $2bdxdz$,

$$\frac{df}{dx} + \frac{dS}{dz} = 0 \quad \text{. . . . . (5.19)}$$

But, by (5.10)

$$f = \frac{z}{I}M \quad \text{. . . . . (5.20)}$$

Therefore, since I is constant,

$$\frac{dS}{dz} = -\frac{z}{I}\frac{dM}{dx} = \mp\frac{z}{I}F,$$

and consequently

$$S = \left(\frac{z^2}{2I} - \frac{c^2}{2I}\right)F \quad \text{. . . . . . . (5.21)}$$

the constant being adjusted so as to make S = 0 when $z = c$. Writing A for the area of the section, we get

$$I = \tfrac{1}{3}c^2 A \quad \text{. . . . . (5.22)}$$

and therefore

$$S = \frac{3(z^2 - c^2)}{2c^2}\frac{F}{A} \quad \text{. . . . . . . (5.23)}$$

Neglecting the sign the maximum value of this shear stress occurs where $z = 0$ and its value is

$$S_0 = \frac{3}{2}\frac{F}{A} \quad \text{. . . . . (5.24)}$$

which is once and a half the magnitude of the mean shear stress over the area.

54. Justification of the neglect of the shear strains in beams. We have just shown that our equations for beams lead to the conclusion that the maximum shear stress in a beam of rectangular section is

$$S_0 = \frac{3}{2}\frac{F}{A} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.25)$$

Also the maximum tensional stress given by (5.20) is

$$f_1 = \frac{c}{\frac{1}{3}c^2 A} M = 3\frac{M}{cA} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.26)$$

Then

$$\frac{S_0}{f_1} = \frac{cF}{2M} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.27)$$

Now F is a force of the same order as the loads on the beam, while M is of the order lF where l is the length of the beam. Of course there may be points where M is zero or very small but the strains due to these small values of M are themselves negligible, and over practically the whole of the beam M is of the order lF, and the deflection of the beam is due to bending moments of this order. Then $\frac{S_0}{f_1}$ is of the order $\frac{c}{l}$, and for most beams this is a small fraction. For the ideal thin rod it is a negligible fraction. Now since the shear stress is of smaller order than the tensional stress it follows that the shear strains are also of smaller order than the tensional strains; and finally the displacements due to the shear strains are of smaller order than those due to tensional strains. But if we neglect altogether the shear strains we arrive back at the condition where there is no shearing force, and therefore a constant bending moment. This justifies us in using equation (5.8) even when M is not constant. It should be remembered, however, that the preceding argument is based on the assumption that the depth is small compared with the length. The greater the ratio $c:l$ is the less accurate do our results become.

A precisely similar argument to the above could be used for a beam with any shape of cross-section.

55. Uniform beam clamped horizontally at both ends under a uniformly distributed load

Let w be the load per unit length and l the length of the beam.

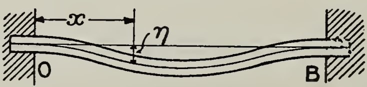

Fig. 28

Let x be measured from the left hand end of the beam, and η from the level of the central line at the ends. We shall write D, D^2, etc,. for $\frac{d}{dx}$, $\frac{d^2}{dx^2}$, etc.

Then, starting from equation (5.14),

$$EID^4\eta = w \quad \ldots \quad (5.28)$$

and integrating four times in succession we get

$$EID^3\eta = wx + A \quad \ldots \quad (5.29)$$

$$EID^2\eta = \tfrac{1}{2}wx^2 + Ax + B \quad \ldots \quad (5.30)$$

$$EID\eta = \tfrac{1}{6}wx^3 + \tfrac{1}{2}Ax^2 + Bx + C \quad \ldots \quad (5.31)$$

$$EI\eta = \tfrac{1}{24}wx^4 + \tfrac{1}{6}Ax^3 + \tfrac{1}{2}Bx^2 + Cx + H \quad \ldots \quad (5.32)$$

The end-conditions clearly are

$$\left.\begin{array}{l} \eta = 0 \\ D\eta = 0 \end{array}\right\} \begin{array}{l} \text{both where } x = 0 \\ \text{and where } x = l \end{array}\Bigg\} \quad \ldots \quad (5.33)$$

The two conditions at the end $x = 0$ give

$$0 = H,$$
$$0 = C.$$

The other two now give

$$0 = \tfrac{1}{24}wl^4 + \tfrac{1}{6}Al^3 + \tfrac{1}{2}Bl^2 \quad \ldots \quad (5.34)$$

$$0 = \tfrac{1}{6}wl^3 + \tfrac{1}{2}Al^2 + Bl \quad \ldots \quad (5.35)$$

Solving these equations for A and B we get

$$A = -\tfrac{1}{2}wl$$
$$B = \tfrac{1}{12}wl^2$$

Then the complete solution is

$$EI\eta = \tfrac{1}{24}wx^4 - \tfrac{1}{12}wlx^3 + \tfrac{1}{24}wl^2x^2$$
$$= \tfrac{1}{24}wx^2(l-x)^2 \quad \ldots \quad (5.36)$$

Also

$$M = EID^2\eta$$
$$= \tfrac{1}{12}w(6x^2 - 6lx + l^2) \quad \ldots \quad (5.37)$$
$$F = -EID^3\eta$$
$$= w(\tfrac{1}{2}l - x) \quad \ldots \quad (5.38)$$

Thus F is zero at the middle of the beam, which is obvious from the symmetry about the middle. Also M is zero where

$$6x^2 - 6lx + l^2 = 0$$

or

$$\frac{x}{l} = \tfrac{1}{2} \pm \frac{\sqrt{3}}{6} \quad \ldots \quad (5.39)$$

These two points are at the same distance $\frac{\sqrt{3}}{6}l$ from the middle of the beam and on opposite sides of the middle.

The maximum deflexion occurs at the middle of the beam and its magnitude is

$$\eta_1 = \frac{w}{24EI}(\tfrac{1}{2}l)^2(\tfrac{1}{2}l)^2$$
$$= \frac{wl^4}{384EI} \quad \ldots \quad (5.40)$$

The bending moments at one end and at the middle are respectively

$$M_0 = \tfrac{1}{12}wl^2 \text{ and } M_1 = -\tfrac{1}{24}wl^2 \quad \ldots \quad (5.41)$$

whence it follows that the greatest stresses are at the ends of the beam.

56. Uniform beam, under a uniformly distributed load, clamped horizontally at one end and supported at the other at the same level as at the clamped end.

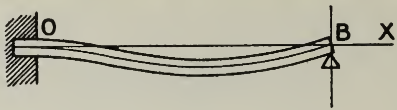

Fig. 29

The origin is taken at the centre of the section of the beam at the clamped end just where it enters the wall.

As in the last question

$$EID^4\eta = w$$
$$EI\eta = \tfrac{1}{24}wx^4 + \tfrac{1}{6}Ax^3 + \tfrac{1}{2}Bx^2 + Cx + H.$$

The end-conditions now are

$$\left.\begin{array}{r}\eta = 0\\ D\eta = 0\end{array}\right\} \text{ where } x = 0 \quad \ldots \quad (5.42)$$
$$\left.\begin{array}{r}\eta = 0\\ D^2\eta = 0\end{array}\right\} \text{ where } x = l \quad \ldots \quad (5.43)$$

Conditions (5.42) give H = 0 and C = 0.

Then conditions (5.43) give

$$0 = \tfrac{1}{24}wl^4 + \tfrac{1}{6}Al^3 + \tfrac{1}{2}Bl^2,$$
$$0 = \tfrac{1}{2}wl^2 + Al + B.$$

The values of A and B satisfying these equations are

$$A = -\tfrac{5}{8}wl,$$
$$B = \tfrac{1}{8}wl^2.$$

It follows that

$$\left.\begin{array}{ll} F = -EID^3\eta = -(wx + A) = w(\tfrac{5}{8}l - x), & \\ \text{and} \quad M = EID^2\eta = w(\tfrac{1}{2}x^2 - \tfrac{5}{8}lx + \tfrac{1}{8}l^2) & \end{array}\right\} \quad \ldots \quad (5.44)$$

The value of F at the supported end of the beam, where $x = l$, is

$$F_1 = -\tfrac{3}{8}wl \quad \ldots \quad (5.45)$$

But the shearing force at the end is equal to the load on the end. The support at the end must be regarded as applying a negative load the magnitude of which we have shown to be $\tfrac{3}{8}wl$ or $\tfrac{3}{8}W$, W being the total load on the beam. Then $\tfrac{5}{8}$ of the weight of the beam must be supported at the clamped end.

The greatest magnitude of the bending moment in the beam, and therefore also the greatest stress f, occurs either at the clamped end, or at the point where DM = 0, that is, where F = 0. But F = 0 where $x = \tfrac{5}{8}l$. The values of the bending moment at $x = 0$ and at $x = \tfrac{5}{8}l$ are respectively

$$\left.\begin{array}{ll} M_0 = \tfrac{1}{8}wl^2 = \tfrac{1}{8}Wl & \\ \text{and} \quad M_1 = -\tfrac{9}{128}wl^2 = -\tfrac{9}{128}Wl & \end{array}\right\} \quad \ldots \quad (5.46)$$

so that M_0 has the greater magnitude.

The maximum deflexion occurs where $D\eta = 0$, that is, where

$$\tfrac{1}{6}x^3 + \tfrac{1}{2}Ax^2 + Bx = 0$$

or

$$x(\tfrac{1}{6}x^2 - \tfrac{5}{16}lx + \tfrac{1}{8}l^2) = 0$$

whence

$$\frac{x}{l} = \frac{15 \pm \sqrt{33}}{16} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5\cdot 47)$$

The positive sign cannot be taken since the corresponding point is not on the beam. Then

$$\frac{x}{l} = \frac{15 - \sqrt{33}}{16} = 0\cdot 5785 \quad . \quad . \quad . \quad . \quad . \quad (5\cdot 48)$$

57. A uniform beam of length *l*, clamped horizontally at one end, under a given concentrated load W and a given couple C at the other end.

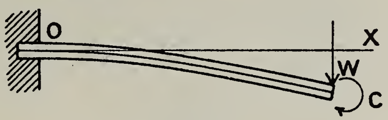

Fig. 30

By taking moments we find that the bending moment at x is

$$M = C + W(l - x) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5\cdot 49)$$

Therefore

$$EID^2\eta = (C + Wl) - Wx \quad . \quad . \quad . \quad . \quad . \quad . \quad (5\cdot 50)$$

Integrating twice we get

$$EID\eta = (C + Wl)x - \tfrac{1}{2}Wx^2 \quad . \quad . \quad . \quad . \quad . \quad (5\cdot 51)$$

$$EI\eta = \tfrac{1}{2}(C + Wl)x^2 - \tfrac{1}{6}Wx^3, \quad . \quad . \quad . \quad . \quad (5\cdot 52)$$

no constants of integration being needed because $\eta = 0$ and $D\eta = 0$ where $x = 0$.

Let θ_1 denote the slope of the beam at the end $x = l$, and η_1 the deflexion at the same point. Then

$$EI \tan\theta_1 = (C + Wl)l - \tfrac{1}{2}Wl^2$$

$$= (C + \tfrac{1}{2}Wl)l \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5\cdot 53)$$

$$EI\eta_1 = (\tfrac{1}{2}C + \tfrac{1}{3}Wl)l^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (5\cdot 54)$$

Since θ_1 is small we may write θ_1 for $\tan\theta_1$. Then solving equations (5·53), (5·54), for C and W in terms of θ_1 and η_1 we get

$$W = \frac{6EI}{l^3}(2\eta_1 - l\theta_1) \quad . \quad . \quad . \quad . \quad . \quad . \quad (5\cdot 55)$$

$$C = -\frac{2EI}{l^2}(3\eta_1 - 2l\theta_1) \quad . \quad . \quad . \quad . \quad . \quad (5\cdot 56)$$

Then

$$C + Wl = \frac{EI}{l^2}(6\eta_1 - 2l\theta_1) \quad . \quad . \quad . \quad . \quad . \quad (5\cdot 57)$$

In terms of η_1 and θ_1 our equation for η becomes

$$\eta = \frac{x^2}{l^2}(3\eta_1 - l\theta_1) - \frac{x^3}{l^3}(2\eta_1 - l\theta_1) \quad . \quad . \quad . \quad (5.58)$$

In particular, if $\theta_1 = 0$ then

$$\eta = \eta_1\left(\frac{3x^2}{l^2} - \frac{2x^3}{l^3}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.59)$$

whereas if $\eta_1 = 0$ we get

$$\eta = l\theta_1\left(\frac{x^3}{l^3} - \frac{x^2}{l^2}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.60)$$

In the general case

$$\begin{aligned} M &= EID^2\eta \\ &= EI\left\{\eta_1\left(\frac{6}{l^2} - \frac{12x}{l^3}\right) - l\theta_1\left(\frac{2}{l^2} - \frac{6x}{l^3}\right)\right\} \\ &= \frac{EI}{l^3}\left\{\eta_1(6l - 12x) - l\theta_1(2l - 6x)\right\} \quad . \quad . \quad . \quad (5.61) \\ F &= -DM \\ &= \frac{6EI}{l^3}\left\{2\eta_1 - l\theta_1\right\} = W \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.62) \end{aligned}$$

This last result is only a verification of (5.55) since F is clearly constant along the beam.

58. Deflexion of a beam due to several loads or several systems of loads.

Suppose that the conditions at each end of a beam are that two of the quantities η, $D\eta$, $D^2\eta$, $D^3\eta$, are zero, and suppose that it is required to find the deflexion η due to a load $(w_1 + w_2)$ per unit length. It is our purpose to prove that

$$\eta = \eta_1 + \eta_2$$

where η_1 is the deflexion due to the load w_1 alone with the given end conditions, and η_2 is the deflexion due to w_2 alone with the same end conditions.

The equations for η_1 and η_2 are

$$EID^4\eta_1 = w_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.63)$$

$$EID^4\eta_2 = w_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.64)$$

Adding these we get

$$EID^4(\eta_1 + \eta_2) = w_1 + w_2 \quad . \quad . \quad . \quad . \quad . \quad (5.65)$$

But if η is the deflexion due to the total load $(w_1 + w_2)$ then

$$EID^4\eta = w_1 + w_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.66)$$

Thus η satisfies the same differential equations as $(\eta_1 + \eta_2)$.

Suppose one of the end-conditions is that

$$D^n\eta = 0 \text{ at that end} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.67)$$

Then

$$\left.\begin{aligned} D^n\eta_1 = 0 \\ D^n\eta_2 = 0 \end{aligned}\right\} \text{ at the same end} \quad . \quad . \quad . \quad . \quad (5.68)$$

Adding these last two equations we get

$$D^n(\eta_1 + \eta_2) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.69)$$

which shows that $(\eta_1 + \eta_2)$ satisfies the same end conditions as η. Thus the total deflexion η satisfies all the equations and end-conditions that $(\eta_1 + \eta_2)$ satisfies Then η must be equal to $(\eta_1 + \eta_2)$.
If one of the end-conditions is that

$$D^n\eta = a \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.70)$$

then the corresponding end conditions of η_1 ane η_2 can be

$$\left.\begin{aligned} D^n\eta_1 &= a \\ D^n\eta_2 &= 0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.71)$$

or we may choose

$$\left.\begin{aligned} D^n\eta_1 &= 0 \\ D^n\eta_2 &= a \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.72)$$

In either case

$$D^n(\eta_1 + \eta_2) = a = D^n\eta \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.73)$$

As a particular case, if one of the end conditions is

$$\eta = a, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.74)$$

Then we can choose

$$\left.\begin{aligned} \eta_1 &= a \\ \eta_2 &= 0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.75)$$

at that end.

The preceding argument can be extended to include concentrated loads. A concentrated load W_1 may be represented by w_1 per unit length provided w_1 is a function of x which is very large over a small range of values of x on opposite sides of the point where the load is applied, and zero everywhere else, and such that,

$$\int w_1 dx = W_1.$$

This device of replacing a load at a single point by an equal load distributed over a small length (or area) of the beam has the advantage that it does actually represent the physical facts, for a finite load cannot be concentrated on a point or on a geometrical line.

A couple applied at any point of a beam can also be included in the preceding argument, for the couple can be regarded as two large concentrated forces acting in opposite directions at a very short distance apart.

What has just been proved is only a particular case of the law, stated in Art 41, namely, that each force produces its own stresses and strains just as if the other forces were not acting.

The proof has been given only for uniform beams but it is equally true for beams with non-uniform sections.

59. Uniform beam clamped horizontally at one end and free at the other, carrying only a concentrated load W.

Let the load be applied at distance a from the clamped end, which is taken as the origin. Then, by taking moments about the point at x, when $x < a$,

$$M = W(a - x); \quad \ldots \quad (5.76)$$

that is,
$$EID^2\eta = W(a - x) \quad \ldots \quad (5.77)$$

Therefore
$$EID\eta = W(ax - \tfrac{1}{2}x^2), \quad \ldots \quad (5.78)$$

and
$$EI\eta = W(\tfrac{1}{2}ax^2 - \tfrac{1}{6}x^3), \quad \ldots \quad (5.79)$$

no constants of integration being needed in consequence of the conditions at the origin.

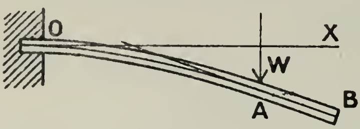

Fig 31

The part AB is under no load and is therefore straight. To find its equation we need the value of η at A and the slope at A. From (5.78) and (5.79) wet get, writing η_1 for η at A,

$$\left.\begin{aligned} EID\eta_1 &= \tfrac{1}{2}Wa^2 \\ EI\eta_1 &= \tfrac{1}{3}Wa^3 \end{aligned}\right\} \text{ at A} \quad \ldots \quad (5.80)$$

Therefore the equation to AB is

$$\eta - \eta_1 = \frac{Wa^2}{2EI}(x - a) \quad \ldots \quad (5.81)$$

or
$$EI(\eta - \eta_1) = \tfrac{1}{2}Wa^2(x - a) \quad \ldots \quad (5.82)$$

whence
$$EI\eta = Wa^2(\tfrac{1}{2}x - \tfrac{1}{6}a) \quad \ldots \quad (5.83)$$

The values of η and $D\eta$ at the free end where $x = l$ are given by

$$EI\eta_2 = Wa^2(\tfrac{1}{2}l - \tfrac{1}{6}a) \quad \ldots \quad (5.84)$$
$$EID\eta_2 = \tfrac{1}{2}Wa^2 \quad \ldots \quad (5.85)$$

60. Uniform beam clamped horizontally at both ends and carrying only a concentrated load.

Let the load be applied at A and let the origin be taken at the end O; let $OA = a$ and $OB = l$.

If the end B were free the deflexion η would be that given in the last problem. Now the wall at B exerts a couple and a force similar to those dealt with in Art 57. If η_1 and θ_1 are the deflexion and slope due to these at the end B then equation (5.58) gives the value of η corresponding to the actions at B alone as

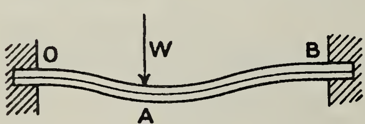

Fig. 32

$$\eta = \frac{x^2}{l^2}(3\eta_1 - l\theta_1) - \frac{x^3}{l^3}(2\eta_1 - l\theta_1) \quad \ldots \quad (5.86)$$

If we take η_1 and θ_1 in this equation as the negative of η_2 and $D\eta_2$ in (5.84) and (5.85) then we get

$$\begin{aligned}\mathrm{EI}\eta &= -\frac{x^2}{l^2}\left\{\mathrm{W}a^2(\tfrac{3}{2}l-\tfrac{1}{2}a)-\tfrac{1}{2}\mathrm{W}a^2l\right\}\\&\quad+\frac{x^3}{l^3}\left\{\mathrm{W}a^2(l-\tfrac{1}{3}a)-\tfrac{1}{2}\mathrm{W}a^2l\right\}\\&= -\mathrm{W}\frac{a^2x^2}{l^2}(l-\tfrac{1}{2}a)+\mathrm{W}\frac{a^2x^3}{l^3}(\tfrac{1}{2}l-\tfrac{1}{3}a) \quad . \ (5.87)\end{aligned}$$

By adding together the deflexion in (5.79) and the deflexion in (5.87) we shall get the deflexion between O and A in the present problems. Writing η_A for the sum of these two deflexions we get

$$\begin{aligned}\mathrm{EI}\eta_A &= \tfrac{1}{6}\frac{\mathrm{W}x^2}{l^3}\left\{(3a^2l-2a^3-l^3)x-6a^2l^2+3a^3l+3al^3\right\}\\&= \tfrac{1}{6}\frac{\mathrm{W}x^2}{l^3}(l-a)^2\left\{3al-(l+2a)x\right\} \quad . \ . \ . \ . \ . \ . \ (5.88)\end{aligned}$$

as the equation between O and A, that is, where $x<a$.

The deflexion between A and B can be obtained by adding the deflexions in (5.83) and (5.87). But it is easier to deduce it from (5.88) by changing a into $(l-a)$ and x into $(l-x)$. Then

$$\mathrm{EI}\eta_B = \tfrac{1}{6}\frac{\mathrm{W}a^2}{l^3}(l-x)^2\left\{(3l-2a)x-al\right\} \quad . \ . \ . \ (5.89)$$

gives the deflexion between A and B.

61. Uniform beam clamped horizontally at both ends under any load.

We can make use of the last result to get the deflexion of a beam clamped at both ends under any load. If there are a number of concentrated loads the deflexion for each load must be calculated and the deflexion added at every point of the beam.

If there is a distributed load w per foot, w being constant or variable, we can write $w da$ for the load at $x=a$, regarding w as a function of a, and then integrate to get the total deflexion at x due to the whole load. Thus writing $w da$ for W in (5.88), and writing $d\eta$ for the deflexion due to $w da$,

$$\mathrm{EI}d\eta = \tfrac{1}{6}\frac{x^2}{l^3}(l-a)^2\left\{3al-(l+2a)x\right\}w\,da \quad . \ . \ . \ (5.90)$$

This is the correct equation when $x<a$, whereas if $x>a$ the equation for $d\eta$ is

$$\mathrm{EI}d\eta = \tfrac{1}{6}\frac{a^2}{l^3}(l-x)^2\left\{(3l-2a)x-al\right\}w\,da \quad . \ . \ . \ (5.91)$$

The total deflexion η is obtained by integrating both sides of (5.90) from $a=x$ to $a=l$, and by integrating (5.91) from $a=0$ to $a=x$ and adding the results. Thus

$$\mathrm{EI}\eta = \int_x^l f(a, x)da + \int_0^x \mathrm{F}(a, x)\,da \quad . \quad . \quad . \quad . \quad (5.92)$$

where $f(a, x)da$ is the expression on the right of (5.90) and $\mathrm{F}(a, x)da$ is the expression on the right of (5.91).

To avoid the labour of working out two different integrals we can express both in the same form. If we write a_1 for $(l-a)$ and x_1 for $(l-x)$ we know that

$$f(a,\ x) = \mathrm{F}(a_1,\ x_1)$$

Also

$$da_1 = -\,da$$

Therefore

$$\int_x^l f(a, x)\,da = -\int_{x_1}^0 \mathrm{F}(a_1, x_1)da_1$$

$$= \int_0^{x_1} \mathrm{F}(a_1, x_1)\,da_1$$

Consequently

$$\mathrm{EI}\eta = \int_0^{x_1} \mathrm{F}(a_1, x_1)\,da_1 + \int_0^x \mathrm{F}(a, x)\,da \quad . \quad . \quad . \quad . \quad (5.93)$$

the function to be integrated being obtained from equation (5.91)

Suppose w is constant. Then

$$\mathrm{EI}\eta = \frac{w}{6}\frac{(l-x)^2}{l^3}\int_0^x \left\{(3a^2l - 2a^3)x - a^3l\right\} da$$

$$+ \frac{w}{6}\frac{(l-x_1)^2}{l^3}\int_0^{x_1}\left\{(3a_1{}^2l - 2a_1{}^3)x_1 - a_1{}^3l\right\} da_1$$

$$= \frac{w}{6}\frac{(l-x)^2}{l^3}\Big[(a^3l - \tfrac{1}{2}a^4)x - \tfrac{1}{4}a^4l\Big]_0^x$$

$$+ \frac{w}{6}\frac{x^2}{l^3}\Big[(a_1{}^3l - \tfrac{1}{2}a_1{}^4)x_1 - \tfrac{1}{4}a_1{}^4l\Big]_0^{x_1}$$

$$= \frac{w}{6}\frac{x_1{}^2x^2}{l^3}\left\{\tfrac{3}{4}lx^2 - \tfrac{1}{2}x^3\right\}$$

$$+ \frac{w}{6}\frac{x_1{}^2x^2}{l^3}\left\{\tfrac{3}{4}lx_1{}^2 - \tfrac{1}{2}x_1{}^3\right\}$$

$$= \frac{w}{6}\frac{x_1{}^2x^2}{l^3}\left\{\tfrac{3}{4}l(x^2 + x_1{}^2) - \tfrac{1}{2}(x^3 + x_1{}^3)\right\}$$

$$= \frac{w}{6}\frac{x_1{}^2x^2}{l^3}\left\{\tfrac{3}{4}l(x^2 + x_1{}^2) - \tfrac{1}{2}l(x^2 - xx_1 + x_1{}^2)\right\}$$

$$= \frac{w}{6}\frac{x_1{}^2x^2}{l^3} \times \tfrac{1}{4}l(x_1 + x)^2$$

$$= \tfrac{1}{24}wx^2(l-x)^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.94)$$

which agrees with the result in (5.36)

The advantage of the last method for a beam clamped at both ends is that the deflexion is obtained, for any system of loading, by means of a pair of integrals without any necessity for the adjusting of constants.

62. Bending moment expressed by integrals.

It is often convenient to get the bending moment by a direct integration for the clamped-clamped beam just as we have already obtained the deflexion; that is, an equation for bending moment similar to the equation (5.92) for η is wanted.

The bending moments in the two parts of the beam in fig. 32 are obtained by differentiating both sides of equations (5.88) and (5.89). Thus

$$M_A = EID^2\eta_A$$
$$= \frac{W}{l^3}(l-a)^2\left\{al-(l+2a)x\right\} \quad . \quad . \quad . \quad . \quad . \quad (5.95)$$
$$M_B = EID^2\eta_B$$
$$= \frac{Wa^2}{l^3}\left\{al-2l^2+(3l-2a)x\right\} \quad . \quad . \quad . \quad . \quad (5.96)$$

the first of these being the bending moment for values of x less than a, and the second for values of x greater than a. Then the bending moment at x due to a distributed load w per unit length at $x=a$ is

$$M = \int_x^l \frac{M_A}{W} wda + \int_0^x \frac{M_B}{W} wda$$
$$= \int_x^l \frac{(l-a)^2}{l^3}\left\{al-(l+2a)x\right\} wda$$
$$+\int_0^x \frac{a^2}{l^3}\left\{al-2l^2+(3l-2a)x\right\} wda \quad . \quad (5.97)$$

Equation (5.97) gives a formula for the bending moment in terms of x, but it can be used very conveniently for finding the bending moment at a particular point by putting for x the abscissa of the particular point. It should be remembered that the x to suit equation (5.97) is measured from one end of the beam.

As an illustration we shall find the bending moment at the middle of a clamped-clamped beam which is symmetrically loaded about the middle, the load per foot at any point being proportional to the distance of that point from the nearest end of the beam. The load curve is in fact, a triangle with its apex at the middle of the beam as shown in fig. 33.

Fig. 33

Here $w=ka$ or $k(l-a)$ according as a is less than or greater than $\frac{1}{2}l$. Then in equation (5.97) we put $x=\frac{1}{2}l$ since the bending moment at the middle is required. Therefore

$$M = \int_{\frac{1}{2}l}^{l} -\frac{(l-a)^2}{2l} kada + \int_{0}^{\frac{1}{2}l} -\frac{a^2}{2l} k(l-a)\,da$$
$$= -\tfrac{5}{384}kl^3 \tfrac{5}{384}kl^3$$
$$= -\tfrac{5}{48}Wl \quad . \quad . \quad . \quad . \quad . \quad . \quad (5\cdot98)$$

where W is the total load $\frac{1}{4}kl^2$.

63. Beams with variable cross-sections.

Provided the cross-section of a beam does not change quickly with the length, or to be precise, provided that all the sections of an unstrained beam containing some straight line, which we may call the length, are bounded by a pair of curves every element of which is only slightly inclined to the length, we may still use some of the equations which we have proved for uniform beams, and which we rewrite here in the forms in which they apply.

$$\frac{dF}{dx} = -w, \quad . \quad . \quad . \quad . \quad . \quad . \quad (5\cdot99)$$

$$\frac{dM}{dx} = -F, \quad . \quad . \quad . \quad . \quad . \quad . \quad (5\cdot100)$$

$$\frac{d^2M}{dx^2} = w, \quad . \quad . \quad . \quad . \quad . \quad . \quad (5\cdot101)$$

$$M = \frac{EI}{R} = EI\frac{d^2\eta}{dx^2}, \quad . \quad . \quad . \quad . \quad . \quad (5\cdot102)$$

$$\frac{f}{z} = \frac{M}{I}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (5\cdot103)$$

When we substitute for M from (5.102) in either of the equations (5.100) and (5.101) we have to remember now that I is a function of x. Consequently

$$E\frac{d^2}{dx^2}\left(I\frac{d^2\eta}{dx^2}\right) = w. \quad . \quad . \quad . \quad . \quad . \quad (5\cdot104)$$

We can always get M by integrating (5.101) twice, and then we can get η by integrating twice both sides of the equation

$$\frac{d^2\eta}{dx^2} = \frac{1}{E}\frac{M}{I}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (5\cdot105)$$

64. Example of a beam with variable section.

A beam of uniform width b and variable depth d rests on supports at its ends, and carries only its own weight. The depth at distance x from the middle is given by

$$d = c\cos\frac{\pi x}{l} \quad . \quad . \quad . \quad (5\cdot106)$$

where l is the length of the beam.

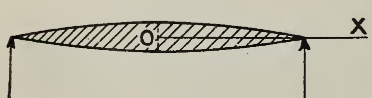

Fig. 34

Such a beam is shown in fig. 34. Here the load w per unit length at x is ϱbd where ϱ is the weight of unit volume of the material Hence

$$\frac{d^2M}{dx^2} = \varrho bd = \varrho bc \cos\frac{\pi x}{l} \quad . \quad . \quad . \quad . \quad . \quad (5\ 107)$$

$$\frac{dM}{dx} = \frac{\varrho bcl}{\pi} \sin\frac{\pi x}{l} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5\ 108)$$

No constant need be added here because M is clearly a minimum at the middle of the beam where $x = 0$. Integrating again

$$M = -\frac{\varrho bcl^2}{\pi^2} \cos\frac{\pi x}{l} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.109)$$

Again no constant is needed because M is zero at the ends where $x = \pm \frac{1}{2} l$, and at these points the cosine is zero.

Now

$$I = \tfrac{1}{12} bd^3 = \tfrac{1}{12} bc^3 \cos^3 \frac{\pi x}{l} \quad . \quad . \quad . \quad . \quad (5.110)$$

and therefore

$$\frac{d^2\eta}{dx^2} = \frac{M}{EI} = \frac{12\varrho l^2}{\pi^2 c^2 E} \sec^2 \frac{\pi x}{l} \quad . \quad . \quad . \quad . \quad . \quad (5.111)$$

Integrating

$$\frac{d\eta}{dx} = -\frac{12\varrho l^3}{\pi^3 c^2 E} \tan\frac{\pi x}{l} + H \quad . \quad . \quad . \quad . \quad (5.112)$$

$$\eta = -\frac{12\varrho l^4}{\pi^4 c^2 E} \log_e \sec \frac{\pi x}{l} + Hx + K \quad . \quad . \quad (5.113)$$

The constant H is zero because $\frac{d\eta}{dx}$ is zero at the middle of the beam. Then to find the constant K we must use the condition that $\eta = 0$ where $x = \frac{l}{2}$. But this gives

$$0 = -\infty + K$$

which makes K equal to $+\infty$ We can shirk this difficulty to some extent by measuring η from the displaced position of the middle of the beam; that is, by taking $\eta = 0$ where $x = 0$. Then $K = 0$. Therefore

$$\eta = -\frac{12\varrho l^4}{\pi^4 c^2 E} \log_e \sec \frac{\pi x}{l} \quad . \quad . \quad . \quad . \quad (5.114)$$

As long as we keep away from the ends of the beam the last value of η is quite reasonable, but at each end we find $\eta = -\infty$, so the difficulty returns. This difficulty will always appear with a beam tapering to a sharp edge at an end where a finite force is applied, the edge being perpendicular to the load as in our present problem.

The correct interpretation is that the beam fails at such an edge. But even where a beam fails the deflexion η should not become infinite. The reason why infinity occurs is because we have used an incorrect expression for curvature, namely $\frac{d^2\eta}{dx^2}$, in our beam-equations.

Since it is only just near the ends that failure occurs this failure could clearly be avoided by adding a little to the depth at and near the ends without adding anything appreciable elsewhere. It is easy to write down mathematical expressions for suitable additions to d, but it is not easy to perform the complete integrations for η when we have got our new expression for d.

We can avoid the infinite value of η in another way. Clearly all our equations apply correctly to a beam of the assumed shape fixed at any two points of its length provided only that the supports at the fixed points apply the calculated shearing forces and bending moments at those points, and give the beam the calculated values of η and $\frac{d\eta}{dx}$. Further, if the supports of the beam, which we originally assumed to be at the ends, were brought a little inwards towards the centre, and the overhanging pieces cut off, then the new state of the beam would differ from the old state in that the bending moment would be zero at the new supports instead of at the old ones. The additional stresses and deflexions of the beam would then be those due to the pair of couples, applied at the new supports, which would neutralise the bending moments existing at those points with the old supports. If the new supports were not far removed from the old then the bending moments that existed at the new supports were small, and therefore the new state of stress would differ little from the old state, and the part at which infinite deflexions and stresses occurred would now have been removed off the beam. We know, in fact, that the old equations give stresses and deflexions slightly too big for the new conditions.

65. The modulus of a section of a beam.

When the bending moment at any section of a beam is known the stress at distance z from the neutral axis of the section is given by

$$f=\frac{z\mathrm{M}}{\mathrm{I}} \quad \text{. . . . .} \quad (5.115)$$

If h is the greatest distance of a point of the area from the neutral axis, that is, the greatest positive or negative value of z, then the greatest stress at the sections is f_1 given by

$$f_1=\frac{h\mathrm{M}}{\mathrm{I}}=\frac{\mathrm{M}}{\mathrm{V}} \quad \text{. . . . . . .} \quad (5.116)$$

where

$$\mathrm{V}=\frac{\mathrm{I}}{h} \quad \text{. . . . .} \quad (5.117)$$

This quantity V is called the *modulus* of the section for bending. Engineers prefer to remember the modulus of a section rather than its moment of inertia. For a circle of radius r we have $I = \frac{1}{4}\pi r^4$, $h = r$, and therefore $V = \frac{1}{4}\pi r^3$ For a rectangle of breadth b and depth d we get $I = \frac{1}{12} b d^3$, $h = \frac{1}{2} d$, and therefore $V = \frac{1}{6} b d^2$. Most engineers' pocket books give the moduli of the sections of such beams as occur in practical engineering.

66. Beams of uniform strength.

If the maximum stress f_1 is the same for all sections of a beam we call it a beam of uniform strength. It is always possible to vary the section of a beam so as to keep f_1 constant whatever the load on the beam may be.

Suppose a beam fixed at one end and free at the other carries a load W at the free end and no other load, the weight of the beam itself being negligible. We shall show how to make the beam of uniform strength.

Measuring x from the load W we find, by taking moments about any section, that

$$M = Wx \quad \ldots\ldots\ldots\ldots \quad (5.118)$$

Then since

$$M = \frac{If_1}{h}$$

we find that

$$\frac{I}{xh} = \frac{W}{f_1} = \text{constant} \quad \ldots\ldots \quad (5.119)$$

There are many ways of making I proportional to xh. Suppose we choose a rectangular section. Then $h = \frac{1}{2}d$ and $I = \frac{1}{12} bd^3$. Therefore we must make bd^2 proportional to x. We could make d constant and b proportional to x. This would be a wedge shaped beam with the edge of the wedge along the line of action of W. Or we might make d^2 proportional to x and b constant. This gives a beam whose vertical section taken along the length of the beam is a parabola. Or again we might take b and d each proportional to $\sqrt[3]{x}$ giving a beam whose cross-section has a constant shape but a variable size.

We might also make the section of the beam circular, in which case r would have to be proportional to $\sqrt[3]{x}$.

67. Beam of uniform strength supporting only its own weight.

A beam fixed at one end and free at the other supports its own weight only and is of uniform strength. Assuming the depth to be constant we shall find the breadth necessary for uniform strength.

Taking the origin at the fixed end

$$\frac{d^2M}{dx^2} = \varrho bd \quad \ldots\ldots\ldots \quad (5.120)$$

Also

$$M = \frac{2If_1}{d}$$

$$= \tfrac{1}{6}bd^2f_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.121)$$

Differentiating (5.121) with respect to x twice, with the condition that b is the only variable on the right

$$\frac{d^2M}{dx^2} = \tfrac{1}{6}d^2f_1\frac{d^2b}{dx^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.122)$$

From (5.120) and (5.122) we get

$$\tfrac{1}{6}d^2f_1\frac{d^2b}{dx^2} = \varrho bd$$

or

$$\frac{d^2b}{dx^2} = \frac{6\varrho}{f_1d}b$$

$$= n^2b \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.123)$$

where

$$n^2 = \frac{6\varrho}{f_1d} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.124)$$

The solution of (5.123) is

$$b = He^{nx} + Ke^{-nx} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.125)$$

Now it is impossible, with this value of b and the corresponding value of M given by (5.121), to make both M and F equal to zero when $x = l$, for these conditions make both H and K zero. We can, however, get something out of our result if we make H zero. Then

$$b = Ke^{-nx}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.126)$$

whence

$$M = \tfrac{1}{6}d^2f_1Ke^{-nx}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.127)$$

and

$$F = -\frac{dM}{dx} = \tfrac{1}{6}nd^2f_1Ke^{-nx} \quad . \quad . \quad . \quad (5.128)$$

Thus the breadth b, the bending moment M, and the shearing force F, are all zero when $x = \infty$. Our solution then gives a beam of infinite length, having only finite stresses. If, however, a beam of length l be taken whose breadth is given by (5.126) and if, at the free end, a shearing force

$$F_1 = \tfrac{1}{6}nd^2f_1Ke^{-nl} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.129)$$

and a bending moment

$$M_1 = \tfrac{1}{6}d^2f_1Ke^{-nl} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.130)$$

be produced by means of externally applied forces, then the stresses in this beam are those given by (5.127) and (5.128). In effect, this amounts to supposing that the actions across the section at $x = l$ due to the weight of the infinite length of beam from $x = l$ to $x = \infty$ are supplied by external forces applied at the section instead of by the weight of this infinite portion of the beam itself, which is supposed to be cut off. If these actions are not applied at all then the stresses in the remaining portion of the beam, from $x=0$ to $x=l$, are actually less than we have calculated. The beam then is not one of

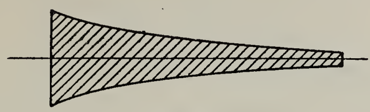

Fig. 35

uniform strength but one in which the maximum stress is everywhere less than f_1. The shape of the horizontal section of the beam is shown in fig. 35.

68. Unsymmetrical bending of a beam.

In all the preceding part of this chapter it has been assumed that the neutral axis of any section of a bent beam was coincident with one of the principal axes of inertia of the section, and therefore that the bending moment acted in the plane perpendicular to this principal axis. We shall now consider what happens when the bending moment is in a plane inclined to both principal axes.

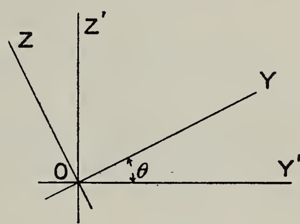

Fig. 36

Let OY′, OZ′, be the principal axes of inertia of a section, and suppose OY is the neutral axis of the section of the bent beam. Then by equations (3.62) and (3.63) there is a couple M about OY and M′ about OZ such that

$$M = EI_y a \qquad (5.131)$$

$$M' = -EI_{yz} a \qquad (5.132)$$

where $a = \frac{1}{R}$ = the curvature of the beam.

The signs attached to the quantities on the right hand sides of (5.131) and (5.132) indicate the directions in which the vectors representing M and M′ must be drawn along OY and OZ respectively according to the right-handed screw system. In this system a couple is represented by a vector perpendicular to its plane and the direction in which this vector is drawn is that direction in which a right-handed screw, whose axis is perpendicular to the couple, would move in a fixed nut if it were under the action of the couple. Thus a pair of equal but opposite couples in the same plane are represented by equal vectors along the same line but drawn in opposite directions.

The couple M is thus represented by a vector along OY, and M′ by a vector of length EaI_{yz} in the negative direction along OZ.

Now let the principal moments of inertia be $I_{y'}$ and $I_{z'}$.

In the theory of moments of inertia it is proved that

$$I_y = I_{y'} \cos^2 \theta + I_{z'} \sin^2 \theta \quad . \quad . \quad . \quad . \quad . \quad (5{\cdot}133)$$

and

$$I_{yz} = (I_{y'} - I_{z'}) \sin \theta \cos \theta \quad . \quad . \quad . \quad . \quad . \quad (5{\cdot}134)$$

θ being the inclination of OY to OY′

Now resolving the couples M and M′ along OY′ and OZ′ respectively we get

$$\begin{aligned} M_{y'} &= EaI_y \cos \theta + Ea\, I_{yz} \sin \theta \\ &= Ea\{I_{y'} \cos^3 \theta + I_{z'} \sin^2 \theta \cos \theta\} \\ &\quad + Ea(I_{y'} - I_{z'}) \sin^2 \theta \cos \theta \\ &= EaI_{y'} \cos \theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5{\cdot}135) \end{aligned}$$

and

$$\begin{aligned} M_{z'} &= EaI_y \sin \theta - Ea\, I_{yz} \cos \theta \\ &= Ea\{I_{y'} \cos^2 \theta \sin \theta + I_{z'} \sin^3 \theta\} \\ &\quad - Ea\,(I_{y'} - I_{z'}) \sin \theta \cos^2 \theta \\ &= EaI_{z'} \sin \theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5{\cdot}136) \end{aligned}$$

These are the two components of the couple about the principal axes.

We are now in a position to find the neutral axis when the axis of the bending moment is given. Let this bending moment be resolved into a pair of couples $M_{y'}$ and $M_{z'}$ about the principal axes of the section. Then

$$\frac{M_{z'}}{M_{y'}} = \frac{I_{z'}}{I_{y'}} \tan \theta$$

or

$$\tan \theta = \frac{M_{z'}}{I_{z'}} \div \frac{M_{y'}}{I_{y'}} \quad . \quad . \quad . \quad . \quad . \quad (5{\cdot}137)$$

Thus the angle θ is known since all the quantities on the right hand side of (5.137) are known. The radius of curvature caused by the couple is R given by

$$\frac{1}{R} = a = \frac{1}{E}\sqrt{\left(\frac{M_{y'}}{I_{y'}}\right)^2 + \left(\frac{M_{z'}}{I_{z'}}\right)^2} \quad . \quad . \quad . \quad . \quad (5{\cdot}138)$$

and the angle which the axis of the resultant couple makes with OY is φ determined by

$$\tan \varphi = \frac{M_{z'}}{M_{y'}}$$

It is worth while to notice that, if we regard the curvature of the central line of the beam as a vector drawn in the direction of the radius of curvature, which, in fig. 36 would be in the direction contrary to OZ, the components of curvature perpendicular to OY′ and OZ′ are $a \cos \theta$ and $a \sin \theta$. Writing these in the forms $a_{y'}$ and $a_{z'}$ equations (5.135) and (5.136) become

$$M_{y'} = Ea_{y'} I_{y'}, \quad \ldots \ldots \quad (5.139)$$
$$M_{z'} = Fa_{z'} I_{z'}, \quad \ldots \ldots \quad (5.140)$$

showing that each component of the bending moment perpendicular to a principal axis produces its own component curvature.

If the cross section of a beam has an axis of symmetry this axis must be one of the principal axes of inertia through the centre of gravity of the section, and the other principal axis is perpendicular to the first. If the cross section has more than two axes of symmetry then each of these is a principal axis, and consequently there is more than one pair of principal axes. In such a case it is proved in the theory of moments of inertia that all axes through the centre of gravity are principal axes and the moments of inertia about these axes are all equal. Such a beam bends, therefore, in the plane of the bending moment. An equilateral triangle, for instance, has three axes of symmetry, namely, the medians of the triangle. A square has four, namely, the two diagonals, and two lines parallel to the sides. A regular polygon of n sides has n axes of symmetry. Then, if the cross-section is a regular polygon the beam bends in the plane of the bending moment, and the amount of curvature produced by a given couple, whose axis is perpendicular to the length of the beam, is the same for all positions of the axis of the couple in the plane of the cross-section. For example, a rod with a square section, fixed in a given way and carrying a given load, has exactly the same deflexion when a pair of sides are vertical and when these sides are inclined at any angle to the vertical. The beam, however, is not equally strong in all positions because the maximum stress depends on h (equation 5.116), which varies as the rod is rotated about its axis, although M and I do not vary.

69. The theorem of three moments for uniform beams. When a beam rests on a number of supports at known levels there is a relation between the bending moments at three successive supports and the relative positions of those supports, a relation which is independent of the conditions outside those supports. This relation is expressed in Clapeyron's theorem of three moments, which we shall now prove for the case of a beam of uniform section.

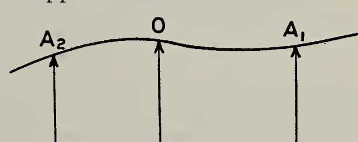

Fig. 37

Let the origin be taken at the middle support so that x and y are both zero at this support. Let the coordinates of the supports A_1 and

A_2 be (l_1, y_1) and $(-l_2, y_2)$. Let the load per unit length along OA_1 be w_1 and along OA_2 let it be w_2, and let η_1, η_2, be the downward deflections in these two regions. Then

$$EI\frac{d^4\eta_1}{dx^4} = w_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.141)$$

We shall have to integrate four times to get η_1, and it will be convenient, since we are regarding w_1 and w_2 as functions of x, and not necessarily constants, to write a short symbol for the fourth integral of w_1 with respect to x. Then let $f(x)$ be a function of x such that

$$w_1 = f^{IV}(x), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.142)$$

and $f(x)$ is that particular fourth integral of w_1 obtained by integrating each time between the limits o and x.

Thus

$$f'''(x) = \int_0^x w_1\,dx \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.143)$$

$$f''(x) = \int_0^x \left\{ \int_0^x w_1\,dx \right\} dx \quad . \quad . \quad . \quad . \quad (5.144)$$

and so on. It follows, by putting $x = 0$, that

$$f'''(0) = 0, \; f''(0) = 0, \; f'(0) = 0, \; f(0) = 0 \quad . \quad . \quad (5.145)$$

Now integrating equation (5.141) once we get

$$EI\frac{d^3\eta_1}{dx^3} = f'''(x) + A_1 . \quad . \quad . \quad . \quad . \quad . \quad (5.146)$$

If we put $x = 0$ in this the left hand side is the negative of the shearing force immediately to the right of O, and the right hand side is A_1. Then $-A_1$ represents the shearing force immediately to the right of O, and this differs from the shearing force immediately to the left of O by the amount of the supporting force at O.

Integrating equation (5.146) once again we get

$$EI\frac{d^2\eta_1}{dx^2} = f''(x) + A_1x + M_0, \quad . \quad . \quad . \quad . \quad (5.147)$$

the constant M_0 being clearly the bending moment immediately to the right of O, which is the same as the bending moment immediately to the left of O. Integrating twice more we get

$$EI\eta_1 = f(x) + \tfrac{1}{6}A_1x^3 + \tfrac{1}{2}M_0x^2 + Bx \quad . \quad . \quad . \quad (5.148)$$

The constant B represents the slope of the beam at the origin. Also, no constant is added at the last integration because $\eta_1 = 0$ when $x = 0$ by our choice of axes.

If $F(x)$ represents the fourth integral of w_2 obtained in the same way as $f(x)$ was obtained from w_1, then four successive integrations of the equation

$$EI\frac{d^4\eta_2}{dx^4} = w_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.149)$$

give

$$EI\eta_2 = F(x) + \tfrac{1}{6}A_2x^3 + \tfrac{1}{2}M_0x^2 + Bx \quad . \quad . \quad . \quad (5.150)$$

M_0 and B being the same constants as in equation (5.148); whereas A_2, being the negative shearing force, is different from A_1.

Putting l_1 for x in (5.147) and (5.148), and writing M_1 for the bending moment and y_1 for η_1 at this point we find

$$M_1 = f''(l_1) + A_1l_1 + M_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.151)$$

$$EI\eta_1 = f(l_1) + \tfrac{1}{6}A_1l_1^3 + \tfrac{1}{2}M_0l_1^2 + Bl_1 \quad . \quad . \quad . \quad (5.152)$$

Eliminating A_1 from these two equations we get

$$M_1l_1^2 - 6EIy_1 = -6f(l_1) + l_1^2f''(l_1) - 2M_0l_1^2 - 6Bl_1 \quad (5.153)$$

The corresponding equation obtained from the span OA_2 is, since $-l_2$ takes the place of l_1,

$$M_2l_2^2 - 6EIy_2 = -6F(-l_2) + l_2^2F''(-l_2) - 2M_0l_2^2 + 6Bl_2 \quad (5.154)$$

Now eliminating B from equations (5.153) and (5.154) we get finally

$$l_1l_2\{M_1l_1 + M_2l_2 + 2(l_1 + l_2)M_0\} - 6EI(l_2y_1 + l_1y_2)$$
$$= -6l_2f(l_1) - 6l_1F(-l_2) + l_1l_2\{l_1f''(l_1) + l_2F''(-l_2)\} \quad (5.155)$$

This is the theorem of three moments for the case we have considered.

As a particular case, suppose w_1 and w_2 are constants, and that the three supports are all at the same level, so that y_1 and y_2 are zero. Then

$$\left.\begin{aligned} f(x) &= \tfrac{1}{24}w_1x^4, \quad F(x) = \tfrac{1}{24}w_2x^4, \\ f^{II}(x) &= \tfrac{1}{2}w_1x^2, \quad F^{II}(x) = \tfrac{1}{2}w_2x^2. \end{aligned}\right\} \quad . \quad . \quad . \quad (5.156)$$

Therefore

$$M_1l_1 + M_2l_2 + 2(l_1 + l_2)M_0 = \tfrac{1}{4}w_1l_1^3 + \tfrac{1}{4}w_2l_2^3 \quad . \quad (5.157)$$

70. Two supports coincident.

At a point where a beam is clamped, that is, where its direction is fixed, we may look upon the beam as having a pair of supports at a zero distance apart, one above and one below the beam. We can therefore use the theorem of three moments to give a relation between the bending moment at a clamped end and the bending moment at the next support. Thus, in fig. 29, we may regard the point A_2 as coincident with O, and A_1 at the end B. Then for this case

$$M_2 = M_0, \quad M_1 = 0,$$
$$l_1 = l, \qquad l_2 = 0,$$
$$w_1 = w_2 = w \text{ (a constant)},$$
$$y_1 = y_2 = 0$$

Then, using equation (5.175), which applies to this case,

$$2l\mathrm{M}_0 = \tfrac{1}{4}wl^3,$$

whence $$\mathrm{M}_0 = \tfrac{1}{8}wl^2$$

which agress with (5.46).

Again, taking the case shown in fig. 28 we get everything as in the last case except that now

$$\mathrm{M}_1 = \mathrm{M}_0.$$

Therefore equation (5.157) gives

$$3\mathrm{M}_0 l = \tfrac{1}{4}wl^3,$$

whence $$\mathrm{M}_0 = \tfrac{1}{12}wl^2,$$

which agrees with (5.41).

We can apply this method to find the unknown bending moments at the ends of a beam clamped horizontally at the same level at both ends under any load whatever provided there is no support between the ends applying an unknown force.

Let the bending moments at the left and right hand ends be M_2 and M_1 respectively, and let us first regard the left hand end as the point at which there are two supports. Then in equation (5.155)

$$\left.\begin{aligned} &\mathrm{M}_2 = \mathrm{M}_0 \\ &l_2 = 0,\ l_1 = l \\ &y_1 = y_2 = 0 \\ &w_1 = w_2 = w = f^{\mathrm{IV}}(x) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (5.158)$$

In order to use equation (5.155) for cases where l_1 or l_2 is infinitesimal it is necessary to divide by this infinitesimal quantity. Dividing by $l_1\ l_2$ at once, the general equation becomes

$$\mathrm{M}_1 l_1 + \mathrm{M}_2 l_2 + 2(l_1 + l_2)\mathrm{M}_0 - 6\mathrm{EI}\left(\frac{y_1}{l_1} + \frac{y_2}{l_2}\right)$$
$$= -\frac{6}{l_1}f(l_1) - \frac{6}{l_2}\mathrm{F}(-l_2) + l_1 f''(l_1) + l_2\mathrm{F}''(-l_2) \quad . \ (5.159)$$

In the present case both y_2 and l_2 are infinitesimal, and consequently $\frac{y_2}{l_2}$ must be regarded as the slope of the beam at the point where M_0 and M_2 act. Since the beam is clamped horizontally this slope is zero, but it is clear that if the beam were fixed in any other direction but the horizontal at the end where M_2 acts we should have to take account of this slope.

The value of $f(l_1)$, expressed as an integral, is

$$f(l_1) = \int_0^{l_1}\left\{\int_0^x\int_0^x\int_0^x w\,dx\,dx\,dx\right\}dx \quad . \quad . \quad . \quad . \quad . \ (5.160)$$

the symbol on the right indicating four successive integrations, the first three from 0 to x and the last from 0 to l_1.

$F(-l_2)$ is a function similar to $f(l_1)$ and it is clear that, if l_2 is small, $F(-l_2)$ is of the fourth order in l_2. Consequently $\frac{1}{l_2}F(-l_2)$ is of the third order in l_2 which vanishes when l_2 vanishes. Moreover, $l_2F''(-l_2)$ obviously vanishes when l_2 vanishes. Therefore equation (5.155) becomes, on the assumption of two supports at the left hand end,

$$l(M_1 + 2M_2) = -\frac{6}{l}f(l) + lf''(l)$$

whence

$$M_1 + 2M_2 = f''(l) - \frac{6}{l^2}f(l) \quad . \quad . \quad . \quad . \quad . \quad (5.161)$$

If we next regard the other end as having two supports and write x_1 for the distance, measured along the beam from that end, that is, $x_1 = (l - x)$, then we shall get another equation exactly similar to (5.161) but with M_1 and M_2 interchanged, and with w expressed in terms of x_1 instead of x. Thus

$$M_2 + 2M_1 = \varphi''(l) - \frac{6}{l^2}\varphi(l) \quad . \quad . \quad . \quad . \quad . \quad (5.162)$$

where the function $\varphi^{IV}(x_1)$ is defined by

$$\varphi^{IV}(x_1) = w = f^{IV}(x) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.163)$$

and

$$\varphi(x_1) = \int_0^{x_1}\int_0^{x_1}\int_0^{x_1}\int_0^{x_1} w\,dx_1\,dx_1\,dx_1\,dx_1 \quad . \quad . \quad . \quad . \quad (5.164)$$

We may express our symbols physically in this way: $\frac{1}{EI}f(l)$ and $f''(l)$ would be the deflexion and the bending moment at the right hand end if, with the same loading, the right hand end were so held that the shearing force, the bending moment, the slope, and the deflexion were all zero at the left hand end. $\frac{1}{EI}\varphi(l)$ and $\varphi''(l)$ are similarly expressed with left and right interchanged in the preceding.

71. Concentrated loads.

A concentrated load or several concentrated loads may be covered by w. It is only necessary to notice that

$$f'''(x) = \int_0^x w\,dx$$

$$= \text{total load from 0 to } x \quad . \quad (5.165)$$

which total load may include any number of concentrated loads.

Also $$f''(x) = \left\{\begin{matrix}\text{bending moment at } x \text{ assuming} \\ \text{that the end } x = 0 \text{ is free}\end{matrix}\right\} \quad . \quad . \quad (5.166)$$

To illustrate the use of equations (5.161) and (5.162) we shall find the bending moments at the ends of the beam in fig. 32. Here, measuring x from the left hand end,

$$\left.\begin{aligned} f''(x) &= 0 \text{ when } x < a \\ &= W(x-a) \text{ when } x > a \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (5.167)$$

Also

$$f'(x) = \int_0^x f''(x)dx$$

$$= 0 \text{ when } x < a$$

and

$$= \int_0^a f''(x)dx + \int_a^x f''(x)dx$$

$$= 0 + \tfrac{1}{2}W(x-a)^2 \text{ when } x > a.$$

Likewise

$$\left.\begin{aligned} f(x) &= 0 \text{ when } x < a \\ &= \tfrac{1}{6}W(x-a)^3 \text{ when } x > a \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (5.168)$$

Therefore equation (5.161) gives

$$M_1 + 2M_2 = W(l-a) - \frac{1}{l^2}W(l-a)^3$$

$$= \frac{W}{l^2}a(l-a)(2l-a) \quad . \quad . \quad . \quad . \quad . \quad (5.169)$$

For equation (5.162) we need only write $(l-a)$ for a in the preceding, and interchange M_1 and M_2. Then

$$M_2 + 2M_1 = \frac{W}{l^2}(l-a)a(l+a) \quad . \quad . \quad . \quad . \quad (5.170)$$

Solving (5.169) and (5.170) for M_1 and M_2 we get

$$\left.\begin{aligned} M_1 &= \frac{W}{l^2}a^2(l-a) \\ M_2 &= \frac{W}{l^2}a(l-a)^2 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.171)$$

M_1 being the bending moment at B, and M_2 the bending moment at O.

72. Transverse forces in different planes acting on a uniform beam.

Let a pair of axes OY, OZ, be parallel to the principal axes of the normal sections of an unstrained beam. Let the force per unit length at any section x be resolved into two components w_1 and w_2 parallel to OY and OZ respectively. Let the components of the bending moment at x be M_1 and M_2 about lines parallel to OZ and OY respectively, and let the corresponding moments of inertia of the section be I_1, I_2. Then by equations (5.139) and (5.140),

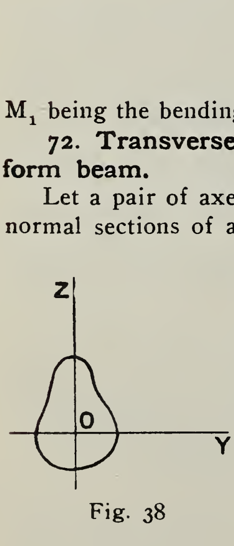

Fig. 38

$$\left.\begin{aligned} M_1 &= EI_1\frac{d^2y}{dx^2} \\ M_2 &= EI_2\frac{d^2z}{dx^2} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.172)$$

y and z being the component displacements of the beam at x.

Differentiating each of these twice we get

$$\left.\begin{aligned}\frac{d^2M_1}{dx^2} &= EI_1\frac{d^4y}{dx^4}\\ \frac{d^2M_2}{dx^2} &= EI_2\frac{d^4z}{dx^4}\end{aligned}\right\} \quad \ldots \ldots \quad (5{\cdot}173)$$

But the relation between M_1 and w_1 is just the same as if M_2 and w_2 were zero, for this relation is obtained by resolving forces in one plane. That is,

$$\frac{d^2M_1}{dx^2} = w_1, \quad \frac{d^2M_2}{dx^2} = w_2 \quad \ldots \ldots \quad (5{\cdot}174)$$

Therefore

$$\left.\begin{aligned}EI_1\frac{d^4y}{dx^4} &= w_1\\ EI_2\frac{d^4z}{dx^4} &= w_2\end{aligned}\right\} \quad \ldots \ldots \quad (5{\cdot}175)$$

Each of these equations can be solved independently of the other. The method of solution is exactly the same for each equation as if the deflexion were entirely in one plane. Then it follows that the usual equations for a beam under forces parallel to one principal axis only can be used for this case, the only thing new being the resolving of the forces parallel to the two principal axes. Since even a concentrated force can be supposed to be a force distributed over a very small length of the beam it is clear that a concentrated force can also be handled by resolving it into two components parallel to OY and OZ, and then treating each component in the same way, as if only that component acted.

Even if the forces on a beam are all in one plane and that plane is not parallel to one of the principal axes we are obliged to use the method of the present article.

73. A particular example.

As an easy example to illustrate the method suppose a rod of length l with a uniform circular section is acted on by a uniform force w per unit length the direction of w at x being the direction of the radius of a helix which wraps once round the whole length of the rod; that is, if OZ is parallel to w at O, then the two components of w at x are

$$\left.\begin{aligned}w_1 &= w\sin\frac{2\pi x}{l}\\ w_2 &= w\cos\frac{2\pi x}{l}\end{aligned}\right\} \quad \ldots \ldots \quad (5{\cdot}176)$$

Suppose the end $x = 0$ is built into a rigid body, and suppose that, at $x = l$, a smooth fixed pin parallel to the z-axis passes through the rod, so that the rod is free to slide along the pin while the pin maintains its direction. Then the boundary conditions are

$$\left.\begin{array}{l} y=0, \quad \dfrac{dy}{dx}=0, \\ z=0, \quad \dfrac{dz}{dx}=0, \end{array}\right\} \text{ where } x=0 \quad . \quad . \quad . \quad . \quad (5.177)$$

$$\left.\begin{array}{l} M_1=0, \quad F_1=0, \\ \dfrac{dz}{dx}=0, \quad F_2=0, \end{array}\right\} \text{ where } x=l. \quad . \quad . \quad . \quad (5.178)$$

The two principal moments of inertia of the section of the rod are equal; let the common value be I. Then

$$\left.\begin{array}{l} EI\dfrac{d^4y}{dx^4}=w\sin\dfrac{2\pi x}{l} \\ EI\dfrac{d^4z}{dx^4}=w\cos\dfrac{2\pi x}{l} \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.179)$$

The solutions of these equations satisfying the conditions at $x=0$ are

$$\left.\begin{array}{l} EIy=\dfrac{l^4}{16\pi^4}w\sin\dfrac{2\pi x}{l}+Ax^3+Bx^2 \\ EIz=\dfrac{l^4}{16\pi^4}w\cos\dfrac{2\pi x}{l}+A_2x^3+B_2x^2 \end{array}\right\} \quad . \quad . \quad (5.180)$$

The conditions for y at $x=l$ give

$$0=-\frac{wl^2}{4\pi^2}\sin 2\pi+6Al+2B$$

$$0=-\frac{wl}{2\pi}\cos 2\pi+6A$$

whence

$$A=\frac{wl}{12\pi}, \quad B=-\frac{wl^2}{4\pi}.$$

Therefore $$EIy=\frac{wl}{4\pi}\left\{\frac{l^3}{4\pi^3}\sin\frac{2\pi x}{l}+\tfrac{1}{3}x^3-lx^2\right\} \quad . \quad . \quad . \quad (5.181)$$

Also, the conditions for z at $x=l$ give

$$0=3A_2l^2+2B_2l,$$

$$0=\frac{wl}{2\pi}\cos 2\pi+6A_2;$$

whence

$$A_2=-\frac{wl}{12\pi},$$

$$B_2=\frac{wl^2}{8\pi}.$$

Therefore $$EIz=\frac{wl}{4\pi}\left\{\frac{l^3}{4\pi^3}\cos\frac{2\pi x}{l}-\tfrac{1}{3}x^3+\tfrac{1}{2}lx^2\right\} \quad . \quad . \quad . \quad (5.182)$$

CHAPTER VI

THIN RODS UNDER TENSION OR THRUST

74. Stresses in a rod under bending moment and tension. In treating of the bending of beams in the last chapter we assumed always that the neutral axis of any section passed through the centre of gravity of that section. When the neutral axis is in this position the resultant of the tensional stresses across the section is a pure

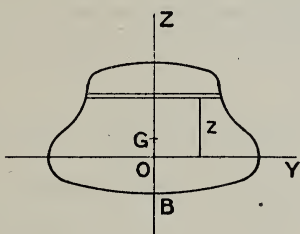

Fig. 39a

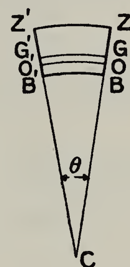

Fig. 39b

couple. Let us now assume that the centre of gravity of a particular section of a bent rod is at a distance r from the neutral axis, the fibres through the centre of gravity being in tension. In fig. 39(a) OY is the neutral axis and G the centre of gravity, the length of OG being r. Let R be the radius of curvature at the point G of the line of fibres through G. Let fig. 39(b) represent a short piece of the rod bounded by two cross sections BOGZ and B′O′G′Z′ which were parallel before strain. In this figure O′O, being on the neutral axis, has its natural length; G′G has a length $R\theta$, and O′O a length $(R-r)\theta$. A thin bundle of fibres parallel to O′O, of cross section dA, and cropping out of the section OYZ at distance z from OY has a strained length $(R-r+z)\theta$. Since its natural length was the same as that of O′O, namely, $(R-r)\theta$, its strain is therefore

$$\frac{z\theta}{(R-r)\theta}=\frac{z}{R-r}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.1)$$

and the tension in these fibres is

$$dT = f dA = \frac{Ez}{R-r} dA \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.2)$$

The total tension across the section is thus

$$T = \int \frac{Ez}{R-r} dA$$
$$= \frac{E}{R-r} \int z dA \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.3)$$

But by a property of the centre of gravity

$$\int z dA = \bar{z} A = r A, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.4)$$

$\bar{z}$ being the z of the centre of gravity of the section. Hence

$$T = \frac{Er}{R-r} A \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.5)$$

Denoting the stress of the fibres through G by f_1 we find

$$f_1 = \frac{Er}{R-r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.6)$$

and therefore

$$T = A f_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.7)$$

thus showing that the *mean* stress across the section is the stress at the centre of gravity.

Again the moment of dT about an axis through G parallel to OY is

$$dM = (z-r) dT$$
$$= (z-r) \frac{Ez}{R-r} dA \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.8)$$

The total moment of all the stresses about the same axis is

$$M = \int \frac{E(z^2 - rz)}{R-r} dA$$
$$= \frac{E}{R-r} \int z^2 dA - \frac{Er}{R-r} \int z dA$$
$$= \frac{E}{R-r} \left\{ I - r(rA) \right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.9)$$

where I_y is the moment of inertia of the section about OY. But, if I denotes the moment of inertia of the section about the parallel axis through G,

$$I_y = I + r^2 A \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.10)$$

Therefore

$$M = \frac{EI}{R-r}$$

In the preceding investigation we have retained $(R-r)$ but there will be very little error in writing R for this since r is certain to be small compared with R. Making this assumption we get

$$f=\frac{Ez}{R} \quad \ldots \ldots \quad (6.11)$$

$$T=\frac{Er}{R}A=f_1A \quad \ldots \ldots \quad (6.12)$$

$$M=\frac{EI}{R} \quad \ldots \ldots \quad (6.13)$$

This shows that the resultant of the tensions across a section which is symmetrical about OZ is the tension T acting through G, together with the couple M, which is the same couple as if the neutral axis passed through the centre of gravity itself. If then, we always take moments about an axis through the centre of gravity of a cross section, the bending moment is always equal to $\frac{EI}{R}$ whether or not there is a tension (or thrust) in addition to the bending couple.

75. To find the position of the neutral axis.

If we are given T and M we can find the position of the neutral axis and therefore the stress at any point. Thus from (6.12) and (6.13)

$$\frac{T}{M}=\frac{rA}{I}$$

or

$$r=\frac{T}{M}\times\frac{I}{A} \quad \ldots \ldots \quad (6.14)$$

which gives the distance of the neutral axis from the centre of gravity. If T is a tension the fibres through the centre of gravity are in tension and therefore the neural axis is on the side of the centre of gravity towards the thrusts; whereas if T is negative, that is, represents a thrust, then the fibres through the centre of gravity are in thrust and the neutral axis is on the tension side of the centre of gravity.

Neglecting r compared with R the stress at distance z from the neutral axis is

$$f=\frac{Ez}{R}=\frac{M}{I}z$$

Let z' be the distance of the same fibres from the axis through G parallel to the neutral axis,

Then

$$f=\frac{M}{I}(z'+r)=\frac{M}{I}z'+\frac{M}{I}r$$

$$=\frac{M}{I}z'+\frac{T}{A} \text{ by (6.14)} \quad \ldots \quad (6.15)$$

The two terms in the expression for f are the stresses due to the bending moment M and the tension T separately, the stress for each of these actions being calculated as if the other were not present.

76. Euler's theory of struts.

Suppose a pair of balancing forces, each of magnitude P, act at the ends of a thin uniform rod of length l, these ends being free from couples. As long as the rod remains straight it is in equilibrium, and it can only fail by crushing, that is, because the compressive stress set up is greater than the material of the rod can stand. But it is a well-known fact that a rod, whose length is thirty or more times its breadth, may bend or buckle long before it fails by crushing. If the forces P are gradually increased from zero, there is, in fact, a particular magnitude of these forces at which the straight state of the rod ceases to be stable, although equilibrium is still possible.

Fig. 40

Suppose the rod is slightly bent under the action of the forces P at the ends O and B (fig. 40). Let Q be any point on the rod and let ON $= x$, NQ $= y$. Let M denote the bending moment at Q. Then by taking moments about Q of the forces acting on the part OQ,

$$M = Py \qquad (6.16)$$

But by equation (5.13)

$$M = \pm EI\frac{d^2y}{dx^2} \qquad (6.17)$$

y being now used instead of η.

From (6.16) and (6.17)

$$EI\frac{d^2y}{dx^2} = \pm Py \qquad (6.18)$$

The ambiguity of sign in (6.18) is removed by the fact that, if y is positive in the figure, then $\frac{d^2y}{dx^2}$ is certainly negative (and vice versa). Hence

$$EI\frac{d^2y}{dx^2} = -Py \qquad (6.19)$$

Let this be written in the form

$$\frac{d^2y}{dx^2} = -n^2y \qquad (6.20)$$

where

$$n^2 = \frac{P}{EI} \qquad (6.21)$$

The solution of (6.20) is

$$y = A \cos nx + B \sin nx \qquad (6.22)$$

The two additional facts that we know about the rod are that

$$\left.\begin{array}{r} y = 0 \text{ where } x = 0 \\ \text{and where } x = l \end{array}\right\} \quad \ldots \ldots \quad (6.23)$$

These conditions give

$$0 = A \cos 0 + B \sin 0$$
$$= A$$

and

$$0 = A \cos nl + B \sin nl$$
$$= B \sin nl \text{ (since } A = 0) \quad \ldots \ldots \quad (6.24)$$

There are two distinct ways in which equation (6.24) can be satisfied; firstly, if $B = 0$, in which case $y = 0$ everywhere, and the rod is straight; secondly, if

$$\sin nl = 0,$$

that is, if

$$nl = \pi, \text{ or } 2\pi, \text{ or } 3\pi, \text{ etc.,} \quad \ldots \quad (6.25)$$

Taking the first of these values of n, we get

$$n = \frac{\pi}{l} \quad \ldots \ldots \quad (6.26)$$

whence

$$P = EI\frac{\pi^2}{l^2} \quad \ldots \ldots \quad (6.27)$$

and

$$y = B \sin \frac{\pi x}{l} \quad \ldots \ldots \quad (6.28)$$

As x varies from 0 to l the angle $\frac{\pi x}{l}$ varies from 0 to π, and therefore the rod takes the form of one half wave of a sine curve, as shown in fig. 40. It is remarkable that the two end-conditions of the rod have determined A and P, and not the two constants of integration A and B, as such conditions usually do. The interpretation of our result is that there is a minimum value of P, given by (6.27), which will bend the rod, and this same value of P will bend the rod into the curve given by (6.28) with any value of the constant B. It should be borne in mind, however, that if B is large and therefore the deflexion y large, then the approximate expression used for curvature in equation (6.17) is not valid. For a large deflection we must use the correct expression for curvature, namely,

$$\pm \frac{\frac{d^2y}{dx^2}}{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}$$

which will give a different value of P from the one we have obtained, a value depending on the amount of deflection (see Art 88), which, however, will approach the value in (6.27) as the deflection is made to approach zero. The value of P we have found, is, in fact, the force at which the rod is unstable in the straight state.

If we take the second value of n from (6.25) we find,

$$P = EI\frac{4\pi^2}{l^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.29)$$

and $$y = B\sin\frac{2\pi x}{l} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.30)$$

This curve is shown in fig. 41 (a)

The rod is in equilibrium in this state but it is less stable equilibrium than in the form of one half wave. The point C must be held in

Fig. 41 a

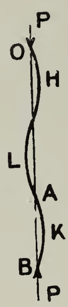

Fig. 41 b

position in the line OB to make it stable, although theoretically no force is needed to keep C in position, just as no force is needed to hold a needle in position when it is standing in equilibrium on its point.

The third value of n gives

$$P = EI\frac{9\pi^2}{l^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.31)$$

$$y = B\ \sin\frac{3\pi x}{l} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.32)$$

and the curve is shown in fig. 41 (b).

Although there are many possible states of equilibrium of the rod under the pair of thrusts P there is only one stable state with the ends at a distance apart less than l and that is the one shown in fig. 40, and the corresponding value of P in (6.27) is often called the breaking load for the rod used as a strut, but it is really the buckling load. The breaking load will be shown later to be greater than the buckling load.

It should be noticed that the half wave OC of the rod in fig. 41 (a) is exactly similar to the whole rod in fig. 40. There is a pure thrust and no bending moment at C and we can get the results for the rod in the state shown in fig. 41 (a) by putting $\frac{1}{2}l$ for l in equations (6.27) and (6.28).

77. Rod clamped at both ends.

Suppose that a strut of length l under the action of balancing forces P is clamped so that the tangents at the ends are in the direction of the P's. In this case there is an unknown couple M_0 acting at each end. The bending moment at distance x from one end O (fig. 42) is

Fig. 42

$$M = M_0 - Py$$

Hence

$$EI\frac{d^2y}{dx^2} = M_0 - Py \quad . \quad . \quad . \quad . \quad (6.33)$$

takes the place of equation (6.19). Writing y_1 for $\left(y - \frac{M_0}{P}\right)$ this last equation becomes

$$EI\frac{d^2y_1}{dx^2} = -Py_1, \quad . \quad . \quad . \quad (6.34)$$

which in the same equation for y_1 as we had for y in (6.19). Again, writing n^2 for $\frac{P}{EI}$, the solution is

$$y_1 = A\cos nx + B\sin nx,$$

whence

$$y = \frac{M_0}{P} + A\cos nx + B\sin nx \quad . \quad . \quad . \quad . \quad (6.35)$$

The end-conditions are now

$$\left.\begin{aligned} y &= 0 \\ \frac{dy}{dx} &= 0 \end{aligned}\right\} \begin{aligned} &\text{both where } x = 0 \\ &\text{and where } x = l \end{aligned}\Bigg\} \quad . \quad . \quad . \quad . \quad . \quad (6.36)$$

These conditions give

$$0 = \frac{M_0}{P} + A \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.37)$$

$$0 = \frac{M_0}{P} + A\cos nl + B\sin nl \quad . \quad . \quad . \quad . \quad (6.38)$$

$$0 = nB \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.39)$$

$$0 = n\{-A\sin nl + B\cos nl\} \quad . \quad . \quad . \quad . \quad (6.40)$$

From the first three of these equations we get

$$0 = A(1 - \cos nl), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.41)$$

and from the last two

$$0 = A\sin nl \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.42)$$

Thus, either A is zero as well as B, or

$$nl = 2\pi, \text{ or } 4\pi, \text{ or } 6\pi, \text{ etc.} \quad . \quad . \quad . \quad . \quad . \quad (6.43)$$

The only stable state with the ends at a less distance apart than l corresponds to the first value of nl. With this value of n

$$P = EI \times \frac{4\pi^2}{l^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.44)$$

and

$$y = \frac{M_0}{P}\left(1 - \cos\frac{2\pi x}{l}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.45)$$

There is one complete wave of the cosine curve on the rod, the distance from one crest to the next crest on the same side, as shown in fig. 42, or similar to the portion from H to K in fig. 41(b).

There are again many possible equilibrium states, one corresponding to each of the infinite number of values of n given by (6.43), but only the first gives any sort of stability except when P is less than its value in (6.29), in which case the straight state is stable. It should be noticed that the load at which the straight state is unstable, when the ends are clamped, is the same as the load that would hold the rod in the form of a complete wave of a sine curve when the ends are not clamped, that is, the load given in (6.29).

78. Strut eccentrically loaded.

Let a pair of balancing forces P act along a line parallel to the line joining the centres of gravity of the end sections of a thin uniform rod of length l, and at distance a from that line. If, in fig. 43, $ON = x$, $NQ = y$, the equation connecting P and y is

Fig. 43

$$EI\frac{d^2y}{dx^2} = P(y + a) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.46)$$

the solution of which is

$$y + a = A\cos nx + B\sin nx \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.47)$$

with the usual value of n.

The end-conditions are that

$$\left.\begin{array}{r} y = 0 \text{ where } x = 0 \\ \text{and where } x = l \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.48)$$

Therefore

$$a = A \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.49)$$

$$a = A\cos nl + B\sin nl \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.50)$$

whence

$$B = a\frac{1 - \cos nl}{\sin nl}$$

$$= a\frac{2\sin^2\frac{1}{2}nl}{2\sin\frac{1}{2}nl\cos\frac{1}{2}nl}$$

$$= a\tan\tfrac{1}{2}nl \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.51)$$

On substituting these values of the constants in (6.47) we get

$$y + a = a\cos nx + a\tan\tfrac{1}{2}nl\sin nx \quad . \quad . \quad . \quad . \quad (6.52)$$

Now there are particular values of n that make $\tan\frac{1}{2}nl$ infinite, and these values make y infinite also. The only interpretation of this mathematical infinity in our equation is that the rod is unstable for the values of n satisfying the equation

$$\tan \tfrac{1}{2} nl = \pm \infty \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.53)$$

that is, the values of n given by

$$\tfrac{1}{2} nl = \frac{\pi}{2}, \ \frac{3\pi}{2}, \ \frac{5\pi}{2}, \text{ etc.} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.54)$$

The load at which instability really appears is the first one, given by

$$\frac{P}{EI} = n^2 = \frac{\pi^2}{l^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.55)$$

exactly the same as if the load were not eccentric. In this case the strut bends under any load, however small, and the deflection is correctly given by (6.52) before instability is reached, but at the particular load given by (6.55) the rod is really unstable, since equilibrium is not now possible, because infinite deflections are certainly not possible.

If the length a is large the stress in the rod may become so great before the load is great enough to produce instability that the rod fails just as a beam would fail under transverse loads. If there is any possibility of failure in this way it will be necessary to make two separate calculations, one to find the safe load assuming instability impossible, the other to find the buckling load assuming that the stresses do not become unsafe before buckling begins. The smaller of these loads is the safe load for the strut.

79. Strut clamped at one end.

Suppose a strut is clamped at one end O, that is, under the action of a force and a couple, so that the line of action of the force is along

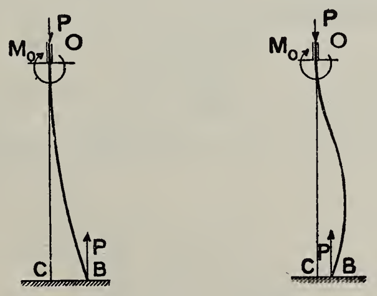

Fig. 44a Fig. 44b

the tangent at that end; and suppose that the other end is in contact with a smooth plane which is perpendicular to the force at the clamped end. Then it is clear, without working out the equations afresh, that the curve is a portion of a sine curve with the clamped end at one of the crests and the other end at one of the points of inflexion. The strut has two stable forms when the straight form is unstable and these forms are shown in figs 44(a) and 44(b).

In fig. 44(a) the rod forms a quarter of a sine wave, and corresponds to the length LA of fig. 41 b. In fig. 44(b) the rod forms three quarters of a sine wave and corresponds to LB in fig. 41(b). In the first case a length $2l$ would form half a wave, and in the latter case a length $\frac{2}{3}l$ forms half a wave. Then the values of P for the two cases are obtained by putting $2l$ for l or $\frac{2}{3}l$ for l in equation (6.27). These values are

$$\left.\begin{aligned} P &= EI\frac{\pi^2}{4l^2} \text{ for Fig. 44 (a)} \\ \text{and} \qquad P &= EI\frac{9\pi^2}{4l^2} \text{ for Fig. 44 (b)} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (6.56)$$

The couple M_0 at O balances the couple formed by the two P's at the ends of the rod, and therefore, in both cases

$$M_0 = CB \times P \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.57)$$

80. The same strut with the end B in the line of the tangent at O.

It is clear that the end B in fig. 44(a) could not be brought into the position C without the action of a transverse force in the direction BC. Let us suppose that a force X is applied at B in the direction BC so as to bring B to C. There must be a balancing force X at O together with a couple Xl to balance the pair of X forces, which couple will only alter the magnitude of M_0. We shall solve this problem to find what difference the force X makes in P.

Fig. 45

The bending moment at distance x from O in the direction of M_0 is

$$M = M_0 - Py + Xx$$

that is,

$$EI\frac{d^2y}{dx^2} = M_0 - Py + Xx$$
$$= P\left\{\frac{M_0}{P} - y + \frac{X}{P}x\right\} \quad . \quad . \quad (6.58)$$

Now let

$$y_1 = y - \frac{M_0}{P} - \frac{X}{P}x$$

Then

$$EI\frac{d^2y_1}{dx^2} = EI\frac{d^2y}{dx^2}$$
$$= -Py_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.59)$$

The solution of this, with n^2 for $\frac{P}{EI}$, is

$$y_1 = A\cos nx + B\sin nx,$$

whence $$y=\frac{M_0}{P}+\frac{X}{P}x+A\cos nx+B\sin nx \quad . \quad . \quad (6.60)$$

The end-conditions are

$$\left.\begin{aligned} y&=0 \\ \frac{dy}{dx}&=0 \end{aligned}\right\} \text{ where } x=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.61)$$

and $$y=0 \text{ where } x=l \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.62)$$

These give

$$0=\frac{M_0}{P}+A \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.63)$$

$$0=\frac{X}{P}+nB \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.64)$$

$$0=\frac{M_0}{P}+\frac{X}{P}l+A\cos nl+B\sin nl \quad . \quad . \quad . \quad (6.65)$$

We have one other condition which we have not yet used, namely that

$$M=0 \text{ where } x=l.$$

Therefore

$$0=M_0+Xl \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.66)$$

Equations (6.63) (6.64) and (6.66) give

$$A=-\frac{M_0}{P}=\frac{Xl}{P}=-nlB, \quad . \quad . \quad . \quad . \quad . \quad (6.67)$$

and therefore (6.65) is equivalent to

$$A\left\{\cos nl-\frac{\sin nl}{nl}\right\}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.68)$$

Assuming that A is not zero we find that

$$\tan nl=nl \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.69)$$

The most direct way of solving this equation is by plotting the graphs

$$\left.\begin{aligned} y&=\tan x \\ \text{and} \quad y&=x \end{aligned}\right| \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.70)$$

and finding the abscissae of their points of intersection. The graphs show that the smallest root corresponds to an angle in the third quadrant and is nearly $\frac{3\pi}{2}$.

The arithmetic is simplified by putting

$$nl=\frac{3\pi}{2}-u.$$

Then equation (6.69) becomes

$$\begin{aligned} \cot u&=\frac{3\pi}{2}-u \\ &=\frac{3\pi}{2} \text{ nearly} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.71) \end{aligned}$$

The first approximation to the root is

$$u_1 = 0\cdot 207$$

A better approximation is the value of u_2 given by

$$\cot u_2 = \frac{3\pi}{2} - u_1$$

$$= 4\cdot 506$$

whence $$u_2 = 0\cdot 218$$

A still better approximation is the value of u_3 satisfying

$$\cot u_3 = \frac{3\pi}{2} - u_2$$

$$= 4\cdot 495$$

The final value, to 4 figures, is

$$nl = 4\cdot 494 = 1\cdot 431\pi \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.72)$$

and therefore $$P = 1\cdot 431^2 \frac{\pi^2}{l^2} EI$$

$$= 2\cdot 048 \frac{\pi^2}{l^2} EI \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.73)$$

This result, it will be seen, does not differ much from the second value of P in equation (6.56).

81. Rankine's empirical formula for struts.

The buckling loads calculated by Euler's method are the loads at which the straight rod becomes unstable assuming that it does not fail in any other way before this buckling occurs. But a very short rod would clearly fail by crushing before it buckled. If f is the maximum intensity of compressive stress that the material will stand without taking permanent set, and if A is the area of the cross section, the short rod will fail when

$$P = Af, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.74)$$

and a very long rod with no couples at the ends would fail when

$$P = EI\frac{\pi^2}{l^2} = EAk^2\frac{\pi^2}{l^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.75)$$

where k is the radius of gyration of the area of the section for the axis through the centre of gravity perpendicular to the plane of bending.

Now if we write P_1 for the value of P given by (6.74) and P_2 for the value given by (6.75) then the equation

$$\frac{1}{P} = \frac{1}{P_1} + \frac{1}{P_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.76)$$

makes $P = P_1$ when $P_2 = \infty$, and $P = P_2$ when $P_1 = \infty$. At these two limits the equation gives the correct values of P, for clearly failure will occur by crushing if l is very small, that is, if P_2 is very great; and failure will occur by buckling if f is very great, that is if P_1 is very great. But it is to be expected that the load at which failure occurs will vary gradually as f or l is varied, and therefore that there is a continuous variation of P from P_1 to P_2. For such reasons Rankine gave equation (6.76) as an empirical formula for any strut, where P_1 is the load that would crush it, and P_2 is the load that would buckle it. For the strut with hinged ends

$$P_2 = EAk^2\frac{\pi^2}{l^2}; \qquad (6.77)$$

for the strut with clamped ends

$$P_2 = 4EAk^2\frac{\pi^2}{l^2}; \qquad (6.78)$$

for the strut clamped at one end and hinged at the other end, the hinge being on the tangent at the clamped end,

$$P_2 = 2{\cdot}048\, EAk^2\frac{\pi^2}{l^2} \qquad (6.79)$$

In all cases

$$P_2 = qEAk^2\frac{\pi^2}{l^2} \qquad (6.80)$$

Then Rankine's formula can be written

$$\begin{aligned}\frac{1}{P} &= \frac{1}{P_1}\left\{1 + \frac{P_1}{P_2}\right\}\\ &= \frac{1}{P_1}\left\{1 + \frac{fl^2}{q\pi^2 Ek^2}\right\}\\ &= \frac{1}{P_1}\left\{1 + \frac{cl^2}{k^2}\right\} \qquad (6.81)\end{aligned}$$

where c denotes the factor $\dfrac{f}{q\pi^2 E}$, which is a constant for struts of the same material with the ends held in the same way. The constant q depends only on the way in which the ends are held.

For a solid rod with a circular section of radius r the value of k^2 is $\frac{1}{4}r^2$, and for a thin tube of circular section $k^2 = \frac{1}{2}r^2$. For a rod of square section, side a, $k^2 = \frac{1}{12}a^2$.

A rod cannot fail by pure buckling if $P_1 > P_2$, that is, if

$$f > qEk^2\frac{\pi^2}{l^2} \qquad (6.82)$$

For a solid circular rod hinged at both ends this inequality becomes

$$f > \tfrac{1}{4}Er^2\frac{\pi^2}{l^2} \qquad (6.83)$$

Taking a steel strut for which the yield point is about 25 tons per square inch, and E about 13000 tons per square inch, the above inequality gives

$$\frac{l^2}{r^2} > \frac{13000\,\pi^2}{4 \times 25}$$

whence

$$\frac{l}{2r} > 18 \qquad (6.84)$$

For a thin tube we get

$$\frac{l}{2r} > 25.5 \qquad (6.85)$$

Then it follows that Euler's rule alone should not be used for a hinged steel strut whose length is not greater than about 30 times its diameter, and it is better to use Rankine's formula for lengths between say, five and forty diameters.

82. Strut subject to a lateral force.

Let us take the case of a uniform strut with hinged ends under a pair of balancing forces P through the centres of gravity of the end sections and a single force Q perpendicular to the strut at distances a and b from the ends, with $(a+b)=l$. The forces at the ends necessary to balance Q are $\frac{b}{l}$ Q and $\frac{a}{l}$ Q as shown

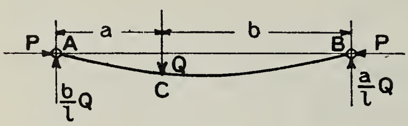

Fig. 46

in fig. 46. Taking the origin at A and denoting quantities in the portion AC by suffix (1), as y_1, M_1, etc., we find that the bending moment in this portion is

$$M_1 = -Py_1 - \frac{bx}{l}Q \qquad (6.86)$$

Therefore

$$EI\frac{d^2y_1}{dx^2} = -Py_1 - \frac{bx}{l}Q$$

$$= -P\left\{y_1 + \frac{Qb}{Pl}x\right\} \qquad (6.87)$$

By putting z_1 for $\left\{y_1 + \frac{Qb}{Pl}x\right\}$ we find that

$$EI\frac{d^2z_1}{dx^2} = EI\frac{d^2y_1}{dx^2} = -Pz_1 \qquad (6.88)$$

Writing n^2 for $\frac{P}{EI}$ as usual, the solution of this is

$$z_1 = A_1 \cos nx + B_1 \sin nx;$$

that is,

$$y_1 = -\frac{Qb}{Pl}x + A_1 \cos nx + B_1 \sin nx \quad . \quad . \quad . \quad . \quad (6.89)$$

We know that $y = 0$ at the end A where $x = 0$. Hence

$$0 = A_1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.90)$$

and therefore

$$y_1 = -\frac{Qb}{Pl}x + B_1 \sin nx \quad . \quad . \quad . \quad . \quad . \quad (6.91)$$

gives the deflexion between A and C.

We can write down the deflexion between C and B by replacing b by a and x by $(l-x)$ in the above. Then, denoting quantities in the region CB by the suffix (2) we get

$$y_2 = -\frac{Qa}{Pl}(l-x) + B_2 \sin n(l-x) \quad . \quad . \quad . \quad (6.92)$$

as the deflexion between C and B.

Since the deflexion and slope are the same for the two portions at the point C we have

$$\left.\begin{aligned} y_1 &= y_2 \\ \frac{dy_1}{dx} &= \frac{dy_2}{dx} \end{aligned}\right\} \text{ at } x = a \quad . \quad . \quad . \quad . \quad . \quad (6.93)$$

and

These give

$$B_1 \sin na - B_2 \sin nb = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.94)$$

and

$$n\left\{ B_1 \cos na + B_2 \cos nb \right\} = \frac{Q}{P} \quad . \quad . \quad . \quad . \quad (6.95)$$

From the last two equations we get

$$B_1 \left\{ \cos na + \cos nb \frac{\sin na}{\sin nb} \right\} = \frac{Q}{nP}$$

whence

$$B_1 \frac{\sin n(a+b)}{\sin nb} = \frac{Q}{nP}$$

and therefore

$$B_1 = \frac{Q}{nP} \frac{\sin nb}{\sin nl},$$

and

$$B_2 = \frac{Q}{nP} \frac{\sin na}{\sin nl} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.96)$$

Consequently

$$y_1 = \frac{Q}{P}\left\{ \frac{\sin nb \sin nx}{n \sin nl} - \frac{b}{l}x \right\} \quad . \quad . \quad . \quad . \quad . \quad (6.97)$$

$$y_2 = \frac{Q}{P}\left\{ \frac{\sin na \sin n(l-x)}{n \sin nl} - \frac{a}{l}(l-x) \right\} \quad (6.98)$$

Also,

$$M_1 = EI\frac{d^2y_1}{dx^2}$$

$$= -EI\frac{nQ}{P}\frac{\sin nb}{\sin nl}\sin nx$$

$$= -\frac{Q}{n}\frac{\sin nb}{\sin nl}\sin nx \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.99)$$

$$M_2 = -\frac{Q}{n}\frac{\sin na}{\sin nl}\sin n(l-x) \quad . \quad . \quad . \quad . \quad . \quad (6.100)$$

It should be observed that M_1 and M_2 are both infinite when $\sin nl = 0$. This means that the rod is unstable for exactly the same value of n, and therefore of P, as if Q did not act. Then the lateral load does not affect the stability, although, of course it produces its effect on the bending moment when P is smaller than the buckling thrust, and the rod may fail as a beam fails, namely, because the tension (or compression) in the fibres exceeds the safe stress.

83. Effect of several lateral forces and of a distributed force. It is easy to show from the differential equation that the total bending moment due to several lateral forces is the sum of the bending moments due to each force separately. Suppose forces Q_1, Q_2, Q_3, act at points distant a_1, a_2, a_3, from the origin end of the strut, and b_1, b_2, b_3 from the other end. Then the bending moment on the origin side of all the loads is

$$M = -\frac{\sin nx}{n \sin nl}\left\{Q_1 \sin nb_1 + Q_2 \sin nb_2 + Q_3 \sin nb_3\right\} \quad . \quad . \quad (6.101)$$

The bending moment on the origin side of Q_2 and Q_3 and on the opposite side of Q_1 is

$$M = -\frac{\sin nx}{n \sin nl}\left\{Q_2 \sin nb_2 + Q_3 \sin nb_3\right\}$$

$$-\frac{\sin n(l-x)}{n \sin nl}Q_1 \sin na_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.102)$$

The general form can be written thus

$$M = -\frac{\sin nx}{n \sin nl}\Sigma Q \sin nb$$

$$-\frac{\sin n(l-x)}{n \sin nl}\Sigma Q \sin na \quad . \quad . \quad (6.103)$$

the sum $\Sigma Q \sin nb$ extending over all the forces for which $a>x$, and the sum $\Sigma Q \sin na$ extending over all the forces for which $a>x$.

We can adapt these at once to give the bending moment for a continuous distribution of lateral force. Suppose that the force on the element of strut between $x=a$ and $x=(a+da)$ is wda. This takes the place of Q in equation (6.103). Thus

$$\begin{aligned} M = & -\frac{\sin nx}{n \sin nl}\int_x^l \sin nb \cdot wda \\ & -\frac{\sin n(l-x)}{n \sin nl}\int_0^x \sin na \cdot wda \quad . \quad . \quad (6.104) \end{aligned}$$

Since $(a+b)=l$ it follows that $db=-da$ and the limits for b corresponding to x and l for a are $(l-x)$ and o. Hence

$$\begin{aligned} M = & \quad \frac{\sin nx}{n \sin nl}\int_{l-x}^0 \sin nb \cdot wdb \\ & -\frac{\sin n(l-x)}{n \sin nl}\int_0^x \sin na \cdot wda \\ = & -\frac{\sin nx}{n \sin nl}\int_0^{l-x} w \sin nbdb \\ & -\frac{\sin n(l-x)}{n \sin nl}\int_0^x w \sin nada \quad . \quad . \quad . \quad (6.105) \end{aligned}$$

As a particular case suppose w is constant. Then

$$\begin{aligned} M = & -w\frac{\sin nx}{n^2 \sin nl}\left\{1-\cos n(l-x)\right\} \\ & -w\frac{\sin n(l-x)}{n^2 \sin nl}\left\{1-\cos nx\right\} \\ = & -\frac{wEI}{P \sin nl}\left\{\begin{array}{l}\sin nx+\sin n(l-x) \\ -\sin nx \cos n(l-x)-\cos nx \sin n(l-x)\end{array}\right\} \\ = & -\frac{wEI}{P \sin nl}\left\{\sin nx+\sin n(l-x)-\sin nl\right\} \\ = & -\frac{wEI}{P \sin nl}\left\{2 \sin \tfrac{1}{2}nl \cos n(x-\tfrac{1}{2}l)-\sin nl\right\} \\ = & -\frac{wEI}{P}\left\{\frac{\cos nx'}{\cos \frac{1}{2}nl}-1\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.106) \end{aligned}$$

where $x'=(x-\frac{1}{2}l)$, which is the abscissa measured from the middle of the strut.

84. Another method for a distributed lateral force.

There is another distinct method of arriving at the bending moment in a strut under a distributed lateral force. Let M′ be the bending moment at x due to a distributed lateral force w per foot at x when P does not act. Then the total bending moment at x is

$$M=-Py+M' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad 6.107)$$

Differentiating this twice we get

$$\frac{d^2M}{dx^2}=-P\frac{d^2y}{dx^2}+\frac{d^2M'}{dx^2} \quad . \quad . \quad . \quad . \quad . \quad (6.108)$$

But since M′ is the bending moment on the assumption that only lateral forces act we are entitled to use equation (5.7) That is

$$\frac{d^2M'}{dx^2}=w \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.109)$$

Also

$$EI\frac{d^2y}{dx^2}=M \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.110)$$

Therefore equation (6.108) becomes

$$\frac{d^2M}{dx^2}=-\frac{P}{EI}M+w \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.111)$$

or

$$\frac{d^2M}{dx^2}+n^2M=w \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.112)$$

In this equation w must be regarded as a function of x, unless it is a constant. Let $f(x)$ be written for w. Methods of solving this are given in books on "Differential Equations". One method is given in the appendix. It is shown therein that the solution of (6.112) when $w=f(x)$ is

$$M=A\cos nx+B\sin nx+\frac{1}{n}\int_0^x \sin n(x-u)\,f(u)\,du \quad . \quad (6.113)$$

The conditions at the hinged ends, namely

$$\left.\begin{array}{l} M=0 \text{ where } x=0 \\ \text{and where } x=l \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.114)$$

give

$$0=A \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.115)$$

and

$$0=B\sin nl+\frac{1}{n}\int_0^l \sin n(l-u)\,f(u)\,du \quad . \quad . \quad (6.116)$$

This last equation gives

$$B=-\frac{1}{n\sin nl}\int_0^l \sin n(l-u)\,f(u)\,du \quad . \quad . \quad . \quad (6.117)$$

and therefore

$$M=-\frac{\sin nx}{n\sin nl}\int_0^l \sin n(l-u)\,f(u)\,du$$
$$+\frac{1}{n}\int_0^x \sin n(x-u)\,f(u)\,du \quad . \quad . \quad . \quad . \quad (6.118)$$

It can be shown that this last result agrees with (6.104) if we write u for a and $(l-u)$ for b in that equation.

Instead of using equation (6.118) or (6.105) it is often easier to solve equation (6.112) at once when w is an easy function of x. Moreover it must be borne in mind that equations (6.118) and (6.105) apply only to the case of a strut with hinged ends. For any other case, say the case of clamped ends, it would be necessary to find afresh the equation for M. This could be done by the method of Art 83, that is, by finding the effect of a single force Q and then getting the result for a distributed force by integration; or it could be done by adjusting the constants in (6.113) to satisfy the end conditions.

85. Strut clamped at the ends with uniformly distributed lateral force.

For this case we will use (6.112) directly. It is understood that y and $\frac{dy}{dx}$ are both zero at the ends and that the forces P act along the x-axis. Since w is constant the solution of (6.112) is

$$M = A\cos nx + B\sin nx + \frac{w}{n^2} \quad . \quad . \quad . \quad . \quad (6.119)$$

Let s represent the slope at any point, s_0 and s_1 the slopes at the ends. Then, taking the origin at one end,

$$\begin{aligned} M &= EI\frac{d^2y}{dx^2} \\ &= EI\frac{ds}{dx} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.120) \end{aligned}$$

Therefore

$$\begin{aligned} \int_0^x M dx &= EI\{s - s_0\} \\ &= EIs \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.121) \end{aligned}$$

since $s_0 = 0$. Also

$$\int_0^l M dx = EIs_1 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.122)$$

Similarly we find

$$\begin{aligned} \int_0^l EIs dx &= EI\int_0^l \frac{dy}{dx}dx \\ &= EI\{y_1 - y_0\} = 0 \quad . \quad . \quad . \quad (6.123) \end{aligned}$$

Equations (6.122) and (6.123) are the end-conditions for the present case. With the value of M from (6.119) condition (6.122) gives

$$\frac{1}{n}\left(A\sin nl + B - B\cos nl\right) + \frac{wl}{n^2} = 0, \quad . \quad . \quad (6.124)$$

and equation (6.121) gives, as the general value of EIs,

$$EIs = \frac{1}{n}\left(A\sin nx + B - B\cos nx\right) + \frac{wx}{n^2} \quad . \quad . \quad (6.125)$$

Now condition (6.123) gives

$$\frac{1}{n^2}\left\{A - A\cos nl + Bnl - B\sin nl\right\} + \frac{1}{2}\frac{wl^2}{n^2} = 0 \quad (6.126)$$

Equations (6.124) and (6.126) determine A and B in terms of n and l. The values of A and B obtained by solving these equations are

$$\left.\begin{aligned} B &= -\frac{wl}{2n} \\ A &= -\frac{wl}{2n}\cot\tfrac{1}{2}nl \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.127)$$

Then the final value of the bending moment is

$$\begin{aligned} M &= \frac{w}{2n^2}\left\{2 - nl\cot\tfrac{1}{2}nl\cos nx - nl\sin nx\right\} \\ &= \frac{w}{2n^2}\left\{2 - \frac{nl\cos n\,(x-\frac{1}{2}l)}{\sin\frac{1}{2}nl}\right\} \\ &= \frac{EIw}{2P}\left\{2 - \frac{nl\cos nx'}{\sin\frac{1}{2}nl}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.128) \end{aligned}$$

where $x' = (x - \frac{1}{2}l)$, which is the abscissa measured from the middle of the strut.

It is worth while to verify the result in (6.128) by taking the particular case when P is zero, that is, when n is zero. This should give the same value of M as in (5.37) where we dealt with the transverse load only. Starting from the first form for M in equation (6.128) and expanding in powers of n as far as n^2 in the bracket, since n is to be zero finally, we get

$$\begin{aligned} M &= \frac{w}{2n^2}\left\{2 - \frac{nl\cos\frac{1}{2}nl}{\sin\frac{1}{2}nl}\cos nx - nl\sin nx\right\} \\ &= \frac{w}{2n^2}\left\{2 - \frac{nl\,(1-\frac{1}{8}n^2l^2)(1-\frac{1}{2}n^2x^2)}{\frac{1}{2}nl\,(1-\frac{1}{24}n^2l^2)} - nl\times nx\right\} \\ &= \frac{w}{2n^2}\left\{2 - 2\,(1-\tfrac{1}{8}n^2l^2 - \tfrac{1}{2}n^2x^2 + \tfrac{1}{24}n^2l^2) - n^2lx\right\} \\ &= \frac{w}{2}\left\{x^2 - lx + \tfrac{1}{6}l^2\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.129) \end{aligned}$$

which agrees with (5.37)

86. Tie-rod under lateral forces.

A tie-rod differs from a strut in being subjected to tension instead of thrust. That is, the force P in strut problems must have its sign changed. If we still write n^2 for $\frac{P}{EI}$ then the results for tie-rods can be deduced from the results for struts by writing $-n^2$ for n^2, that is by writing $n\sqrt{-1}$ for n. All that is necessary to turn the results into real form is then a knowledge of the algebra of imaginary quantities.

Let us change the result in (6.105) so that it will apply to a tie-rod with hinged ends instead of a strut. Writing i for $\sqrt{-1}$ and substituting *in* for *n* in that equation we get

$$M = -\frac{\sin inx}{in \sin inl}\int^{l-x} w \sin inb db - \frac{\sin in(l-x)}{in \sin inl}\int_0^x w \sin ina da \quad . \quad . \quad (6.130)$$

But it is shown books dealing with imaginary quantities that

$$\sin i\theta = i \sinh \theta$$

Therefore

$$M = -\frac{\sinh nx}{n \sinh nl}\int_0^{l-x} w \sinh nb db - \frac{\sinh n(l-x)}{n \sinh nl}\int_0^x w \sinh na da \quad . \quad . \quad (6.131)$$

When the value of w is given it may be found easier in any particular case not to change the sines into hyperbolic sines till the integrations in (6.105) have been performed, for the reason that circular functions are more familiar than hyperbolic functions, and therefore it is easier to integrate circular than hyperbolic functions. Thus, using the result in (6.106), since

$$\cos i\theta = \cosh \theta,$$

the corresponding result for a tie-rod is

$$M = +\frac{wEI}{P}\left\{\frac{\cosh nx'}{\cosh \frac{1}{2} nl} - 1\right\} = -\frac{wEI}{P}\left\{1 - \frac{\cosh nx'}{\cosh \frac{1}{2} nl}\right\} \quad . \quad . \quad . \quad . \quad (6.132)$$

Again the result in (6.128), adapted to a tie-rod, is

$$M = -\frac{EIw}{2P}\left\{2 - \frac{inl \cosh nx'}{i \sinh \frac{1}{2} nl}\right\} = -\frac{EIw}{2P}\left\{2 - \frac{nl \cosh nx'}{\sinh \frac{1}{2} nl}\right\} \quad . \quad . \quad . \quad . \quad (6.133)$$

87. Tie-rod carrying a single lateral load.

If the forces P are reversed in fig. 46 then the results in equation (6.97) and (6.98) become, when *in* is written for *n* and —P for P,

$$y_1 = \frac{Q}{P}\left\{\frac{b}{l}x - \frac{\sinh nb \sinh nx}{n \sinh nl}\right\} \quad . \quad . \quad . \quad . \quad . \quad (6.134)$$

$$y_2 = \frac{Q}{P}\left\{\frac{a}{l}(l-x) - \frac{\sinh na \sinh n(l-x)}{n \sinh nl}\right\} \quad . \quad . \quad (6.135)$$

88. The Elastica.

In the problem of the strut we have throughout used an inaccurate expression for the radius of curvature. This makes very little error provided that the slope of the strut at every point is small. This assumption led to the conclusion that the same thrust would produce either large or small deflexions. But, of course, we are not entitled to draw any conclusions from our equation concerning what happens when y is not small precisely because our differential equation involves the assumption that y is small. A very thin rod is capable of bending so that its slope is large, but to investigate the state of such a rod we must use the correct formula for curvature. When only a pair of balancing forces with or without a pair of balancing couples act at the ends of a naturally straight rod (as in the case of the struts in figs 41, 42) the form of the curve of the rod is called an *elastica*. There are many different forms of elastica. Two forms are shown in figs 47(a) and 47(b)

Let s denote the arc of the curve of an elastica measured from any convenient point on the curve, φ the angle which ds makes with

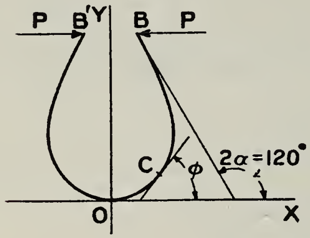

Fig. 47a

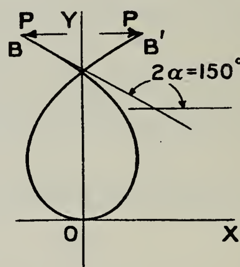

Fig. 47 b

the line of the forces P. Then, with the axes as shown in figs 47(a), 47(b), the equation for the bending moment at C is

$$\frac{EI}{R} = M = P(y_1 - y)$$

where y_1 is the ordinate of B, and y the ordinate of C. This equation may be written

$$EI\frac{d\varphi}{ds} = P(y_1 - y) \qquad (6.136)$$

The sign on the left hand side depends on the directions in which φ and s are measured. The sign suits the case of fig. 47(a) where the arc OC is taken as s.

Differentiating through (6.136) with respect to s we get

$$EI\frac{d^2\varphi}{ds^2} = -P\frac{dy}{ds} = -P\sin\varphi$$

whence

$$\frac{d^2\varphi}{ds^2} = -\frac{P}{EI}\sin\varphi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.137)$$

This last equation is independent of the direction in which s is measured; for if ds were measured from C towards O then one sign would be changed in (6.136) but this would right itself in (6.137) because then $\frac{dy}{ds}$ would be sin φ. Equation (6.137) is, then, the general equation for all forms of elastica. It is useful to compare this equation with the equation for the motion of a simple pendulum of length l. When the pendulum makes an angle φ with the downward vertical the equation of motion is

$$\frac{d^2\varphi}{dt^2} = -\frac{g}{l}\sin\varphi$$

Thus the pendulum problem is precisely the same as the elastica problem, the arc s in the case of the elastica corresponding to the time t for the pendulum, the curvature to the angular velocity, and the constant $\frac{P}{EI}$ to the constant $\frac{g}{l}$. The strut with very little curvature corresponds to the pendulum swinging through a small arc. The complete period of the pendulum in this case is

$$t = 2\pi\sqrt{\frac{l}{g}}$$

This complete period corresponds to the length of a complete wave on the strut; that is, to the length $2l$ for the hinged strut. This gives

$$2l = 2\pi\sqrt{\frac{EI}{P}}$$

whence

$$P = EI\frac{\pi^2}{l^2}$$

as in equation (6.27).

Fig 47(a) corresponds to the case of a pendulum oscillating from about 120° on one side to 120° on the other side of the vertical.

89. The general case.

To solve (6.137) in the general case requires elliptic functions, but we can get the useful results by means of series. Let u be written for the curvature $\frac{d\varphi}{ds}$. Then equation (6.137) gives

$$\frac{du}{ds} = -\frac{P}{EI}\sin\varphi = -n^2\sin\varphi$$

where

$$n^2 = \frac{P}{EI}$$

that is

$$\frac{du}{d\varphi}\frac{d\varphi}{ds} = -n^2 \sin\varphi,$$

or

$$u\frac{du}{d\varphi} = -n^2 \sin\varphi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.138)$$

Integrating this we get

$$\tfrac{1}{2}u^2 = +n^2\cos\varphi + H \quad . \quad . \quad . \quad . \quad . \quad (6.139)$$

We shall now work out the problem shown in figs 47(*a*) and 47(*b*) where the whole range of φ from the force P at one side to the force P on the other is less than 2π. Let the value of φ at B be 2α, where 2α is, as we know, less than π. At the point B we know also that $(y_1 - y)$ is zero and therefore, by equation (6.136), that u is also zero. Thus one end-condition is that

$$u = 0 \text{ where } \varphi = 2\alpha \quad . \quad . \quad . \quad . \quad . \quad (6.140)$$

This determines the constant H in (6.139), and the result is

$$u^2 = 2n^2(\cos\varphi - \cos 2\alpha) \quad . \quad . \quad . \quad . \quad . \quad (6.141)$$

or

$$\left(\frac{d\varphi}{ds}\right)^2 = 2n^2\left(1 - 2\sin^2\frac{\varphi}{2} - 1 + 2\sin^2\alpha\right)$$

$$= 4n^2\left(\sin^2\alpha - \sin^2\frac{\varphi}{2}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.142)$$

Therefore

$$\frac{ds}{d\varphi} = \frac{1}{2n}\frac{1}{\sqrt{\sin^2\alpha - \sin^2\frac{\varphi}{2}}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.143)$$

Writing s_1 for the length of the arc AB, we find that

$$s_1 = \frac{1}{2n}\int^{2\alpha}\frac{d\varphi}{\sqrt{\sin^2\alpha - \sin^2\frac{\varphi}{2}}} \quad . \quad . \quad . \quad . \quad . \quad (6.144)$$

To work out this integral put

$$\sin\frac{\varphi}{2} = \sin\alpha\sin\theta \quad . \quad . \quad . \quad . \quad . \quad (6.145)$$

Then

$$\tfrac{1}{2}\cos\frac{\varphi}{2}d\varphi = \sin\alpha\cos\theta d\theta \quad . \quad . \quad . \quad . \quad (6.146)$$

and

$$\sqrt{\sin^2\alpha - \sin^2\frac{\varphi}{2}} = \sin\alpha\cos\theta \quad . \quad . \quad . \quad (6.147)$$

Also the limits for θ are 0 and $\frac{\pi}{2}$ corresponding to 0 and 2α for φ.

Then $$s_1 = \frac{1}{n}\int^{\frac{\pi}{2}} \frac{d\theta}{\cos\frac{\varphi}{2}} = \frac{1}{n}\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - \sin^2\alpha\,\sin^2\theta}} \quad . \quad . \quad (1.148)$$

Now by the binomial theorem

$$\frac{1}{\sqrt{1-x}} = (1-x)^{-\frac{1}{2}}$$

$$= 1 + \tfrac{1}{2}x + \frac{1\cdot 3}{2\cdot 4}x^2 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^3 + \quad . \quad (6.149)$$

Hence

$$ns_1 = \int_0^{\frac{\pi}{2}} \left\{ 1 + \tfrac{1}{2}\sin^2\alpha\,\sin^2\theta + \frac{1\cdot 3}{2\cdot 4}\sin^4\alpha\,\sin^4\theta + \ldots\ldots \right\} d\theta \quad (6.150)$$

But it is proved in works on the integral calculus that

$$\int_0^{\frac{\pi}{2}} \sin^{2n}\theta d\theta = \frac{2n-1}{2n}\cdot\frac{2n-3}{2n-2}\ldots\ldots\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} \quad . \quad . \quad (6.151)$$

In particular

$$\int_0^{\frac{\pi}{2}} \sin^2\theta a\theta = \frac{1}{2}\cdot\frac{\pi}{2}$$

$$\int_0^{\frac{\pi}{2}} \sin^4\theta d\theta = \frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}$$

Therefore

$$ns_1 = \frac{\pi}{2}\left\{ 1 + \frac{1^2}{2^2}\sin^2\alpha + \frac{1^2\cdot 3^2}{2^2\cdot 4^2}\sin^4\alpha + \ldots \right\} \quad . \quad . \quad (6.152)$$

Writing l for the whole length of the rod, which is also $2s_1$,

$$l\sqrt{\frac{P}{EI}} = \pi\left\{ 1 + \frac{1^2}{2^2}\sin^2\alpha + \frac{1^2\cdot 3^2}{2^2\cdot 4^2}\sin^4\alpha + \ldots \right\} . \quad . \quad (6.153)$$

If we take only the first term on the right we get the usual value of P for a strut with hinged ends. The remaining terms in the series are thus the corrections to the result for the strut. The series of powers of $\sin^2\alpha$ is convergent for all values of α less than $\frac{\pi}{2}$. We see from our present result that P is always actually greater than in the usual strut theory, and that P increases as α increases, 2α being the angle of slope at the end of the strut. If, for example, 2α is 20^0, then

$$l\sqrt{\frac{P}{EI}} = \pi\left\{ 1 + \frac{1}{4}\sin^2 10^0 \right\} = 1\cdot 0075\,\pi, \quad . \quad . \quad (6.154)$$

so that

$$P = 1\cdot 015\,\frac{\pi^2}{l^2}EI \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.155)$$

which is only $1\frac{1}{2}$ °/₀ greater than the ordinary strut theory gives.

If $2\alpha = 90^\circ$, in which case the ends of the elastica are perpendicular to the forces P, as shown in fig. 48, then

$$l\sqrt{\frac{P}{EI}} = \pi\left\{1 + \frac{1}{4}\sin^2 45^\circ + \frac{9}{64}\sin^4 45^\circ + \ldots\right\}$$
$$= \pi \times 1 \cdot 180 \quad \ldots \ldots \quad (6.156)$$

whence

$$P = 1 \cdot 392 \frac{\pi^2}{l^2} EI \quad \ldots \ldots \quad (6.157)$$

which is 39 % greater than the force that will just start the buckling of the same rod.

If we denote by P_0 the force that will just buckle a rod of length l with hinged ends, we see now that, as the ends are pressed nearer and

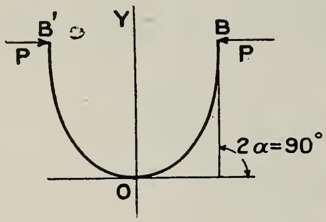

Fig. 48

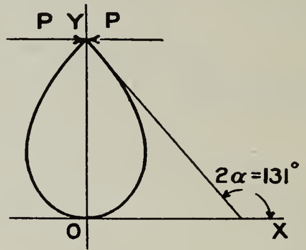

Fig. 49

nearer together, the force P increases, and its value when each end has been turned through the angle 2α is

$$P = S^2 P_0$$

where S denotes the series in brackets in (6.153). It can be shown that the ends come together as in fig. 49 when 2α is about 131°, and then P is approximately $2 \cdot 28\, P_0$. As the angle 2α approaches 180° the series S approaches infinity, and therefore P approaches infinity. But at the same time the loop shown in fig. 47 (b) diminishes to infinitesimal dimensions and the rest of the rod straightens out. Long before this stage is reached any rod would break. Fig. 50 shows the curve obtained by plotting the series for S against $\sin\alpha$.

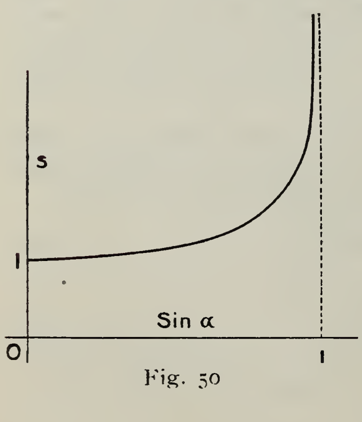

Fig. 50

Although it is usual to regard the buckling load of a strut as the load at which failure begins, yet it is possible that the strut may be well within the elastic limit for small values of the angle α. Failure may occur first on either the compression or the tension side according to the shape of the cross section and the strength in compression and tension. Let us assume that it occurs on the compression side first, and let h be the greatest distance of a point of the section through the middle of a strut from the axis passing through the centre of gravity of the section, and parallel to the neutral axis. Then the maximum stress on the compression side is, by the same reasoning as for (6.15),

$$f=\frac{P}{A}+\frac{E}{R}h,$$

this being the maximum stress in the middle section. But, for the middle section, where the stress due to bending is greatest, $\varphi=0$, and therefore

$$\frac{1}{R}=\frac{d\varphi}{ds}=2n\sin\alpha.$$

Also $$P=n^2EI=n^2EAk^2,$$

where k is the radius of gyration of the section corresponding to I. Hence

$$f=En^2k^2+2Enh\sin\alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.158)$$

If f is given the value of n corresponding to this f is the value at the intersection of the two curves obtained by plotting n against $\sin\alpha$ from equations (6.153) and (6.158). In equation (6.153) n must be written for $\sqrt{\frac{P}{EI}}$.

90. Strut with variable cross section.

We have so far assumed that our struts had uniform cross sections, but in practice it is not uncommon to vary the cross section so as to save material, as for example in the case of the struts between the wings of an aeroplane.

The equation from which the buckling load has to be determined is still

$$EI\frac{d^2y}{dx^2}=-Py \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.159)$$

but now I is not constant. Writing this last equation in the form

$$I=-\frac{P}{E}y\div\frac{d^2y}{dx^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.160)$$

we can use this to calculate I corresponding to any given form of curve assumed for the central line of the strut. If, as is usual, y is measured from the line joining the ends then the assumed curve must make $y=0$ at the two ends $x=0$ and $x=l$.

Suppose, for instance, that $y = \mathrm{B} \sin \frac{\pi x}{l}$.

Then
$$\mathrm{I} = -\frac{\mathrm{P}}{\mathrm{E}} \mathrm{B} \sin \frac{\pi x}{l} \div -\frac{\pi^2}{l^2} \mathrm{B} \sin \frac{\pi x}{l}$$
$$= \frac{l^2}{\pi^2} \frac{\mathrm{P}}{\mathrm{E}} \quad \ldots \ldots \ldots \quad (6.161)$$

giving a constant value of I, and giving at the same time the value of P in terms of I.

Again suppose the curve is a parabola passing through the ends; that is, $y = \mathrm{B}x(l - x)$. Then

$$\mathrm{I} = -\frac{\mathrm{P}}{\mathrm{E}} \mathrm{B}x(l-x) \div -2\mathrm{B}$$
$$= \frac{\mathrm{P}}{2\mathrm{E}} x(l-x) \quad \ldots \ldots \ldots \quad (6.162)$$

If lB is a small fraction the curve $y = \mathrm{B}x(l - x)$, which is really a parabola, is practically coincident with a circular are between the limits $x = 0$ and $x = l$, the radius of the circle being $\frac{1}{2\mathrm{B}}$.

It should be observed that the moment of inertia I in the last case is proportional to y. If follows then that this rod must be much thinner at the ends than at the middle. Suppose, for simplicity, that the section of the rod is circular of radius r. In this case $\mathrm{I} = \frac{1}{4}\pi r^4$ and therefore equation (6.162) gives

$$r^4 = \frac{2\mathrm{P}}{\pi \mathrm{E}} x(l-x) \quad \ldots \ldots \ldots \quad (6.163)$$

which determines the radius of the section at any point of the rod.

Let us next suppose that the rod has any form of section subject to the condition that the section has a pair of perpendicular axes of symmetry, one of which is in the plane of bending, and let us suppose also that all the sections are similar figures. Let b denote the maximum breadth of any section, and b_1 the corresponding dimension for the middle section. Then, since the sections are all similar,

$$\mathrm{I} = \mathrm{C}b^4$$

where C is the same constant for every section. Therefore equation (6.162) gives

$$\mathrm{C}b^4 = \frac{\mathrm{P}}{2\mathrm{E}} x(l-x); \quad \ldots \ldots \ldots \quad (6.164)$$

and at the middle section, where $x = \frac{1}{2} l$,

$$\mathrm{C}b_1^4 = \frac{\mathrm{P}l^2}{8\mathrm{E}} \quad \ldots \ldots \ldots \quad (6.165)$$

By division from the last two equations

$$\left(\frac{b}{b_1}\right)^4=\frac{4x(l-x)}{l^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.166)$$

which gives the breadth of any section in terms of the breadth at the middle and the abscissa of the section. The breadth of an actual strut could not strictly be made according to equation (6.166), for this equation makes the strut taper to points at the ends, whereas its section must be great enough at the ends to stand the stress due to the thrust P distributed uniformly across the section.

91. The strut with variable cross section (continued).

The problem of finding the buckling load for a given strut with variable cross-section depends on the solution of a differential equation which can seldom be expressed in terms of finite algebraic, trigonometric, or logarithmic functions. The solution usually involves Bessel

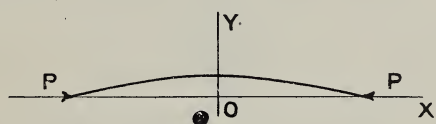

Fig. 51 a

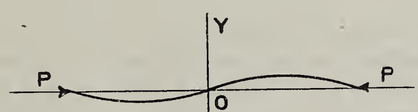

Fig. 51 b

functions, or still less known functions expressible only by infinite series. There are, however, certain algebraic values of I which lead to finite algebraic values of y. We shall now consider some of these.

Suppose we are still dealing with a strut with hinged ends. It is convenient to take the origin at the middle of the straight line joining the ends as shown in figs. 51(a) and 51(b). For such rods as we shall deal with there are two distinct cases to consider, in the first of which $\frac{dy}{dx}=0$ where $x=0$, and in the second $y=0$ where $x=0$. We shall deal firstly with the case where $\frac{dy}{dx}=0$ at the origin.

The equation for y is

$$EI\frac{d^2y}{dx^2}=-Py \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.167)$$

Now suppose

$$I = I_0\left(1 - \frac{x^2}{a^2}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.168)$$

where a denotes half the length of the rod, l being the whole length. Clearly I_0 is the value of I at the middle of the rod. Also let us write u for $\frac{x}{a}$. Then $u = \pm 1$ at the ends, and

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{adu}\left(\frac{dy}{adu}\right)$$

$$= \frac{1}{a^2}\frac{d^2y}{du^2}$$

Then the equation for y in terms of u is

$$EI_0(1 - u^2)\frac{1}{a^2}\frac{d^2y}{du^2} = -Py \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.169)$$

whence

$$(1 - u^2)\frac{d^2y}{du^2} = -n^2y. \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.170)$$

where

$$n^2 = \frac{Pa^2}{EI_0} = \frac{Pl^2}{4EI_0} \quad . \quad . \quad . \quad . \quad (6.171)$$

Equation (6.170) can be solved in series, and this series will terminate under certain conditions which we shall discover. For the case where y is a maximum when $x = 0$, that is, when $u = 0$, it is clear that y must have the same value for equal positive and negative values of u. Then y must involve only even powers of u. Let us assume, therefore, that

$$y = y_0(1 + c_2u^2 + c_4u^4 + c_6u^6 + \ldots)$$

Then

$$\frac{d^2y}{dx^2} = y_0(2c_2 + 4.3c_4u^2 + 6.5c_6u^4 + \ldots)$$

Substituting in equation (6.170) we get

$$y_0(1 - u^2)(2c_2 + 4.3c_4u^2 + 6.5c_6u^4 + \ldots)$$
$$= -n^2y_0(1 + c_2u^2 + c_4u^4 + c_6u^6 + \ldots) \quad . \quad . \quad . \quad (6.172)$$

Equating coefficients of like powers of u we get, after taking out the common factor y_0,

$$\left.\begin{aligned} 2c_2 &= -n^2 \\ 4.3c_4 - 2c_2 &= -n^2c_2 \\ 6.5c_6 - 4.3c_4 &= -n^2c_4 \\ 8.7c_8 - 6.5c_6 &= -n^2c_6 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (6.173)$$

These give

$$\left.\begin{aligned} c_2 &= -\frac{1}{\lfloor 2}n^2 \\ c_4 &= \frac{n^2(n^2-2)}{\lfloor 4} \\ c_6 &= -\frac{n^2(n^2-2)(n^2-12)}{\lfloor 6} \\ c_8 &= \frac{n^2(n^2-2)(n^2-12)(n^2-30)}{\lfloor 8} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (6.174)$$

Therefore

$$y = y_0\left\{1 - \frac{n^2}{\lfloor 2}u^2 + \frac{n^2(n^2-2)}{\lfloor 4}u^4 - \frac{n^2(n^2-2)(n^2-12)}{\lfloor 6}u^6 + \ldots\right\} \quad (6.175)$$

Now we can make this series terminate by properly choosing the value of n^2. For instance, if we take

$$n^2 = 2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.176)$$

that is,

$$\frac{Pl^2}{EI_0} = 8 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.177)$$

we find

$$y = y_0(1-u^2) = y_0\left(1 - \frac{4x^2}{l^2}\right) \quad . \quad . \quad . \quad . \quad . \quad (6.178)$$

This value of y is equal to zero at the ends, so that our solution satisfies all the conditions of the problem. Indeed the value of I in this case is, allowing for the difference of origin, exactly the same as the one in (6.162), and the present value of y is the same as the value of y from which equation (6.162) is deduced Our present problem is, in fact, the previous problem attacked from the opposite end.

The value of y in (6.175) can also be made to terminate, and at the same time to vanish at the ends of the strut, if we take

$$n^2 = 12 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.179)$$

or

$$\frac{Pl^2}{EI_0} = 48 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.180)$$

Then

$$\begin{aligned} y &= y_0(1 - 6u^2 + 5u^4) \\ &= y_0(1-u^2)(1-5u^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.181) \end{aligned}$$

This value of y vanishes not only at the ends but also at two other points given by

$$5u^2 = 1$$

or

$$x = \pm\frac{\sqrt{5}}{10}l = \pm 0{\cdot}2236l \quad . \quad . \quad . \quad . \quad . \quad (6.182)$$

The form of the curve is shown in fig. 52. The distance A′A is equal to $\frac{1}{\sqrt{5}}\,l$. Let us write l_1 for this length. Then, from equation (6.180),

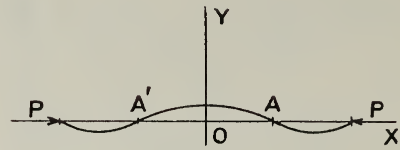

Fig. 52

$$\frac{Pl_1^{\,2}}{EI_0}=\frac{48}{5}=9{\cdot}6 \quad \ldots\ldots\ldots\ldots \quad (6.183)$$

$$I=I_0\left(1-\frac{4}{5}\frac{x^2}{l_1^{\,2}}\right) \quad \ldots\ldots\ldots \quad (6\ 184)$$

and

$$y=y_0\left(1-\frac{4}{5}\frac{x^2}{l_1^{\,2}}\right)\left(1-4\frac{x^2}{l_1^{\,2}}\right) \quad \ldots\ldots \quad (6.185)$$

The piece A′A would be in equilibrium if the forces P were applied at A and A′, so that the last three equations may be regarded as applying to a rod of length l_1. It should be observed how near the number $9{\cdot}6$ in equation (6.183) is to the number π^2 for a uniform rod. This is to be expected, because the value of I at A or A′ is $\frac{4}{5}\,I_0$, which is not very much less than I_0. The portion A′A is thus very nearly a uniform rod.

By taking

$$n^2=30 \quad \ldots\ldots\ldots\ldots \quad (6.186)$$

or

$$\frac{Pl^2}{EI_0}=120 \quad \ldots\ldots\ldots\ldots \quad (6.187)$$

we get

$$\begin{aligned} y&=y_0\,(1-15u^2+35u^4-21u^6) \\ &=y_0\,(1-u^2)(1-14u^2+21u^4) \quad \ldots\ldots \quad (6.188) \end{aligned}$$

so that y vanishes, not only at the ends, but also where

$$1-14u^2+21u^4=0$$

that is, where

$$\begin{aligned} u^2&=+\frac{1}{3}\pm\frac{2}{3\sqrt{7}} \\ &=0{\cdot}5853 \text{ or } 0{\cdot}0813 \\ u&=\pm0{\cdot}765 \text{ or } \pm0{\cdot}285. \end{aligned}$$

whence

$$\left.\begin{aligned} x&=\pm0{\cdot}765a \\ \text{or}\quad x&=\pm0{\cdot}285a \end{aligned}\right\} \quad \ldots \quad (1.189$$

Thus the strut crosses the x-axis four times between the ends, and the length of the middle bay, corresponding to A′A in fig. 52 above, is 0.285 l. Denoting this by l_1 as before, we get

$$\mathrm{I} = \mathrm{I}_0\left\{1 - \frac{4\times(0.285x)^2}{l_1^2}\right\}$$
$$= \mathrm{I}_0\left\{1 - \frac{0 \cdot 325x^2}{l_1^2}\right\}; \quad \ldots \ldots \ldots (6.190)$$

$$\frac{\mathrm{P}l_1^2}{\mathrm{EI}_0} = 9 \cdot 76 \quad \ldots \ldots \ldots (6.191)$$

$$y = y_0\left(1 - \frac{0 \cdot 325x^2}{l_1^2}\right)\left(1 - \frac{0 \cdot 556x^2}{l_1^2}\right)\left(1 - \frac{4x^2}{l_1^2}\right). \quad . \ (6.192)$$

By the preceding process we can clearly get an infinite number of possible curves and the corresponding thrusts that will hold the rod, whose section has the given moment of inertia, in the form of these curves. Our method will give us an odd number of bays on the strut, corresponding to an odd number of half-wave lengths on the strut of constant section. If we now find the solutions giving an even number of bays we shall have done for the given rod what was done by Euler's method for the strut of constant section.

92. The same strut with an even number of bays.

We have now to get a solution of (6.170) which will make y zero where $u = 0$. Then, since y must change sign with u, we assume that

$$y = r(u + c_3u^3 + c_5u^5 + c_7u^7 + \ldots .) \quad . \ . \ . \ (6.193)$$

Proceeding to find the values of the c's exactly as before, we get finally

$$y = r\left\{u - \frac{n^2}{\underline{|3}}u^3 + \frac{n^2(n^2 - 2.3)}{\underline{|5}}u^5 - \frac{n^2(n^2 - 2.3)(n^2 - 4.5)}{\underline{|7}}u^7 + \ldots (6.194)\right.$$

The smallest value of n^2 (omitting the case where $n^2 = 0$) which makes this series terminate, is

$$n^2 = 6, \quad \ldots \ldots \ldots (6.195)$$

whence

$$\frac{\mathrm{P}l^2}{\mathrm{EI}_0} = 24 \quad \ldots \ldots \ldots (6.196)$$

Then

$$y = ru(1 - u^2)$$
$$= 2r\frac{x}{l}\left(1 - \frac{4x^2}{l^2}\right) \quad \ldots \ldots \ldots (6.197)$$

which makes y zero only at the middle and ends.

The second solution is obtained by taking

$$n^2 = 20, \quad \ldots \ldots \ldots (6.198)$$

and therefore $$\frac{Pl^2}{EI_0} = 80 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.199)$$

Then

$$\begin{aligned} y &= ru(1 - \tfrac{10}{3}u^2 + \tfrac{7}{3}u^4) \\ &= ru(1 - u^2)(1 - \tfrac{7}{3}u^2) \quad . \quad . \quad . \quad . \quad (6.200) \end{aligned}$$

which makes y zero at the middle, at the ends, and at the points where

$$\begin{aligned} x &= \pm\tfrac{1}{2}\sqrt{\tfrac{3}{7}}\,l \\ &= \pm 0{\cdot}3273\,l \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.201) \end{aligned}$$

Further solutions can be obtained by taking $n^2 = 6.7$ or 8.9 etc., and the corresponding values of y will be zero at the middle and ends, and at four, six, etc. other points respectively.

Because the value of I reduces to zero at the ends of the strut that we have just been considering, it might be expected that the bending stress at the end would be large. But the stresses at the ends due to bending are actually zero. Let h be the greatest distance of a point on a cross section of the strut from the neutral axis. Then the maximum stress due to bending at that section is

$$f = \frac{hM}{I} = \frac{hPy}{I} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.202)$$

Now at the ends I vanishes because it contains the factor $1 - \dfrac{4x^2}{l^2}$. But y also contains the same factor, and therefore $\dfrac{y}{I}$ is a finite quantity at the ends. Moreover in a reasonably constructed strut, h will be zero when I is zero, and therefore f will be zero. It follows then that the tapered strut that we have dealt with would be quite good enough to stand the bending stresses. It would, however, fail to stand the crushing stress at the ends. In practice, then, a tapered strut must have a section A at the ends such that $\dfrac{P}{A}$ is a safe working compressive stress.

93. Uniform vertical rod, clamped at the lower end, under a distributed load.

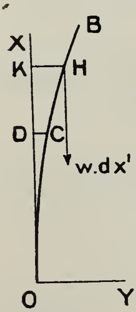

Fig. 53

Let the coordinates of the point C referred to the axes shown (fig. 53) be (x, y), and the co-ordinates of H (x', y'). Let w denote the load per unit length at H. We must regard w as a function of x'; say $w = f(x')$. The moment, about C, of the weight wdx' at H, in $wdx'(y' - y)$. Then total bending moment at C is

$$M = \int_x^l w(y' - y)dx' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.203)$$

the whole length of the rod being l. We can write this in the form

$$M = \int_l^x w(y - y')dx'$$

Differentiating both sides of this equation with respect to the upper limit x we get

$$\frac{dM}{dx} = \Big[w(y - y')\Big]_{x'=x} + \int_l^x w\frac{dy}{dx}dx'$$

$$= w(y' - y') + \frac{dy}{dx}\int_l^x wdx'$$

$$= -\frac{dy}{dx}\int_x^l wdx' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.204)$$

In the preceding differentiations it has to be borne in mind that y is a function of x but not of x', and y' is a function of x' but not of x. Likewise $\frac{dy}{dx}$ is not a function of x', and can therefore be treated as a constant in integrating with respect to x'.

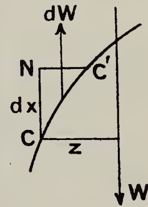

Fig. 54

It is worth while to deduce equation (6.204) without differentiating an integral. Let C′ be a point on the rod at $(x + dx,\ y + dy)$, and let the total load above C be W, and suppose its line of action passes at distance z from C. Let dW be the increase in W due to the increase dx in x. Then dW is negative and its magnitude is the weight of CC′. The bending moments at C and C′ are denoted by M and M + dM. Then

$$M = zW$$

$$M + dM = (z - C'N)W - \tfrac{1}{2}C'N\,dW$$

Therefore

$$dM = -C'N\{W + \tfrac{1}{2}dW\}$$

$$= -dy\{W + \tfrac{1}{2}dW\};$$

whence

$$\frac{dM}{dx} = -\frac{dy}{dx}(W + \tfrac{1}{2}dW)$$

$$= -W\frac{dy}{dx} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.205)$$

in the limit when dx and dW vanish. This is the same result as in equation (6.204) since $\int_x^l wdx'$ represents W.

Now using the value of M in terms of y and x, namely,

$$M = EI\frac{d^2y}{dx^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.206)$$

equation (6.205) gives

$$EI\frac{d^3y}{dx^3} = -W\frac{dy}{dx}$$

Writing p for $\frac{dy}{dx}$ this becomes

$$\mathrm{EI}\frac{d^2p}{dx^2}=-\mathrm{W}p \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.207)$$

In this equation the symbol W, which denotes the total load above C, may include any number of concentrated loads. In particular it may include a load at the top end, and the equation will, of course, remain correct if the end load is the only load, in which case w vanishes everywhere but near the top and there w is very large, so that $\int wdx'$ over a very small length is equal to the finite load at the end.

94. The load per unit length assumed constant.

If w is constant and there is no load on the end, then $\mathrm{W}=w(l-x)$. Therefore

$$\mathrm{EI}\frac{d^2p}{dx^2}=-w(l-x)p$$

whence

$$\frac{d^2p}{dx^2}=-c(l-x)p \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.208)$$

where

$$c=\frac{w}{\mathrm{EI}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.209)$$

Let $l-x=z$, thus measuring z from the free end. Then

$$\begin{aligned}\frac{d^2p}{dx^2}&=\frac{d}{dx}\left(\frac{dp}{dx}\right)\\&=-\frac{d}{dz}\left(-\frac{dp}{dz}\right)\\&=\frac{d^2p}{dz^2}\end{aligned}$$

and therefore

$$\frac{d^2p}{dz^2}=-czp \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.210)$$

We can express p in terms of Bessel functions, or we can solve the equation for p in powers of z directly from the differential equation. We shall use the latter method. Let

$$p=a_0+a_1z+a_2z^2+a_3z^3+a_4z^4+\quad . \quad . \quad (6.211)$$

Then

$$\frac{d^2p}{dz^2}=\qquad 2a_2+3.2\ a_3z+4.3\ a_4z^2+\ .$$

Hence equation (6.210) gives

$$\begin{aligned}&2a_2+3.2\ a_3z+4.3\ a_4z^2+5.4\ a_5z^3+\\&=-c\left\{a_0z+a_1z^2+a_2z^3+a_3z^4+\ .\ .\ .\ .\ .\right\}\end{aligned}$$

from which, by equating coefficients of like powers of z, we get

$$2a_2 = 0,$$

$$3.2a_3 = -ca_0, \qquad a_3 = -\frac{c}{3.2}a_0$$

$$4.3a_4 = -ca_1, \qquad a_4 = -\frac{c}{4.3}a_1$$

$$5.4a_5 = -ca_2, \qquad a_5 = 0$$

$$6.5a_6 = -ca_3, \qquad a_6 = -\frac{c}{6.5}\times\left(-\frac{c}{3.2}\right)a_0 = \frac{c^2}{2.3.5.6}a_0$$

$$7.6a_7 = -ca_4, \qquad a_7 = \frac{c^2}{3.4.6.7}a_1.$$

Thus we get

$$p = a_0\left(1 - \frac{cz^3}{2.3} + \frac{c^2z^6}{2.3.5.6} - \ldots\ldots\right) + a_1\left(z - \frac{cz^4}{3.4} + \frac{c^2z^7}{3.4.6.7} - \ldots\ldots\right) \quad . \quad . \quad . \quad (6.212)$$

To determine the constants a_0 and a_1 we must use the conditions

$$p = 0 \quad \text{where } z = l$$

and $M = 0$, that is, $\dfrac{dp}{dz} = 0$, where $z = 0$.

The second of these conditions gives

$$a_1 = 0,$$

and then the first gives

$$0 = 1 - \frac{cl^3}{6} + \frac{c^2l^6}{180} - \frac{c^3l^9}{12960} + \quad . \quad . \quad . \quad . \quad (6.213)$$

Equation (6.213) is satisfied by an infinite number of values of cl^3. The series is similar in type to a cosine series. Each of the roots of our present equation corresponds to one of the possible equilibrium forms of the rod, and there is an infinite number of these forms, just as there is an infinite number of forms of equilibrium of the uniform strut. Instability begins when cl^3 has the smallest value which will satisfy (6.213). Writing the equation in the form

$$cl^3 = 6 + \frac{(cl^3)^2}{30} - \frac{(cl^3)^3}{2160} + \quad . \quad . \quad . \quad . \quad . \quad (6.214)$$

we see that there is a root not very much greater than 6. When cl^3 is 6 the third term on the right is 0·1, so it is clear that we shall get a good value of cl^3 by solving the quadratic

$$(cl^3)^2 - 30cl^3 + 180 = 0$$

8*

The smallest root of the quadratic is

$$cl^3 = 15 - 3\sqrt{5} = 8\cdot 3.$$

Take, as a first approximation,

$$cl^3 = 8.$$

Then, using (6.214), a better approximation is

$$cl^3 = 6 + \frac{8^2}{30} - \frac{8^2}{270} + \frac{8^3}{270 \times 11 \times 12}$$
$$= 7\cdot 9.$$

The root appears to be near 8. To get a better approximation let

$$f(z) = 6 - z + \frac{z^2}{30} - \frac{z^3}{2160} + \frac{z^4}{2160.11.12}.$$

Then $$f'(z) = -1 + \frac{z}{15} - \frac{z^2}{720} + \frac{z^3}{2160.11.3}$$

Let the smallest root of $f(z) = 0$ be $(8 + u)$ in which we know that u is small. Therefore the equation

$$f(z) = 0$$

becomes, on neglecting powers of u beyond the first,

$$f(8) + uf'(8) = 0,$$

whence an approximate value of u is

$$u = -\frac{f(8)}{f'(8}$$
$$= -\frac{-0\cdot 0898}{-0\cdot 05486} = -0\cdot 164.$$

Therefore

$$cl^3 = 8 + u$$
$$= 7\cdot 84 \text{ approximately,}$$

or $$\frac{wl^3}{EI} = 7\cdot 84 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.215)$$

If w is given the critical length at which instability begins is

$$l = \left(7\cdot 84 \frac{EI}{w}\right)^{\frac{1}{3}}$$
$$= 1\cdot 99\left(\frac{EI}{w}\right)^{\frac{1}{3}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.216)$$

If l is given, the critical value of w is

$$w = 7\cdot 84 \frac{EI}{l^3}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.217)$$

In the preceding work w is the load per unit length. Suppose the load is the weight of the rod itself. Now let A denote the area of the cross-section, ϱ the weight of a cubic unit of the material, and k the radius of gyration of the cross-section. Then, since $w = \varrho A$ and $I = k^2 A$ the preceding results can be written

$$l = 1 \cdot 99 \left(\frac{Ek^2}{\varrho} \right)^{\frac{1}{3}}$$

and

$$\varrho = 7 \cdot 84 \frac{Ek^2}{l^3}.$$

Suppose a solid steel rod with a circular section, whose diameter is two inches, is clamped at the lower end and held vertically. Taking $E = 31 \times 10^6$ pounds per square inch and $\varrho = 0{\cdot}285$ pounds per cubic inch, the straight form is stable provided

$$l < 1 \cdot 99 \left(\frac{31 \times 10^6 \times 1^2}{4 \times 0 \cdot 285} \right)^{\frac{1}{3}} \text{ inches}$$

or

$$< 50 \text{ feet}$$

If the diameter were $\frac{1}{10}$ of an inch then, since k is proportional to the diameter, the critical length would be

$$l = \left(\frac{1}{20} \right)^{\frac{2}{3}} \times 50 \text{ feet}$$

$$= 6 \cdot 8 \text{ feet.}$$

95. The extension and tension of a rod due to lateral displacements.

When the central line of a rod, which was originally straight, is bent into a curve, the whole lentgth of that curve is greater than the length of the straight line unless the ends of the rod are allowed to move nearer together as the rod takes the curvilinear form. For example, a beam built into rigid supports at the ends must increase the length of its central line when any load is put on it. Of course no supports are so rigid that they will not yield to forces applied to them, so that in every case the ends of a rod under lateral forces do move a little nearer together. Nevertheless the supports may be in some cases so stiff that they yield only a negligible fraction of the increase of length of the rod. We shall deal only with the case where the yield of the supports is negligible, and assume therefore that the increase of length of the central line is the difference between the length of the curve into which it is bent and the shortest distance between the ends.

Let the x-axis be taken on the line joining the centres of the end sections of the rod, and let (x, y) denote the coordinates of any point on the curve of the central line in the bent state. Then, if the origin is taken at one end of the rod, the strained length of the central line is

$$s = \int_0^l \left\{ (dx)^2 + (dy)^2 \right\}^{\frac{1}{2}}$$

$$= \int_0^l \left\{ 1 + \left(\frac{dy}{dx} \right)^2 \right\}^{\frac{1}{2}} dx \quad . \quad . \quad . \quad . \quad . \quad (6.218)$$

Now we shall deal ouly with the cases where $\frac{dy}{dx}$ is everywhere small, as in all beam problems. Then neglecting all powers of $\frac{dy}{dx}$ beyond the cube,

$$s = \int_0^l \left\{ 1 + \tfrac{1}{2}\left(\frac{dy}{dx}\right)^2 \right\} dx$$

$$= l + \tfrac{1}{2}\int_0^l \left(\frac{dy}{dx}\right)^2 dx.$$

Therefore, the increase of length is

$$s - l = \tfrac{1}{2}\int_0^l \left(\frac{dy}{dx}\right)^2 dx \quad \ldots \ldots \quad (6.219)$$

Then the longitudinal strain of the central line is

$$\frac{s-l}{l} = \frac{1}{2l}\int_0^l \left(\frac{dy}{dx}\right)^2 dx$$

It follows that there is a tensional stress in the fibres which pass through the centres of gravity of any section the magnitude of which stress is

$$f_0 = \frac{E}{2l}\int_0^l \left(\frac{dy}{dx}\right)^2 dx \quad \ldots \ldots \quad (6.220)$$

If A denotes the whole section of the rod the whole tension across the section is

$$P = f_0 A = \frac{EA}{2l}\int_0^l \left(\frac{dy}{dx}\right)^2 dx \quad \ldots \ldots \quad (6.221)$$

when the origin is taken at one end of the rod. But if the origin is taken at the middle of the rod, and if $2b$ is the length of the rod,

$$P = f_0 A = \frac{EA}{4b}\int_{-b}^{b} \left(\frac{dy}{dx}\right)^2 dx \quad \ldots \ldots \quad (6.222)$$

96. Beam with ends attached to rigid supports.

To find the deflexion of a beam when the ends are held at a fixed distance apart we can use equation (6.111) provided we put —P for P. Thus

$$\frac{d^2 M}{dx^2} - \frac{P}{EI} M = w$$

or

$$\frac{d^2 M}{dx^2} - n^2 M = w \quad \ldots \ldots \quad (6.223)$$

This equation gives M, and therefore also y, when the boundary conditions are known. The substitution of the value of y in (6.221) or (6.222) will then give an equation for P.

We shall investigate the problem of a beam with pinned ends held at a distance apart equal to the natural length of the beam when a uniform load w per unit length is applied to the beam.

Equation (6.223) can be written in the form

$$\frac{d^2 M}{dx^2} - n^2\left(M + \frac{w}{n^2}\right) = 0 \quad . \quad . \quad . \quad . \quad (6.224)$$

When v is put for $\left(M + \frac{w}{n^2}\right)$ this becomes

$$\frac{d^2 v}{dx^2} - n^2 v = 0$$

the solution of which is

$$v = A \cosh nx + B \sinh nx;$$

that is,

$$M = -\frac{w}{n^2} + A \cosh nx + B \sinh nx \quad . \quad . \quad . \quad (6.225)$$

Now let x be measured from the middle of the beam. Then the conditions that M is zero at both ends, where $x = \pm b$, give

$$0 = -\frac{w}{n^2} + A \cosh nb + B \sinh nb,$$

$$0 = -\frac{w}{n^2} + A \cosh nb - B \sinh nb;$$

whence

$$B = 0,$$

$$A = \frac{w}{n^2 \cosh nb}.$$

Therefore

$$M = \frac{w}{n^2}\left(\frac{\cosh nx}{\cosh nb} - 1\right), \quad . \quad . \quad . \quad . \quad . \quad (6.226)$$

or

$$EI\frac{d^2 y}{dx^2} = \frac{w}{n^2}\left(\frac{\cosh nx}{\cosh nb} - 1\right)$$

Integrating this twice we get

$$EIy = \frac{w}{n^4}\left(\frac{\cosh nx}{\cosh nb} - \tfrac{1}{2}n^2x^2 + Hx + K\right)$$

Since y is zero at both ends the equations for H and K are

$$0 = \frac{\cosh nb}{\cosh nb} - \tfrac{1}{2}n^2b^2 + Hb + K,$$

$$0 = \frac{\cosh nb}{\cosh nb} - \tfrac{1}{2}n^2b^2 - Hb + K,$$

from which

$$H = 0,$$
$$K = \tfrac{1}{2}n^2b^2 - 1.$$

Therefore $$\mathrm{EI}y = \frac{w}{n^4}\left\{\frac{\cosh nx}{\cosh nb} - 1 + \tfrac{1}{2}n^2(b^2 - x^2)\right\} \quad . \quad . \quad . \quad (6.227)$$

Consequently

$$\mathrm{EI}\frac{dy}{dx} = \frac{w}{n^3}\left(\frac{\sinh nx}{\cosh nb} - nx\right)$$

With this value of $\frac{dy}{dx}$ equation (6.222) now gives

$$\mathrm{P} = \frac{\mathrm{EA}}{4b}\int_{-b}^{b}\frac{w^2}{\mathrm{E}^2\mathrm{I}^2n^6}\left\{\frac{\sinh^2 nx}{\cosh^2 nb} - 2nx\frac{\sinh nx}{\cosh nb} + n^2x^2\right\}dx$$

$$= \frac{w^2\mathrm{A}}{2n^6b\mathrm{EI}^2}\left[\frac{\sinh 2nx - 2nx}{4n\cosh^2 nb} - 2\frac{nx\cosh nx - \sinh nx}{n\cosh nb} + \tfrac{1}{3}n^2x^3\right]_0^b$$

$$= \frac{w^2\mathrm{A}}{4n^7b\mathrm{EI}^2}\left\{\tanh nb - nb\,\mathrm{sech}^2\, nb - 4nb + 4\tanh nb + \tfrac{2}{3}n^3b^3\right\}$$

$$= \frac{w^2\mathrm{A}}{4n^7b\mathrm{EI}^2}\left\{\tfrac{2}{3}z^3 - 5z + 5\tanh z + z\tanh^2 z\right\} \quad . \quad . \quad . \quad . \quad . \quad (6.228)$$

where $z = nb$.

Now writing $n^2\mathrm{EI}$ for P and $k^2\mathrm{A}$ for I in the last equation, and then expressing n in terms of z, we get

$$4\frac{\mathrm{E}^2\mathrm{A}^2k^6}{w^2b^8}z^9 = \tfrac{2}{3}z^3 - 5z + 5\tanh z + z\tanh^2 z \quad . \quad (6.229)$$

This equation gives the value of z, and therefore of n and P. When the value of n found from this last equation is substituted in (6.227) the deflexion of the beam is determined.

Let us examine the two extreme cases, when z is small, and when z is large.

The case where z is small. When z is small $\tanh z$ can be expanded in powers of z. Thus

$$\tanh z = \frac{z + \frac{z^3}{\underline{|3}} + \frac{z^5}{\underline{|5}} +}{1 + \frac{z^2}{\underline{|2}} + \frac{z^4}{\underline{|4}} +}$$

$$= z - \frac{1}{3}z^3 + \frac{2}{15}z^5 - \frac{17}{315}z^7 + \frac{62}{2835}z^9 -$$

When c is written for $\frac{wb^4}{\mathrm{EA}k^3}$ equation (6.229) becomes

$$\frac{4}{c^2}z^9 = 2\left\{\frac{17}{315}z^7 - 2\frac{62}{2835}z^9 + \right\},$$

whence

$$z^2 = c^2\left(\frac{17}{630} - \frac{62}{2835}z^2 + \right) \quad . \quad . \quad . \quad . \quad (6.230)$$

The first approximation gives

$$z^2 = \frac{17}{630}c^2,$$

and the second

$$z^2 = \frac{17}{630}c^2 - \frac{62c^2}{2835} \times \frac{17}{630}c^2$$

$$= \frac{17}{630}c^2\left(1 - \frac{62}{2835}c^2\right) \quad . \quad . \quad . \quad . \quad . \quad (6.231)$$

The second approximation to P is therefore

$$P = EI\frac{z^2}{b^2} = \frac{17}{630}\frac{EIc^2}{b^2}\left(1 - \frac{62}{2835}c^2\right) \quad . \quad . \quad . \quad . \quad (6.232)$$

If we are dealing with a beam of rectangular section it is quite clear that the maximum stress is the greatest tension in the middle section. If f_1 denotes the tensional stress due to the bending moment alone and f the total stress due to P and M together, then, writing $2h$ for the depth of the beam, we get

$$f = f_1 + \frac{P}{A} = f_1 + f_0$$

where

$$f_1 = -\frac{hM}{I} = \frac{hw}{n^2 I}\left(1 - \frac{\cosh nx}{\cosh nb}\right)$$

$$= \frac{hw}{n^2 k^2 A}\left(1 - \operatorname{sech} z\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.233)$$

at the middle of the beam.

Therefore

$$f = \frac{hwb^2}{k^2 A}\frac{1}{z^2}\left(1 - \operatorname{sech} z\right) + \frac{Ek^2}{b^2}z^2$$

$$= \frac{hwb^2}{k^2 A}\left\{\frac{1 - \operatorname{sech} z}{z^2} + \frac{EAk^4}{wb^4 h}z^2\right\}$$

$$= \frac{hwb^2}{k^2 A}\left\{\frac{1 - \operatorname{sech} z}{z^2} + \frac{k}{h}\frac{z^2}{c}\right\} \quad . \quad . \quad . \quad . \quad . \quad (6.234)$$

With the assumption that z is small we find

$$\operatorname{sech} z = \left(1 + \frac{z^2}{2} + \frac{z^4}{24} + \frac{z^6}{720} + \right)^{-1}$$

$$= 1 - \frac{z^2}{2} + \frac{5z^4}{24} - \frac{61z^6}{720} +$$

Therefore

$$f = \frac{hwb^2}{k^2 A}\left\{\frac{1}{2} + \left(\frac{k}{hc} - \frac{5}{24}\right)z^2 + \frac{61}{720}z^4 - \right\} \quad . \quad . \quad . \quad (6.235)$$

By means of equation (6.231) this becomes

$$f=\frac{hwb^2}{k^2A}\left\{\frac{1}{2}+\frac{17}{630}\left(\frac{k}{hc}-\frac{5}{24}\right)\left(c^2-\frac{62}{2835}c^4\right)+\frac{61}{720}\left(\frac{17}{630}c^2\right)^2\right\} \quad (6.236)$$

For very small values of c we may neglect the terms containing c^4 and then we get

$$f=\frac{hwb^2}{k^2A}\left\{\frac{1}{2}+\frac{17}{630}\left(\frac{kc}{h}-\frac{5}{24}c^2\right)\right\} \quad . \quad . \quad . \quad . \quad (6.237)$$

The case where z is large. Let us now turn to the case where z is large. The value of $\tanh z$ for quite moderate values of z is very nearly unity. Therefore equation (6.229) becomes approximately

$$\begin{aligned}\frac{4z^9}{c^2}&=\frac{2}{3}z^3-5z+5+z\\&=\tfrac{2}{3}z^3-4z+5\end{aligned}$$

whence

$$z^6=\frac{1}{6}c^2\left(1-\frac{6}{z^2}+\frac{15}{2z^3}\right). \quad . \quad . \quad . \quad . \quad (6.238)$$

We shall carry the solution of this to a second approximation and shall therefore neglect the last term in the bracket. Then

$$\begin{aligned}z&=\left(\frac{c^2}{6}\right)^{\frac{1}{6}}\left(1-\frac{6}{z^2}\right)^{\frac{1}{6}}\\&=\left(\frac{c^2}{6}\right)^{\frac{1}{6}}\left(1-\frac{1}{z^2}\right) \text{ nearly}\end{aligned}$$

The first approximation is

$$z=\left(\frac{c^2}{6}\right)^{\frac{1}{6}}$$

and the second

$$z=\left(\frac{c^2}{6}\right)^{\frac{1}{6}}-\left(\frac{6}{c^2}\right)^{\frac{1}{6}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.239)$$

To this degree of approximation

$$P=EI\frac{z^2}{b^2}=\frac{EAk^2}{b^2}\left\{\left(\frac{c^2}{6}\right)^{\frac{1}{3}}-2\right\} \quad . \quad . \quad . \quad . \quad . \quad (6.240)$$

Also, at the middle of the beam,

$$f_1=-\frac{hM}{I}=\frac{hwb^2}{k^2z^2A}\left(1-\operatorname{sech} z\right) \quad . \quad . \quad . \quad . \quad (6.241)$$

But

$$\operatorname{sech} z=\frac{1}{\cosh z}=\frac{2}{e^z+e^{-z}}=\frac{2}{e^z} \text{ nearly}$$

Therefore

$$f_1=\frac{hwb^2}{k^2z^2A}\left(1-2e^{-z}\right)$$

$$=\frac{hwb^2}{k^2z^2A}\left(1-2e^{-m}\right)\text{ nearly} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.242)$$

where

$$m=\left(\frac{c^2}{6}\right)^{\frac{1}{6}}=\left(\frac{1}{6}\right)^{\frac{1}{6}}\left(\frac{wb^4}{EAk^3}\right)^{\frac{1}{3}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.243)$$

Therefore the total maximum stress is

$$\begin{aligned} f&=\frac{P}{A}+f_1\\ &=\frac{Ek^2}{b^2}z^2+\frac{hwb^2}{k^2A}\left(1-2e^{-m}\right)\\ &=\frac{Ek^2}{b^2}m^2+\frac{hwb^2}{k^2A}\text{ approximately}\\ &=\left(\frac{Ew^2b^2}{6A^2}\right)^{\frac{1}{3}}+\frac{hwb^2}{k^2A} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.244)\end{aligned}$$

This last equation gives the most important terms in the maximum stress when z is large, that is, when c^2 is large.

It is worth while to notice the form of the curve of the beam when z is large. We may write equation (6.227) in the form

$$EIy=\frac{wb^2}{n^2}\left\{\frac{1}{2}\left(1-\frac{x^2}{b^2}\right)-\frac{1}{z^2}\left(1-\frac{\cosh nx}{\cosh z}\right)\right\} \quad . \quad . \quad . \quad (6.245)$$

from which it is easily seen that, when z is large, the approximate equation for y is

$$\begin{aligned} EIy&=\frac{wb^2}{2n^2}\left(1-\frac{x^2}{b^2}\right)\\ &=\frac{wb^4}{2z^2}\left(1-\frac{x^2}{b^2}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.246)\end{aligned}$$

This can also be written in the form

$$y=\frac{wb^2}{2P}\left(1-\frac{x^2}{b^2}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.247)$$

Thus the limiting form of the beam for very large values of w is circular.

97. An approximate method for a uniform beam under tension.

From the single example that has just been worked out it is apparent that the accurate method is very cumbersome. To avoid the labour of this method we shall now indicate how approximate results can be got which are good enough for most practical cases.

When we multiply both sides of (6.223) by ydx and integrate over the length of the beam we get

$$\int_{-b}^{b}y\frac{d^2M}{dx^2}dx-n^2\int_{-b}^{b}yM\,dx=\int_{-b}^{b}wy\,dx \quad . \quad . \quad . \quad . \quad (6.248)$$

Now integration by parts gives

$$\int y\frac{d^2\mathrm{M}}{dx^2}dx = y\frac{d\mathrm{M}}{dx} - \frac{dy}{dx}\mathrm{M} + \int \mathrm{M}\frac{d^2y}{dx^2}dx$$

Therefore

$$\int_{-b}^{b} y\frac{d^2\mathrm{M}}{dx^2}dx = \left[y\frac{d\mathrm{M}}{dx} - \mathrm{M}\frac{dy}{dx}\right]_{-b}^{b} + \int_{-b}^{b}\mathrm{M}\frac{d^2y}{dx^2}dx$$

But the integrated terms are zero at either pinned or clamped ends. Consequently

$$\int_{-b}^{b} y\frac{d^2\mathrm{M}}{dx^2}dx = \int_{-b}^{b}\mathrm{M}\frac{d^2y}{dx^2}dx$$

$$= \mathrm{EI}\int_{-b}^{b}\left(\frac{d^2y}{dx^2}\right)^2 dx \quad . \quad . \quad . \quad . \quad (6.249)$$

Again

$$\int_{-b}^{b} y\mathrm{M}dx = \int_{-b}^{b}\mathrm{EI}y\frac{d^2y}{dx^2}dx$$

$$= \left[\mathrm{EI}y\frac{dy}{dx}\right]_{-b}^{b} - \mathrm{EI}\int_{-b}^{b}\left(\frac{dy}{dx}\right)^2 dx$$

$$= -\mathrm{EI}\int_{-b}^{b}\left(\frac{dy}{dx}\right)^2 dx \quad . \quad . \quad . \quad . \quad . \quad (6.250)$$

Therefore equation (6.248) becomes, when P is written for $\mathrm{EI}n^2$,

$$\mathrm{EI}\int_{-b}^{b}\left(\frac{d^2y}{dx^2}\right)^2 dx + \mathrm{P}\int_{-b}^{b}\left(\frac{dy}{dx}\right)^2 dx = \int_{-b}^{b} wy\,dx \quad . \quad . \quad (6.251)$$

where P itself is given by (6.222), that is, by

$$\mathrm{P} = \frac{\mathrm{EA}}{4b}\int_{-b}^{b}\left(\frac{dy}{dx}\right)^2 dx \quad . \quad . \quad . \quad . \quad . \quad (6.252)$$

The last two equations are, of course, quite accurate if the correct expression for y is used in the integrals; but the real value of these equations is due to the fact that the equations remain nearly true if any expression for y be used which makes the assumed curve for the beam not widely different from the real curve. In fact, if we use for y an expression which looks very different from the true expression, provided that it gives a curve of the same general character as the real curve, the resulting error in P and in the maximum deflexion is quite small. A few examples are given below. The method consists in assuming an expression for the deflexion involving one unknown constant, and then using (6.251) and (6.252) to determine this constant.

98. Beam pinned at both ends under a uniform load. Since y and M are zero at both ends a good expression for y is

$$y = a\cos\frac{\pi x}{2b} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.253)$$

a being the deflexion at the middle of the beam, a quantity which has to be determined by means of equations (6.251) and (6.252).

Then

$$\int_{-b}^{b}\left(\frac{dy}{dx}\right)^2 dx = 2\int_0^b \frac{\pi^2 a^2}{4b^2}\sin^2\frac{\pi x}{2b}dx = \frac{\pi^2 a^2}{4b}$$

$$\int_{-b}^{b}\left(\frac{d^2y}{dx^2}\right)^2 dx = 2\int_0^b \frac{\pi^4 a^2}{16b^4}\cos^2\frac{\pi x}{2b}dx$$

$$= \frac{\pi^4 a^2}{16b^3}$$

$$\int_{-b}^{b} wy\,dx = 2\int_0^b wa\cos\frac{\pi x}{2b}\,dx$$

$$= \frac{4wab}{\pi}$$

Therefore equation (6.251) gives

$$\mathrm{EI}\frac{\pi^4 a^2}{16b^3} + \mathrm{EA}\frac{\pi^4 a^4}{64b^3} = \frac{4wab}{\pi}$$

Now let d denote the depth of the beam, and let us assume that the section is rectangular. Then

$$\mathrm{I} = \tfrac{1}{12}\mathrm{A}d^2$$

and therefore

$$\mathrm{EA}\frac{\pi^4 a^2 d^2}{192b^3} + \mathrm{EA}\frac{\pi^4 a^4}{64b^3} = \frac{4wab}{\pi},$$

whence

$$\frac{wb^4}{\mathrm{EA}d^3} = \frac{\pi^5}{256}\left\{\frac{1}{3}\frac{a}{d} + \left(\frac{a}{d}\right)^3\right\}$$

$$= 0{\cdot}398\frac{a}{d} + 1{\cdot}195\left(\frac{a}{d}\right)^3, \quad . \quad . \quad . \quad (6.254)$$

which is the equation for a.

As another example we may take the usual deflexion for the beam under a uniform load when there is no tension. When x is measured from the middle of the beam this deflexion can be written in the form

$$y = \frac{a}{5b^4}\left(5b^4 - 6b^2x^2 + x^4\right) \quad . \quad . \quad . \quad . \quad (6.255)$$

where a denotes the deflexion at the middle, as in the last example. In this case equation (6.251) gives

$$\frac{wb^4}{\mathrm{EA}d^3} = \frac{2}{5}\frac{a}{d} + 2\left(\frac{4\times 34}{5\times 35}\right)^2\left(\frac{a}{d}\right)^3$$

$$= 0{\cdot}400\frac{a}{d} + 1{\cdot}208\left(\frac{a}{d}\right)^3 \quad . \quad . \quad . \quad . \quad (6.256)$$

The two coefficients on the right hand side of this equation differ from the corresponding coefficients in equation (6.254) by only one half per cent and one per cent respectively.

When a is much smaller than d equation (6.256) gives a nearly perfect result, for in that case the deflexion differs very little from that obtained on the assumption of no tension. It is useful, as well as interesting, to apply our present method to the other extreme case, namely, the case where the tension is much more important than the rigidity in supporting the load. If the rigidity had no effect at all the curve of the beam would be circular; that is, we should have

$$y = a\left(1 - \frac{x^2}{b^2}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.257)$$

a parabolic curve which is approximately circular for small values of y. The substitution of this expression in (6.251) gives

$$\frac{wb^4}{\text{EA}d^3} = \frac{1}{3}\frac{a}{d} + \frac{8}{9}\left(\frac{a}{d}\right)^3. \quad . \quad . \quad . \quad . \quad . \quad (6.258)$$

The fact that (6.256) and (6.258), which represent the two extreme cases, do not disagree very greatly, shows that the present method is a good one. Equation (6.257) should not be used for a beam that has any appreciable rigidity. In nearly all cases the two earlier equations will give better results.

99. Rod clamped at both ends under a uniform load.

The deflection for a clamped beam when there is no tension is

$$y = a\left(1 - \frac{x^2}{b^2}\right)^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.259)$$

Another expression for y which makes y and $\frac{dy}{dx}$ zero at both ends is

$$y = \tfrac{1}{2}a\left(1 + \cos\frac{\pi x}{b}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.260)$$

When these values of y are used in equation (6.251) the resulting equations are

$$\frac{wb^4}{\text{EA}d^3} = 2\frac{a}{d} + 1{\cdot}39\left(\frac{a}{d}\right)^3 \quad . \quad . \quad . \quad . \quad . \quad (6.261)$$

$$\frac{wb^4}{\text{EA}d^3} = 2{\cdot}03\frac{a}{d} + 1{\cdot}52\left(\frac{a}{d}\right)^3. \quad . \quad . \quad . \quad . \quad (6.262)$$

which are again in close agreement.

It should be remembered that all the preceding results by the approximate method, from equation (6.254) onwards, have been obtained on the assumption that the beams had uniform rectangular sections. For any other sections the proper value of I must be used.

100. Approximate method of finding the buckling thrust of a strut with any cross-section.

The differential equation from which the buckling thrust has to be determined is

$$EI\frac{d^2y}{dx^2} = -Py \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.263)$$

Multiplying this by $\frac{d^2y}{dx^2}$ and integrating from one end to the other of the rod we get, taking the ends at $x = 0$ and $x = l$,

$$\int_0^l EI\left(\frac{d^2y}{dx^2}\right)^2 dx = -P\int_0^l y\frac{d^2y}{dx^2}dx$$
$$= -P\left[y\frac{dy}{dx}\right]_0^l + P\int_0^l \left(\frac{dy}{dx}\right)^2 dx. \quad . \quad (6.264)$$

Now if the ends are pinned y must be measured from the line through the ends in order that equation (6.263) should be true. If the ends are clamped y is measured from the line joining the points of inflexion on the curve. In both cases either y or $\frac{dy}{dx}$ is zero at each end. Consequently the integrated term in (6.264) is zero. Therefore

$$\int_0^l EI\left(\frac{d^2y}{dx^2}\right)^2 dx = P\int_0^l \left(\frac{dy}{dx}\right)^2 dx$$

whence

$$P = \frac{\int_0^l EI\left(\frac{d^2y}{dx^2}\right)^2 dx}{\int_0^l \left(\frac{dy}{dx}\right)^2 dx} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.265)$$

Now if the value of y in one of the possible equilibrium forms of the strut is used in this equation the equation gives correctly the corresponding value of P. The advantage of this equation, however, lies in the fact that quite good values of P can be found from only approximate values of y. We give a proof of this below.

Let the possible forms of equilibrium of the rod be given by the curves

$$\left.\begin{array}{l} y = A_1y_1, \\ y = A_2y_2, \\ y = A_3y_3, \\ \text{etc. . .} \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.266)$$

and let P_n be the value of P corresponding to y_n. In the case of a uniform rod with pinned ends y_1, y_2, y_3, are $\sin\frac{\pi x}{l}$, $\sin\frac{2\pi x}{l}$, $\sin\frac{3\pi x}{l}$.

Now the values of y in equations (6.266) are the only correct values of y to substitute in (6.265). But suppose we use an incorrect value. Let us suppose that we use

$$y = A_1y_1 + A_2y_2 + A_3y_3 + \ldots \quad . \quad . \quad . \quad . \quad (6.267)$$

This is a series like a Fourier series, and it is possible to represent any function of x between $x=0$ and $x=l$ by such a series. The functions y_1, y_2, etc. are such that

$$\int_0^l \frac{y_n y_m}{EI}\,dx = 0,\ (m \neq n) \quad . \quad . \quad . \quad . \quad (6.268)$$

which result is included in (6.274) below. If therefore we multiply throughout equation (6.267) by $\dfrac{y_n\,dx}{EI}$ and integrate from 0 to l we get

$$\int_0^l \frac{y y_n}{EI}\,dx = A_n \int_0^l \frac{y_n^2}{EI}\,dx \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.269)$$

If y is a *given* function of x, and if the functions y_1, y_2, etc., are known, this last equation gives A_n. Thus the coefficients in the expansion of any function of x in terms of y_1, y_2, etc., can be determined.

Writing D for $\dfrac{d}{dx}$, we find from equation (6.265)

$$P = \frac{\int_0^l EI\{A_1 D^2 y_1 + A_2 D^2 y_2 + A_3 D^2 y_3 + \ldots\}^2\,dx}{\int_0^l \{A_1 Dy_1 + A_2 Dy_2 + A_3 Dy_3 + \ldots\}^2\,dx} \quad . \quad (6.270)$$

Now
$$\int_0^l EID^2 y_n D^2 y_m\,dx = \int_0^l - P_n y_n D^2 y_m\,dx$$
$$= -P_n\Big[y_n D y_m\Big]_0^l + P_n \int_0^l D y_m D y_n\,dx.$$

The integrated term is zero at both limits since it is understood that y_n and y_m satisfy the boundary conditions as well as the differential equation. Therefore

$$\int_0^l EID^2 y_m D^2 y_n dx = P_n \int_0^l D y_m D y_n dx \quad . \quad . \quad . \quad (6.271)$$

By the same method we get

$$\int_0^l EID^2 y_m D^2 y_n dx = P_m \int_0^l D y_m D y_n dy \quad . \quad . \quad . \quad (6.272)$$

Therefore, by subtraction,

$$0 = (P_n - P_m)\int_0^l D y_m D y_n dx$$

whence, if n is not equal to m,

$$\int_0^l D y_m D y_n dx = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.273)$$

Now it follows from (6.271) and (6.273) that, if m is not equal to n,

$$\int_0^l EID^2 y_m D^2 y_n dx = 0 \quad . \quad . \quad . \quad . \quad . \quad (6.274)$$

We have now proved that the integrals of all the terms except the squared terms in the numerator and denominator of the fraction on the right of equation (6.270) are zero. This equation therefore becomes

$$P = \frac{\int_0^l EI\{A_1^2(D^2y_1)^2 + A_2^2(D^2y_2)^2 + \ldots\} dx}{\int_0^l \{A_1^2(Dy_1)^2 + A_2^2(Dy_2)^2 + \ldots\} dx} \quad . \quad . \quad (6.275)$$

Now equation (6.271) is still true if $m = n$. Therefore

$$\int_0^l EI(D^2y_n)^2 dx = P_n \int_0^l (Dy_n)^2 dx.$$

Let us write

$$c_n^2 = \int_0^l (Dy_n)^2 dx \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.276)$$

Then

$$P = \frac{A_1^2c_1^2P_1 + A_2^2c_2^2P_2 + \ldots}{A_1^2c_1^2 + A_2^2c_2^2 + \ldots} \quad . \quad . \quad . \quad . \quad (6.277)$$

Now suppose we are seeking the smallest buckling load, which we have denoted by P_1. We can usually make a fairly good guess at an approximate expression for y for this case. The absolutely correct expression is, of course,

$$y = Ay_1,$$

but we may find it very difficult to get the exact expression for y_1 in terms of x. If our guess is a good one then we have taken

$$y = A_1y_1 + A_2y_2 + A_3y_3 +$$

where A_2y_2, A_3y_3, etc., are all small in comparison with A_1y_1. In that case $A_2^2c_2^2$, $A_3^2c_3^2$, etc., are still smaller in comparison with $A_1^2c_1^2$. But

$$P = \frac{A_1^2c_1^2P_1\left\{1 + \frac{A_2^2c_2^2}{A_1^2c_1^2}\frac{P_2}{P_1} + \ldots\right\}}{A_1^2c_1^2\left\{1 + \frac{A_2^2c_2^2}{A_1^2c_1^2} + \ldots\right\}}$$

$$= P_1 \frac{1 + \frac{A_2^2c_2^2}{A_1^2c_1^2}\frac{P_2}{P_1} + \ldots}{1 + \frac{A_2^2c_2^2}{A_1^2c_1^2} + \ldots} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.278)$$

Now since P_1 is the smallest of the forces P_1, P_2, P_3, etc., it follows that the approximate value of P given by the last equation is greater than the true value P_1 unless all the A's except A_1 vanish, in which case it is equal to P_1. Therefore the true value of the smallest buckling thrust is the least value of P given by (6.265) whatever function of x is used for y.

If we take care that the value of y we use in equation (6.265) satisfies the boundary conditions, then it is true, whatever these boundary conditions are, that the least buckling load is the least value of P given by that equation. Moreover, if we want any buckling load except the smallest we need only take a value of y which has the known characteristics of the ordinate corresponding to this load. Thus if we want the second load we have to take an expression for y which crosses the x-axis once between the ends, and find the least value of P for such a y.

An excellent method of arriving at a close approximation to the buckling load that we are seeking is to use an expression for y which satisfies the boundary conditions and which has one constant in it which we can vary at will without altering the desired characteristics in y. We shall now apply the method to a few cases, and we shall take first cases where we know the result for the sake of testing the method.

101. Illustrative examples.

(a) Uniform strut pinned at both ends.

It is slightly easier to take the origin at the middle instead of at

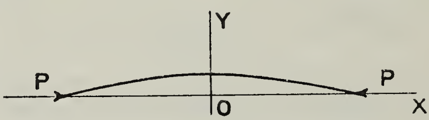

Fig. 55

one end. Then let $2a$ denote the length, and let the origin be taken at the mid point of the line joining the ends.

Let us take

$$y = k(a^2 - x^2)(na^2 + x^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.279)$$

This value of y vanishes at the ends $x = \pm a$. Moreover, since it involves only even powers of x it represents a curve symmetrical about OY, which we know must be a characteristic of the correct curve. The parameter n can be varied for the purpose of making P as small as possible for this form of y. The constant k will introduce a factor k^2 into numerator and denominator of the fraction in (6.265) which gives P. We may therefore take $k = 1$ without affecting the result.

Now

$$y = na^4 - (n - 1)a^2x^2 - x^4$$

It will save trouble in writing to use a single letter for $(n - 1)$. Therefore we write m for $(n - 1)$. Then

$$y = (m + 1)a^4 - ma^2x^2 - x^4 \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.280)$$

Substituting this value of y in (6.265) we get

$$P = \frac{\int_{-a}^{a} EI(2ma^2 + 12x^2)^2 dx}{\int_{-a}^{a} (2ma^2x + 4x^3)^2 dx}$$

$$= \frac{2EIa^5\left(4m^2 + 16m + \frac{144}{5}\right)}{2a^7\left(\frac{4}{3}m^2 + \frac{16}{5}m + \frac{16}{7}\right)} \quad . \quad . \quad . \quad . \quad (6.281)$$

Therefore, writing x for $\frac{Pa^2}{21EI}$, we get

$$x = \frac{5m^2 + 20m + 36}{35m^2 + 84m + 60}, \quad . \quad . \quad . \quad . \quad . \quad (6.282)$$

whence

$$(35x - 5)m^2 + (84x - 20)m + (60x - 36) = 0 \quad . \quad . \quad (6.283)$$

We have to choose m so that x has its least value. Now the values of m given by (6.283) will be real provided

$$(84x - 20)^2 > 4(35x - 5)(60x - 36) \quad . \quad . \quad . \quad . \quad (6.284)$$

This then must give the range of values of x that are possible for real values of m. The least value of x is what we are seeking. This least value is the least root of the equation

$$(84x - 20)^2 = 4(35x - 5)(60x - 36),$$

which is equivalent to

$$(42x - 10)^2 = (70x - 10)(30x - 18),$$

whence

$$21x^2 - 45x + 5 = 0.$$

This least rood is

$$x = \frac{45 - \sqrt{1605}}{42}$$

$$= \frac{4 \cdot 9375}{42}$$

Therefore

$$\frac{P}{EI} = \frac{21x}{a^2} = \frac{84x}{l^2}$$

$$= \frac{9 \cdot 875}{l^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.285)$$

The correct value of the coefficient is π^2, which equals 9·8696. Thus the error is less than one in a thousand.

Strut with a variable section.

As a second example we take the case of a pinned strut the moment of inertia of the section of which is

$$I = I_0 \cos \frac{\pi x}{2a} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6\ 286)$$

at distance x from the middle, $2a$ being written for l, the length of the strut. Thus I is I_0 at the middle and zero at the ends.

The expression for P is, since the ends are at $x = \pm a$,

$$P = \frac{\int_{-a}^{a} EI\left(\frac{d^2y}{dx^2}\right)^2 dx}{\int_{-a}^{a}\left(\frac{dy}{dx}\right)^2 dx}$$

If we take

$$y = a^2 - x^2 \qquad (6.287)$$

$$P = \frac{\int_{-a}^{a} EI_0 \cos\frac{\pi x}{2a}\, 4\, dx}{\int_{-a}^{a} 4x^2 dx}$$

$$= \frac{6\,EI_0}{\pi a^2} = \frac{24\,EI_0}{\pi l^2}$$

$$= 7 \cdot 640\,\frac{EI_0}{l^2} \qquad (6.288)$$

Next suppose we take the same expression for y as we took for the uniform strut, namely

$$y = (m+1)a^4 - ma^2x^2 - x^4 \qquad (6.289)$$

Then

$$P = \frac{\int_{-a}^{a} EI(2ma^2 + 12x^2)^2 dx}{\int_{-a}^{a} (2ma^2x + 4x^3)^2 dx}$$

$$= EI_0\frac{\int_{-a}^{a} \cos\frac{\pi x}{2a}(m^2a^4 + 12ma^2x^2 + 36x^4)dx}{\int_{-a}^{a}(m^2a^4x^2 + 4ma^2x^4 + 4x^6)\,dx}$$

Since we are dealing with an even function of x the integrals have the same values over each half of the strut. Consequently we need integrate only over half of the strut. Therefore

$$P = EI_0\,\frac{\int_{0}^{a} \cos\frac{\pi x}{2a}(m^2a^4 + 12ma^2x^2 + 36x^4)\,dx}{\int_{0}^{a}(m^2a^4x^2 + 4ma^2x^4 + 4x^6)\,dx}$$

$$= \frac{24\,EI_0}{\pi^5a^2}\;\frac{\frac{\pi^4}{12}m^2 + (\pi^4 - 8\pi^2)m + 3\pi^4 - 144\pi^2 + 1152}{\frac{1}{3}m^2 + \frac{4}{5}m + \frac{4}{7}} \qquad (6.290)$$

The minimum value of the expression representing P for different values of m is

$$P = 7 \cdot 634 \frac{EI_0}{l^2} \quad \text{. . . . . (6.291)}$$

This differs very little from the result in (6.288). The smallness of this difference indicates that the form of the curve given by (6.287) is a fairly good approximation to the true form of the strut.

CHAPTER VII

TORSION OF RODS. SAINT VENANT'S THEORY

102. Rod with a uniform twist per unit length.

Suppose that a uniform isotropic rod is fixed at one end, and that any section of the rod at distance z from the fixed end is twisted through an angle τz, so that τ is the angle of twist per unit length. We shall assume that all elements of the rod of equal length are strained in exactly the same way. This means that the stresses and strains are independent of z. For the present we make no assumptions about the displacement w, which is in the direction of z, except that w is independent of z. Our object is, however, to try if the assumed displacements, together with some yet undiscovered value of w, will reduce the shear stresses over any cross-section to a pure couple, and at the same time give no forces on the sides of the rod.

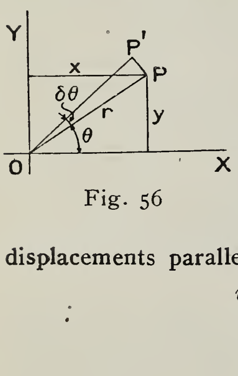

Fig. 56

We shall suppose that we are dealing with a short length of the rod, so that we may regard τz as a small angle. In fig. 56 the angle POP′ is τz, P′ being the displaced position of P. Regarding PP′ as a straight line of length $r\tau z$ we see from the figure that the component displacements parallel to the axes OX, OY, are

$$u = -\text{PP}' \sin\theta = -r\tau z \sin\theta$$
$$= -\tau y z \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.1)$$
$$v = +\text{PP}' \cos\theta$$
$$= \tau x z \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.2)$$

Our assumption concerning w is that w is not a function of z, that is,

$$w = f(x, y)$$

The above values of the displacements give

$$\frac{\partial u}{\partial x} = 0, \quad \frac{\partial v}{\partial y} = 0, \quad \frac{\partial w}{\partial z} = 0, \quad . \quad . \quad . \quad . \quad . \quad (7.3)$$

whence it follows that

$$P_1 = 0, \; P_2 = 0, \; P_3 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.4)$$

Moreover,

$$\begin{aligned} S_3 &= n\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right) \\ &= n(-\tau z+\tau z) \\ &= 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7{\cdot}5) \end{aligned}$$

103. The boundary conditions.

Let us now consider the boundary conditions. We have to make the action on the sides of the prism everywhere zero. Let HN fig. 57(a) be the normal to the surface of the rod at a point H, and

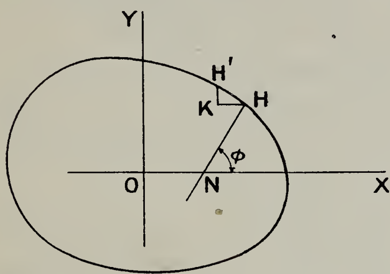

Fig. 57a

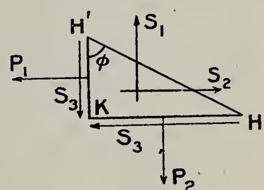

Fig. 57b

let HH′ represent an element of the surface which is assumed to have a length dz in the z-direction. The element HH′K is shown enlarged in fig. 57(b), and all the stresses on this element are shown which are parallel to the xy plane except the stresses on the triangular face opposite to HH′K. These stresses not shown we may denote by S'_1 and S'_2; they act in the directions opposite to the directions of S_1 and S_2 respectively, and, since they act at $z-dz$, we get

$$S'_1 - S_1 = \frac{\partial S_1}{\partial z}(-dz) = -\frac{\partial S_1}{\partial z}\,dz$$

$$S'_2 - S_2 = -\frac{\partial S_2}{\partial z}\,dz$$

Now in order to get the boundary conditions we shall not at first suppose that the stresses on the surface are zero. Let the component stresses parallel to OX and OY on the element of boundary HH′ be F_1 and F_2. Then, resolving parallel to OX and OY in turn, for the equilibrium of the element, we get

$$dz\{P_1 \times KH' + S_3 \times KH\} + \tfrac{1}{2}KH \times KH'(S_2' - S_2) = F_1 dz \times HH',$$

and

$$dz\{P_2 \times KH + S_3 \times KH'\} + \tfrac{1}{2}KH \times KH'(S_1' - S_1) = F_2 dz \times HH'.$$

These give

$$F_1 = P_1 \cos\varphi + S_3 \sin\varphi - \tfrac{1}{2}HH' \sin\varphi\cos\varphi\,\frac{\partial S_2}{\partial z}$$

$$F_2 = P_2 \sin\varphi + S_3 \cos\varphi - \tfrac{1}{2}HH' \sin\varphi \cos\varphi \frac{\partial S_1}{\partial z}.$$

Now, in consequence of (7.4) and (7.5), and because HH′ is an infinitesimal length, these equations make F_1 and F_2 identically zero.

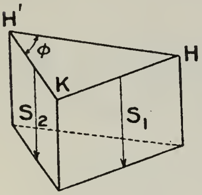

Fig. 58

We have still another boundary condition to satisfy since the forces parallel to the z-axis must be in equilibrium. On the faces KH′ and KH the stresses S_2 and S_1 act, both perpendicular to the plane of the triangle and both in the same direction. Therefore, for equilibrium,

$$dz\{S_2 \times KH' + S_1 \times KH\} = 0;$$

that is, $S_2 \times HH' \cos\varphi + S_1 \times HH' \sin\varphi = 0,$

whence $S_2 \cos\varphi + S_1 \sin\varphi = 0$ . . . . (7.6)

This is our third boundary condition.

Now

$$S_2 = n\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$
$$= n\left(\frac{\partial w}{\partial x} - \tau y\right) \quad . \quad . \quad . \quad . \quad (7.7)$$
$$S_1 = n\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)$$
$$= n\left(\frac{\partial w}{\partial y} + \tau x\right) \quad . \quad . \quad . \quad . \quad (7.8)$$

Therefore equation (7.6) becomes, after division by n,

$$\left(\frac{\partial w}{\partial x} - \tau y\right)\cos\varphi + \left(\frac{\partial w}{\partial y} + \tau x\right)\sin\varphi = 0 \quad . \quad . \quad . \quad (7.9)$$

Since this last equation is a boundary condition the coordinates x and y refer to a point on the boundary. Then let H (fig. 57 a) be the point (x, y) and H′ the point $(x + dx, y + dy)$, and let the length HH′ be denoted by ds. Then

$$dx = -KH = -ds \sin\varphi;$$

that is, $\sin\varphi = -\dfrac{dx}{ds}$

Likewise $\cos\varphi = +\dfrac{dy}{ds}$ . . . . (7.10)

Substituting these values in (7.9) this equation becomes

$$\frac{\partial w}{\partial x}\frac{dy}{ds} - \frac{\partial w}{\partial y}\frac{dx}{ds} - \tau\left(x\frac{dx}{ds} + y\frac{dy}{ds}\right) = 0 \quad . \quad . \quad . \quad (7.11)$$

Let us assume that another function, ψ, of x and y exists, such that

$$\left.\begin{aligned}\frac{\partial\psi}{\partial y}&=\frac{\partial w}{\partial x},\\ \frac{\partial\psi}{\partial x}&=-\frac{\partial w}{\partial y},\end{aligned}\right\} \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.12)$$

and leave the justification of this step till a little later. Then (7.11) may be written

$$\frac{\partial\psi}{\partial y}\frac{dy}{ds}+\frac{\partial\psi}{\partial x}\frac{dx}{ds}-\tau\left(x\frac{dx}{ds}+y\frac{dy}{ds}\right)=0 \quad . \quad . \quad . \quad . \quad (7.13)$$

which is equivalent to

$$\frac{d\psi}{ds}-\tfrac{1}{2}\tau\frac{d}{ds}(x^2+y^2)=0 \quad . \quad . \quad . \quad . \quad . \quad (7.14)$$

Integrating this last equation with respect to s, and keeping in mind that our result is true only along the boundary, we get

$$\psi-\tfrac{1}{2}\tau(x^2+y^2)=\text{a constant} \quad . \quad . \quad . \quad . \quad . \quad (7.15)$$

This last equation may be taken as our boundary condition.

104. Equations for internal equilibrium.

We have not yet used the equations of internal equilibrium, the equations (2.28), (2.29), and (2.30). Since

$$\begin{aligned}\triangle&=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\\ &=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.16)\end{aligned}$$

and since we are assuming that all the body forces and accelerations are zero, the first two of these equations are satisfied because all the terms vanish. The third equation reduces to

$$n\nabla^2 w=0,$$

or, since w is not a function of z,

$$\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.17)$$

This is the equation we needed to justify the assumption in (7.12). In trying to find ψ to satisfy both equations in (7.12) we were imposing too many conditions on ψ. In fact ψ is almost completely determined by one of the equations (7.12), and the equation (7.17) is really implied in the two equations (7.12). For

$$\left.\begin{aligned}\frac{\partial^2 w}{\partial x^2}&=\frac{\partial^2\psi}{\partial x\partial y}\\ \frac{\partial^2 w}{\partial y^2}&=-\frac{\partial^2\psi}{\partial x\partial y}\end{aligned}\right\} \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.18)$$

and therefore

$$\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}=0$$

Having now justified ψ we have to find what equation it satisfies so as to determine the function. From (7.12) we get

$$\frac{\partial^2 \psi}{\partial y^2}=\frac{\partial^2 w}{\partial x\,\partial y}$$

$$\frac{\partial^2 \psi}{\partial x^2}=-\frac{\partial^2 w}{\partial x\,\partial y}$$

Therefore

$$\frac{\partial^2 \psi}{\partial x^2}+\frac{\partial^2 \psi}{\partial y^2}=0, \quad \ldots \ldots \quad (7.19)$$

an equation exactly similar to the one for w. The function ψ has to be found to satisfy equation (7.19) at all points within the boundary of the cross-section, and to satisfy equation (7.15) at all points on the boundary. Then w is found by means of (7.12).

105. Solution of the equations.

A special method has been developed for solving equations such as (7.19), which is particularly useful when a function w which satisfies (7.12) is also needed. This method is given below.

In the boundary condition (7.15) the constant may be taken as zero without loss of generality. If the constant is not zero let it be C. Then if we write ψ_1 for $(\psi - C)$ the new function ψ_1 will satisfy all the equations that ψ has to satisfy, of which (7.19) and (7.15) are the most important, and equation (7.15) becomes

$$\psi_1 - \tfrac{1}{2}\tau(x^2+y^2)=0$$

Then we may regard the constant as zero in (7.15) and retain the symbol ψ instead of ψ_1.

Let us assume that w and ψ are real functions of x and y such that

$$w+i\psi=f(x+iy) \quad \ldots \ldots \quad (7.20)$$

where i denotes $\sqrt{-1}$ and f indicates any function. This means that w is the real part and ψ is the part multiplied by i, in the expanded form of $f(x+iy)$. Then we can prove that w and ψ, given by equation (7.20), satisfy also the two equations (7.12), as well as equations (7.17) and (7.19) which follow from (7.12). Let z be written for $(x+iy)$. Then

$$w+i\psi=f(z) \quad \ldots \ldots \quad (7.21)$$

Therefore

$$\frac{\partial w}{\partial x}+i\frac{\partial \psi}{\partial x}=f'(z)\frac{\partial z}{\partial x}$$

$$=f'(z) \quad \ldots \ldots \quad (7.22)$$

and

$$\frac{\partial w}{\partial y}+i\frac{\partial \psi}{\partial y}=f'(z)\frac{\partial z}{\partial y}$$
$$=if'(z) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.23)$$

Consequently

$$\frac{\partial w}{\partial y}+i\frac{\partial \psi}{\partial y}=i\left(\frac{\partial w}{\partial x}+i\frac{\partial \psi}{\partial x}\right)$$
$$=i\frac{\partial w}{\partial x}-\frac{\partial \psi}{\partial x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.24)$$

Equating real and imaginary parts of the two sides of this equation we get

$$\frac{\partial w}{\partial y}=-\frac{\partial \psi}{\partial x},$$
$$\frac{\partial \psi}{\partial y}=\frac{\partial w}{\partial x},$$

which agree with equations (7.12), from which (7.17) and (7.19) follow. Then we can get values of w and ψ satisfying the differential equations (7.12) merely by taking any function of $(x+iy)$ in (7.20) and equating w and $i\,\psi$ to the real and imaginary parts of the function. Then the boundary condition (7.15) gives us the equation of the cross section of the rod to which the solution applies.

106. Resultant action on a section of the rod.

We have still to prove that the action over a section is a pure couple. We can do this by showing that the component forces on the whole area in the directions of the coordinate axes are both zero.

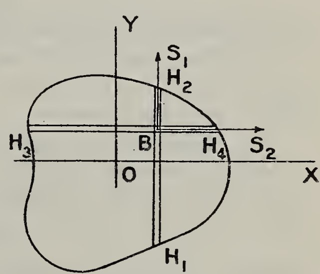

Fig. 59

Let dA denote an element of area of the cross section x, y, on which the component shear forces S_1 and S_2 act. Then the total component action on the section in the direction OX is

$$\int S_2 dA=\int n\left(\frac{\partial w}{\partial x}-\tau y\right)dA$$
$$=\int\int n\left(\frac{\partial \psi}{\partial y}-\tau y\right)dx\,dy \quad . \quad . \quad . \quad . \quad 7.25)$$

Let y_1 and y_2 denote the values of y at H_1 and H_2 in fig. 59, and ψ_1, and ψ_2 the values of ψ at the same points. Then

$$\int_{y_1}^{y_2}\left(\frac{\partial \psi}{\partial y}-\tau y\right)dy=\left[\psi-\tfrac{1}{2}\tau y^2\right]_{y_1}^{y_2}$$
$$=(\psi_2-\tfrac{1}{2}\tau y_2{}^2)-(\psi_1-\tfrac{1}{2}\tau y_1{}^2)$$

But, by equation (7.15), at a point on the boundary of the section,

$$\psi - \tfrac{1}{2}\tau y^2 = \tfrac{1}{2}\tau x^2$$

Therefore

$$\int_{y_1}^{y_2}\left(\frac{\partial\psi}{\partial y} - \tau y\right) dy = \tfrac{1}{2}\tau x^2 - \tfrac{1}{2}\tau x^2$$
$$= 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.26)$$

x being the same at H_1 and H_2.

The x-component force on the strip H_1H_2 of width δx is

$$\delta x \times \int_{y_1}^{y_2}\left(\frac{\partial\psi}{\partial y} - \tau y\right) dy = 0 \text{ by } (7.26)$$

Therefore the x-component force on the whole area is zero, since it is zero on every strip.

The total y-component force is

$$\int S_1 dA = \iint n\left(-\frac{\partial\psi}{\partial x} + \tau x\right) dx\, dy \quad . \quad . \quad . \quad . \quad (7.27)$$

Integrating this time from H_3 to H_4 where x has the values x_3 and x_4, and ψ the values ψ_3 and ψ_4, we get

$$\int_{x_3}^{x_4}\left(-\frac{\partial\psi}{\partial x} + \tau x\right) dx = (-\psi_4 + \tfrac{1}{2}\tau x_4{}^2) - (-\psi_3 + \tfrac{1}{2}\tau x_3{}^2)$$
$$= (-\tfrac{1}{2}\tau y^2) - (-\tfrac{1}{2}\tau y^2)$$
$$= 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.28)$$

since y is the same at both ends.

It follows again, just as in the case of the x-component force, that the total y-component force is zero. Then the action on the section must be a couple.

107. Moment of the couple on the section.

Taking moments about O (fig. 59) we find that the moment of the shear forces on the element of area $dxdy$ is

$$(xS_1 - yS_2)\,dxdy$$

Therefore the total moment on the whole area is

$$\iint (xS_1 - yS_2)\, dx\, dy$$
$$= n\iint\left\{\tau(x^2 + y^2) - \left(x\frac{\partial\psi}{\partial x} + y\frac{\partial\psi}{\partial y}\right)\right\} dx\, dy$$
$$= n\tau I - n\iint\left(x\frac{\partial\psi}{\partial x} + y\frac{\partial\psi}{\partial y}\right) dx\, dy \quad . \quad . \quad . \quad . \quad . \quad (7.29)$$

where I denotes the moment of inertia of the area of the section about the axis of z, and the integration extends over every element of area of the section.

The torsion problem for a section bounded by a single closed curve requires, therefore, a value of ψ, as a function of x and y only, to satisfy the equation

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.30)$$

at all points of the cross-section, and to satisfy the condition

$$\psi=\tfrac{1}{2}\tau(x^2+y^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.31)$$

at all points of the boundary of the section. Then the couple or torque is given by

$$Q=n\tau I-n\iint\left(x\frac{\partial\psi}{\partial x}+y\frac{\partial\psi}{\partial y}\right)dx\,dy \quad . \quad . \quad . \quad . \quad (7.32)$$

when each unit length of the rod is twisted through τ radians.

The results we have obtained are sufficient to determine the torque, but it is worth while to express the torque in another form. For this purpose let r, θ, be the polar coordinates of the element of area dA. That is,

$$\left.\begin{aligned} x&=r\cos\theta\\ y&=r\sin\theta\\ r^2&=x^2+y^2 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.33)$$

so that

Let us write ξ for $\psi-\frac{1}{2}\tau(x^2+y^2)$. Then the torque Q in terms of ξ is

$$\begin{aligned} Q&=n\tau I-n\iint\left\{x\left(\frac{\partial\xi}{\partial x}+\tau x\right)+y\left(\frac{\partial\xi}{\partial y}+\tau y\right)\right\}dx\,dy\\ &=n\tau I-n\iint\left(x\frac{\partial\xi}{\partial x}+y\frac{\partial\xi}{\partial y}\right)dx\,dy-n\tau\iint(x^2+y^2)\,dx\,dy\\ &=-n\iint\left(x\frac{\partial\xi}{\partial x}+y\frac{\partial\xi}{\partial y}\right)dx\,dy \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.34) \end{aligned}$$

since

$$\iint(x^2+y^2)\,dx\,dy=I$$

Now

$$\begin{aligned} d\xi&=\frac{\partial\xi}{\partial x}dx+\frac{\partial\xi}{\partial y}dy\\ &=\frac{\partial\xi}{\partial x}(\cos\theta dr-r\sin\theta d\theta)+\frac{\partial\xi}{\partial y}(\sin\theta dr+r\cos\theta d\theta) \end{aligned}$$

If we let r vary and keep θ constant we must put zero for $d\theta$ in the last equation. Then

$$\begin{aligned} \frac{\partial\xi}{\partial r}&=\frac{\partial\xi}{\partial x}\cos\theta+\frac{\partial\xi}{\partial y}\sin\theta\\ &=\frac{x}{r}\frac{\partial\xi}{\partial x}+\frac{y}{r}\frac{\partial\xi}{\partial y} \end{aligned}$$

whence

$$r\frac{\partial\xi}{\partial r}=x\frac{\partial\xi}{\partial x}+y\frac{\partial\xi}{\partial y}$$

In polar coordinates the element of area is the element bounded by the two radii at θ and $(\theta + d\theta)$ and the two circles with radii r and $(r + dr)$, and its magnitude is $r\,d\theta\,dr$. Then it follows that

$$\iint\left(x\frac{\partial\xi}{\partial x}+y\frac{\partial\xi}{\partial y}\right)dx\,dy=\iint r\,\frac{\partial\xi}{\partial r}r\,d\theta\,dr \quad . \quad . \quad . \quad (7\cdot35)$$

In the integral on the right the limits for r for the section shown in fig. 60 are 0 and r_1, and the limits for θ are 0 and 2π.

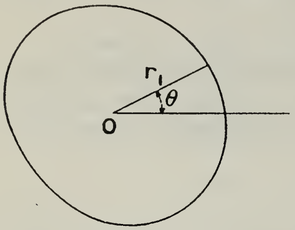

Fig. 60

Now integration by parts gives

$$\int_0^{r_1} r^2\frac{\partial\xi}{\partial r}dr=\Big[r^2\xi\Big]_0^{r_1}-2\int_0^{r_1} r\xi dr$$
$$=0-2\int_0^{r_1} r\xi dr$$

the integrated part being zero at the upper limit because ξ is zero at the boundary, and at the lower limit because r is zero at O.

Therefore

$$Q=-n\iint\left(x\frac{\partial\xi}{\partial x}+y\frac{\partial\xi}{\partial y}\right)dx\,dy$$
$$=2n\iint r\xi dr\,d\theta$$
$$=2n\iint\xi dx\,dy \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7\cdot36)$$

This form is alternative to the one in (7.32).

108. Tubular rod.

The preceding rules require modification when the boundary of the section consists of more than one closed curve, as, for example, when the rod is a tube. Let us consider only the case of a tube whose section has two boundaries, one closed curve inside another closed curve, as in fig. 61. Let $OK = r_0$, $OH = r_1$. The boundary condition for ψ is that $\psi - \frac{1}{2}(x^2 + y^2)$ is constant along any boundary curve. But it need not have, and indeed is very unlikely to have, the same constant value along two different curves bounding the section. The boundary conditions, in fact, for such a section as the one in fig. 61 can be put in the form

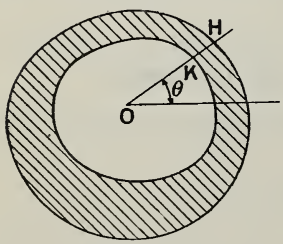

Fig. 61

$$\xi = 0 \text{ over the outer boundary}$$
$$\xi = C \text{ over the inner boundary.}$$

The value over the outer boundary need not be zero; it is zero by choice. But the difference between the values of ξ over the two boundaries is something we cannot choose; this difference depends only on the shape and size of the section and the twist τ.

In (7.35) the limits for r are now r_0 and r_1. Then

$$\int_{r_0}^{r_1} r^2\frac{\partial\xi}{\partial r}\,dr = \Big[r^2\xi\Big]_{r_0}^{r_1} - 2\int_{r_0}^{r_1} r\xi\,dr$$

$$= -r_0{}^2\mathrm{C} - 2\int_{r_0}^{r_1} r\xi\,dr \quad . \quad . \quad . \quad . \quad (7.37)$$

In this case

$$\mathrm{Q} = n\mathrm{C}\int_0^{2\pi} r_0{}^2 d\theta + 2n\int_0^{2\pi}\int_{r_0}^{r_1} r\xi\,d\theta\,dr$$

$$= 2n\mathrm{C}\mathrm{A}_0 + 2n\iint\xi\,dx\,dy \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.38)$$

where A_0 denotes the area enclosed by the inner boundary of the section, for $\frac{1}{2}r_0{}^2\,d\theta$ is the area of a triangular element with its vertex at O and its base on the inner boundary.

We have now got the complete theory of the torsion of thin rods whose sections have one or two bounding curves. It should be noticed that the interpretation of the displacements expressed by (7.1) and (7.2) is that the points originally on the axis of z remain on that axis, and that every cross-section of the rod except the fixed one is twisted about the z-axis. But the sections do not usually remain plane, for this would require that w should be constant, whereas the theory shows that ψ, and therefore also w, is not usually constant. In fact, the only section for which ψ can be constant is the circular section. This means that the different cross sections, which were originally plane, become slightly curved surfaces, all sections being distorted in the same way since w is not a function of z. It is clear that our theory does not apply to sections near a fixed end of a rod, nor to sections near where an external torque is applied. The constraints may be such at a fixed end that w is forced to be very small, as when the rod is a protruding piece from a much larger body of the same material. The preceding theory of torsion is valid, then, at sections of a twisted rod which are so far from the ends that the forces applied at the ends have no appreciable effect there.

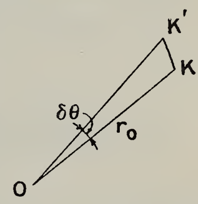

Fig. 62

109. Rod with circular section.

Suppose that the equation to the boundary of the section of the rod is

$$x^2 + y^2 = a^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.39)$$

Combining this with equation (7.31), which is also true at the boundary, it follows that

$$\psi = \tfrac{1}{2}\tau a^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.40)$$

over the boundary of the section. But a constant value of ψ also satisfies equation (7.30). Then

$$\psi = \tfrac{1}{2}\tau a^2$$

satisfies all the conditions of the problem, and the torque is, by (7.32),

$$\begin{aligned} Q &= n\tau I \\ &= \tfrac{1}{2} n\tau\pi a^4 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.41) \end{aligned}$$

Equations (7.12) show that $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ are each zero and therefore w is either zero or a constant. If w is zero for one section then it is zero for all, and it is certainly zero for a fixed section. If we assume w is a constant it does not materially alter the solution. It only gives a bodily displacement to the whole rod in the direction of the z-axis. In any case the plane sections of the rod remain plane after the strain. The circular section is the only section that is not distorted into a curved surface by a torque.

The preceding method can be used for a circular tube whose inner and outer radii are b and a. In this case, since ψ is constant over the whole area, it has the same value over each boundary curve.

The torque in this case is

$$Q = n\tau I = \tfrac{1}{2} n\tau\pi(a^4 - b^4).$$

110. Elliptic Section.

The differential equation (7.30) is satisfied by

$$\psi = A(x^2 - y^2) + C \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.42)$$

With this value of ψ the boundary condition (7.31) is

$$\tfrac{1}{2}\tau(x^2 + y^2) - A(x^2 - y^2) = C$$

or

$$x^2(\tfrac{1}{2}\tau - A) + y^2(\tfrac{1}{2}\tau + A) = C \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.43)$$

If $\frac{1}{2}\tau > A$ this is the equation to an ellipse and if the equation to the ellipse be written in the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.44)$$

then

$$a^2 = \frac{C}{\frac{1}{2}\tau - A}$$

and

$$b^2 = \frac{C}{\frac{1}{2}\tau + A};$$

whence
$$A = \tfrac{1}{2}\tau\frac{a^2-b^2}{a^2+b^2},$$
and
$$C = \tau\frac{a^2b^2}{a^2+b^2}.$$
Therefore
$$\psi = \frac{\tau}{2(a^2+b^2)}\left\{(x^2-y^2)(a^2-b^2)+2a^2b^2\right\} \quad . \quad . \quad . \quad (7.45)$$
and
$$\psi - \tfrac{1}{2}\tau(x^2+y^2) = \frac{\tau a^2b^2}{a^2+b^2}\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right). \quad . \quad . \quad . \quad (7.46)$$
For the ellipse
$$I = \tfrac{1}{4}\pi ab(a^2+b^2).$$
Also
$$x\frac{\partial\psi}{\partial x}+y\frac{\partial\psi}{\partial y} = \frac{a^2-b^2}{a^2+b^2}\tau(x^2-y^2) \quad . \quad . \quad . \quad . \quad . \quad (7.47)$$
Therefore (7.32) gives
$$\begin{aligned} Q &= \tfrac{1}{4}n\tau\pi ab(a^2+b^2) \\ &\quad - \frac{a^2-b^2}{a^2+b^2}n\tau\iint(x^2-y^2)dxdy \\ &= \tfrac{1}{4}n\tau\pi ab(a^2+b^2) - \frac{a^2-b^2}{a^2+b^2}(I_y - I_x) \quad . \quad . \quad . \quad (7.48) \end{aligned}$$
where I_x and I_y denote the moments of inertia of the ellipse about the axes of x and y respectively. The values of these are
$$\left.\begin{aligned} I_x &= \tfrac{1}{4}\pi ab^3 \\ I_y &= \tfrac{1}{4}\pi a^3b \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.49)$$
Finally then
$$\begin{aligned} Q &= \tfrac{1}{4}n\tau\pi ab\left\{a^2+b^2-\frac{(a^2-b^2)^2}{a^2+b^2}\right\} \\ &= \pi n\tau\frac{a^3b^3}{a^2+b^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.50) \end{aligned}$$
The components of the shear stress are

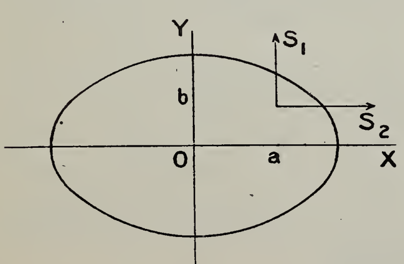

Fig. 63

$$\begin{aligned} S_1 &= n\left(\frac{\partial w}{\partial y}+\tau x\right) \\ &= n\left(-\frac{\partial\psi}{\partial x}+\tau x\right) \\ &= \frac{2b^2}{a^2+b^2}n\tau x \quad . \quad . \quad . \quad (7.51) \end{aligned}$$
$$\begin{aligned} S_2 &= n\left(\frac{\partial\psi}{\partial y}-\tau y\right) \\ &= -\frac{2a^2}{a^2+b^2}n\tau y \quad . \quad . \quad (7.52) \end{aligned}$$

The resultant shear stress is the vector sum of S_1 and S_2, and the maximum magnitude of this resultant is the value of S_2 at the end of the minor axis where $y = -b$. This maximum is

$$S' = \frac{2a^2b}{a^2 + b^2} n\tau \quad . \quad . \quad . \quad (7.53)$$

The magnitude of the resultant stress at any point (x, y) is

$$S = \frac{2n\tau}{a^2 + b^2} \sqrt{b^4x^2 + a^4y^2} \quad . \quad . \quad . \quad (7.54)$$

The most convenient way to express the resultant stress at a point of the boundary is by means of the eccentric angle defined by

$$x = a \cos \varphi, \quad y = b \sin \varphi \quad . \quad . \quad . \quad (7.55)$$

Then the stress at the boundary is

$$S = \frac{2n\tau ab}{a^2 + b^2} \sqrt{b^2 \cos^2 \varphi + a^2 \sin^2 \varphi}$$

$$= \frac{2n\tau ab}{a^2 + b^2} \sqrt{a^2 - (a^2 - b^2) \sin^2 \varphi} \quad . \quad . \quad . \quad (7.56)$$

the greatest value of which clearly occurs where $\sin \varphi = 0$, and its value is S' given in equation (7.53).

The value of w corresponding to ψ for the elliptic section is most easily obtained by observing that ψ is got by taking $f(x + iy)$ to be

$$f(x + iy) = iA(x + iy)^2 + iC \quad . \quad . \quad . \quad (7.57)$$

Then

$$w + i\psi = iA(x^2 - y^2) + iC - 2Axy \quad . \quad . \quad . \quad (7.58)$$

Therefore

$$w = -2Axy$$

$$= -\frac{a^2 - b^2}{a^2 + b^2} \tau xy \quad . \quad . \quad . \quad (7.59)$$

This shows that, in the quadrants where x and y are both positive or both negative, points of the cross-section are displaced in the direction of ZO, and in the other two quadrants in the direction of OZ, and that the contour lines of the distorted sections are rectangular hyperbolas with the axes of x and y as asymptotes.

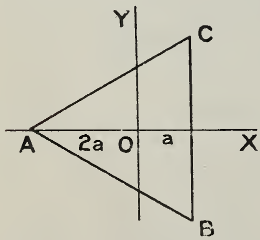

Fig. 64

The values of ψ and w found for the complete ellipse apply equally to a tube whose inner boundary is an ellipse similar to the outer boundary. This follows from the fact that $\psi - \frac{1}{2}\tau(x^2 + y^2)$ is constant over the inner, as well as over the outer, boundary.

111. Section in the form of an equilateral triangle.

Denoting the height of the triangle by

$3a$, the equation to its three sides in the position shown in fig. 64 is

$$(x - a)(x + 2a - y \tan 60^0)(x + 2a + y \tan 60^0) = 0$$

or

$$x^3 - 3xy^2 + 3a(x^2 + y^2) - 4a^3 = 0 \quad . \quad . \quad . \quad . \quad . \quad (7.60)$$

Now if we take $iA(x + iy)^3 + iC$ for $f(x + iy)$ in (7.20) we get

$$w + i\psi = iA(x + iy)^3 + iC$$
$$= A(y^3 - 3x^2y) + iA(x^3 - 3xy^2) + iC \quad (7.61)$$

Therefore

$$\psi = A(x^3 - 3xy^2) + C \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.62)$$

The boundary condition gives

$$A(x^3 - 3xy^2) - \tfrac{1}{2}\tau(x^2 + y^2) - C = 0 \quad . \quad . \quad . \quad . \quad (7.63)$$

This must be the equation to the boundary of the section of the rod to which ψ applies, and it will be equivalent to (7.60) provided

$$aA = -\tfrac{1}{6}\tau = \frac{C}{4a^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.64)$$

Then the value of ψ to suit this triangular section is

$$\psi = -\frac{1}{6}\frac{\tau}{a}(x^3 - 3xy^2) - \tfrac{2}{3}a^2\tau$$
$$= \frac{1}{6}\frac{\tau}{a}(3xy^2 - x^3 - 4a^3) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.65)$$

For this section

$$I = a^2 \times (\text{area of triangle})$$
$$= 3\sqrt{3}a^4 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.66)$$

Also

$$\iint\left(x\frac{\partial\psi}{\partial x} + y\frac{\partial\psi}{\partial y}\right)dxdy = \iint\frac{1}{2}\frac{\tau}{a}(3xy^2 - x^3)dxdy. \quad . \quad (7.67)$$

The values of y along the lines AB and AC are

$$y = \pm\frac{1}{\sqrt{3}}(x + 2a) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.68)$$

and these are the limits for y in the integral (7.67). Then between these limits

$$\int(3xy^2 - x^3)dy = \Big[xy^3 - x^3y\Big]_{-\frac{x+2a}{\sqrt{3}}}^{\frac{x+2a}{\sqrt{3}}}$$
$$= \frac{2x(x + 2a)}{\sqrt{3}}\left\{\tfrac{1}{3}(x + 2a)^2 - x^2\right\}$$
$$= \frac{4}{3\sqrt{3}}(4a^3x + 6a^2x^2 - x^4) \quad . \quad . \quad . \quad (7.69)$$

The limits for x are $-2a$ and $+a$. Consequently the double integral in (7.67) becomes

$$\frac{2}{3\sqrt{3}}\frac{\tau}{a}\int_{-2a}^{a}(4a^3x+6a^2x^2-x^4)dx$$
$$=\frac{2}{3\sqrt{3}}\frac{\tau}{a}\Big[2a^3x^2+2a^2x^3-\tfrac{1}{5}x^5\Big]_{-2a}^{a}$$
$$=\frac{6\sqrt{3}}{5}\tau a^4 \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.70)$$

Hence the torque on the rod is

$$Q=n\tau\times3\sqrt{3}a^4-\frac{6\sqrt{3}}{5}na^4\tau$$
$$=\frac{9\sqrt{3}}{5}na^4\tau$$
$$=\tfrac{3}{5}n\tau I \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.71)$$

The shear stresses are

$$S_1=n\left(\tau x-\frac{\partial\psi}{\partial x}\right)$$
$$=\tfrac{1}{2}n\tau\left(2x+\frac{x^2-y^2}{a}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.72)$$
$$S_2=n\left(\frac{\partial\psi}{\partial y}-\tau y\right)$$
$$=n\tau y\left(\frac{x}{a}-1\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.73)$$

Along the side BC, where $x=a$, these shear stresses become

$$\left.\begin{aligned}S_1&=\tfrac{1}{2}n\tau\frac{3a^2-y^2}{a},\\ S_2&=0\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.74)$$

Then along the side BC the stress S_1 is the resultant stress and this has its maximum value at the middle of the side where $y=0$, this maximum value being $\frac{3}{2}n\tau a$. It is easy to show that this is the maximum shear stress over the section.

It should be remarked that the shear stress is zero at the corners of the triangle, and yet at these points the displacement due to the twist has its greatest value.

112. The rectangular section.

Let the sides of the rectangle be $2a$ and $2b$, and let the origin be at the centre of the rectangle and the coordinate axes parallel to its sides.

Our equations are, as before,

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.75)$$

over the whole rectangle, and

$$\psi=\tfrac{1}{2}\tau(x^2+y^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.76)$$

over the boundary.

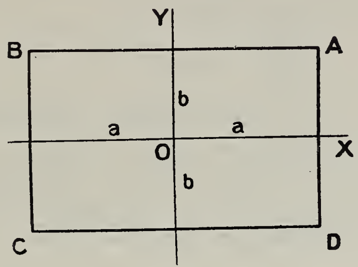

Fig. 65

It is convenient to have only one variable in the boundary condition. Then let

$$\varphi=\psi+\tfrac{1}{2}\tau(x^2-y^2)+C \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.77)$$

whence $$\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}=\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=0.$$

Also $$\varphi=\tau x^2+C$$

over the boundary, and if we take $C=-\tau a^2$ we get

$$\varphi=\tau(x^2-a^2). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.78)$$

as the condition over the boundary. Thus the boundary condition is

$\varphi=0$ over the sides AD, BC.

$\varphi=\tau(x^2-a^2)$ over the sides AB, CD.

The problem before us, then, is to find φ such that

$$\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.79)$$

at every point of the area of the rectangle, and

$$\varphi=0 \text{ where } x=\pm a, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.80)$$
$$\varphi=\tau(x^2-a^2) \text{ where } y=\pm b \quad . \quad . \quad . \quad . \quad (7.81)$$

However y is involved in φ it must be involved in such a way that the terms all vanish when $x=\pm a$. Terms such as $\eta\cos\dfrac{(2n+1)\pi x}{2a}$, where n is an integer and η a function of y only, satisfy this condition. Let us then try to satisfy equation (7.79) by such a function. If

$$\varphi=\eta\cos mx \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.82)$$

then

$$\frac{\partial^2\varphi}{\partial x^2} = -m^2\eta\cos mx$$

and

$$\frac{\partial^2\varphi}{\partial y^2} = \frac{d^2\eta}{dy^2}\cos mx.$$

Therefore equation (7.79) gives

$$\frac{d^2\eta}{dy^2} - m^2\eta = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.83)$$

whence

$$\eta = A\cosh my + B\sinh my \quad . \quad . \quad . \quad . \quad (7.84)$$

On account of the symmetry about the axis of x the constant B must be zero, for sinh my is an odd function of y, that is, changes sign with y, whereas cosh my, being an even function, has the same values for equal positive and negative values of y.

Then

$$\eta = A\cosh my \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.85)$$

and

$$\varphi = A\cosh my\cos mx. \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.86)$$

will suit our present problem. But as it stands this value of φ cannot be made to satisfy the boundary condition where $y = \pm b$. To satisfy this condition we need to sum such terms as those in (7.86), the values of m being chosen to satisfy the boundary condition over the other pair of sides.

For convenience let $x = \frac{2a\theta}{\pi}$. Then one solution satisfying the first boundary condition is

$$\varphi = A_n\cosh\frac{(2n+1)\pi y}{2a}\cos(2n+1)\theta; \quad . \quad . \quad . \quad (7.87)$$

and a more general solution is

$$\varphi = \sum_{n=0}^{n=\infty} A_n\cosh\frac{(2n+1)\pi y}{2a}\cos(2n+1)\theta \quad . \quad . \quad . \quad (7.88)$$

The boundary condition (7.81) in terms of θ is

$$\varphi = \frac{4\tau a^2}{\pi^2}\left(\theta^2 - \frac{\pi^2}{4}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.89)$$

If we put $y = \pm b$ in (7.88) we get

$$\varphi = \sum_{n=0}^{n=\infty} A_n\cosh\frac{(2n+1)\pi b}{2a}\cos(2n+1)\theta \quad . \quad . \quad . \quad (7.90)$$

If we now expand $\frac{4\tau a^2}{\pi^2}\left(\theta^2 - \frac{\pi^2}{4}\right)$ in a Fourier series we can make these last two equations agree. The Fourier expansion in cosines of the necessary type is

$$\frac{4\tau a^2}{\pi^2}\left(\theta^2-\frac{\pi^2}{4}\right)=-\frac{32\tau a^2}{\pi^3}\left\{\cos\theta-\frac{1}{3^3}\cos 3\theta+\frac{1}{5^3}\cos 5\theta\right\} \quad (7.91)$$

To make (7.90) agree with (7.91) we must have

$$\left.\begin{aligned} A_1\cosh\frac{\pi b}{2a} &= -\frac{32\tau a^2}{\pi^3} \\ A_3\cosh\frac{3\pi b}{2a} &= +\frac{1}{3^3}\frac{32\tau a^2}{\pi^3}\end{aligned}\right\} \quad \ldots\ldots (7.92)$$

and so on.

Then the general value of φ satisfying all the conditions of the problem is

$$\varphi=-\frac{32\tau a^2}{\pi^3}\left\{\frac{\cosh\frac{\pi y}{2a}}{\cosh\frac{\pi b}{2a}}\cos\frac{\pi x}{2a}-\frac{1}{3^3}\frac{\cosh\frac{3\pi y}{2a}}{\cosh\frac{3\pi b}{2a}}\cos\frac{3\pi x}{2a}+\ldots\right\}. \quad (7.93)$$

Then, by equation (7.77),

$$\psi=\varphi-\tfrac{1}{2}\tau(x^2-y^2)-C, \quad \ldots\ldots (7.94)$$

whence

$$x\frac{\partial\psi}{\partial x}+y\frac{\partial\psi}{\partial y}=x\frac{\partial\varphi}{\partial x}+y\frac{\partial\varphi}{\partial y}-\tau(x^2-y^2) \quad \ldots (7.95)$$

Therefore the torque is

$$\begin{aligned} Q &= n\tau I-n\iint\left(x\frac{\partial\varphi}{\partial x}+y\frac{\partial\varphi}{\partial y}\right)dxdy+n\tau\iint(x^2-y^2)dxdy \\ &= n\tau I+n\tau(I_y-I_x) \\ &\quad -n\iint\left(x\frac{\partial\varphi}{\partial x}+y\frac{\partial\varphi}{\partial y}\right)dxdy \quad \ldots\ldots (7.96)\end{aligned}$$

The limits for x and y in the double integral are $\pm a$ and $\pm b$. Now

$$\begin{aligned}\int_{-a}^{+a}x\frac{\partial\varphi}{\partial x}dx &= \Big[x\varphi\Big]_{-a}^{a}-\int_{-a}^{a}\varphi dx \\ &= 0-\int_{-a}^{a}\varphi dx \text{ by } (7.80) \quad \ldots (7.97)\end{aligned}$$

Therefore

$$\int_{-b}^{b}\int_{-a}^{a}x\frac{\partial\varphi}{\partial x}dxdy=-\int_{-b}^{b}\int_{-a}^{a}\varphi dxdy \quad \ldots (7.98)$$

Also

$$\int_{-b}^{b}y\frac{\partial\varphi}{\partial y}dy=\Big[y\varphi\Big]_{-b}^{b}-\int_{-b}^{b}\varphi dy$$

$$= b\tau(x^2 - a^2) - (-b)\tau(x^2 - a^2)$$
$$- \int_{-b}^{b} \varphi\, dy \quad \text{by (7.81)}$$
$$= 2b\tau(x^2 - a^2) - \int_{-b}^{b} \varphi\, dy \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.99)$$

Therefore

$$\int_{-a}^{a}\int_{-b}^{b} y\frac{\partial\varphi}{\partial y} dy\, dx = 2b\tau\int_{-a}^{a}(x^2 - a^2)\, dx - \int_{-a}^{a}\int_{-b}^{b} \varphi\, dx\, dy$$
$$= -\tfrac{8}{3}ba^3\tau - \int_{-a}^{a}\int_{-b}^{b} \varphi\, dx\, dy \quad . \quad . \quad . \quad . \quad (7.100)$$

Thus we get

$$Q = n\tau(I - I_x + I_y) + \tfrac{8}{3}ba^3 n\tau + 2n\iint \varphi\, dx\, dy$$
$$= \tfrac{16}{3}a^3 bn\tau + 2n\iint \varphi\, dx\, dy \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.101)$$

A typical term in φ is of the form A cosh my cos mx. Now

$$\int_{-b}^{b} \cosh my \cos mx\, dy = \frac{2}{m} \sinh mb \cos mx, \quad . \quad . \quad (7.102)$$

and therefore

$$\int_{-a}^{a}\int_{-b}^{b} \cosh my \cos mx\, dy\, dx = \frac{4}{m^2} \sinh mb \sin ma \quad . \quad (7.103)$$

The values of ma for the successive terms in φ are $\frac{\pi}{2}$, $\frac{3\pi}{2}$, $\frac{5\pi}{2}$, etc, and therefore the corresponding values of $\sin ma$ are $+1$, -1, $+1$, etc.

Finally then

$$Q = \tfrac{16}{3}a^3 bn\tau - \frac{4^5 a^4 n\tau}{\pi^5}\left\{\tanh r + \frac{1}{3^5}\tanh 3r + ..\right\} \quad . \quad (7.104)$$

where

$$r = \frac{\pi b}{2a} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.105)$$

In the preceding expression for Q it is more convenient to take $2a$ as the shorter side of the rectangle. The reason for this is that tanh r approaches unity very quickly as r increases. Thus

$$\tanh r = \frac{e^r - e^{-r}}{e^r + e^{-r}} = \frac{1 - e^{-2r}}{1 + e^{-2r}}$$
$$\tanh 3 = \frac{1 - e^{-6}}{1 + e^{-6}} = \frac{1 - \frac{1}{400}}{1 + \frac{1}{400}}$$
$$= 1 - \tfrac{1}{200} \text{ approximately}$$

$$\tanh \pi = 1 - \tfrac{1}{268}$$

$$\tanh \frac{3\pi}{2} = 1 - \tfrac{1}{6190}$$

These show how quickly $\tanh r$ approaches unity. It is clear that, if $b > 3a$, the series of hyperbolic tangents can differ only in the fourth significant figure from

$$1 + \frac{1}{3^5} + \frac{1}{5^5} + \frac{1}{7^5} + \frac{1}{9^5} + \dots \quad . \quad . \quad . \quad (7.106)$$

the sum of which is approximately 1·0045. If, then, $b > 3a$ we may write

$$Q = \frac{16}{3} a^3 b n \tau \left\{ 1 - \frac{192}{\pi^5} \frac{a}{b} \times 1{\cdot}0045 \right\}$$

$$= \frac{16}{3} a^3 b n \tau \left\{ 1 - 0{\cdot}630 \frac{a}{b} \right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (7{\cdot}107)$$

So far we have denoted the dimensions of the cross-section by $2a$ and $2b$. If we now use a_1 and b_1 for these dimensions we get, when $b_1 > 3a_1$,

$$Q = \frac{1}{3} a_1{}^3 b_1 n \tau \left\{ 1 - 0{\cdot}630 \frac{a_1}{b_1} \right\} \quad . \quad . \quad . \quad . \quad (7.108)$$

It is worth while to observe that the coefficient $\frac{1}{3} a_1{}^3 b_1$ is the moment of inertia of the cross-section about one of the longer sides, or four times the moment of inertia of the section about the axis through its centre parallel to the longer sides.

When b is not so great as $3a$ there is no great difficulty in summing the series of hyperbolic tangents. Even for the square section, which is the worst case to calculate, $\tanh 3r$ $\tanh 5r$, etc., are all practically unity. The only term that cannot be regarded as unity is therefore the first, namely $\tanh r$. If, then, b is not less than a,

$$\tanh r + \frac{1}{3^5} \tanh 3r + \frac{1}{5^5} \tanh 5r + \dots$$

$$= \tanh r + 0{\cdot}0045 \text{ approximately} \quad . \quad . \quad . \quad . \quad . \quad (7.109)$$

Hence

$$Q = \frac{16}{3} a^3 b n \tau \left\{ 1 - \frac{192}{\pi^5} \frac{a}{b} (\tanh r + 0{\cdot}0045) \right\} \quad . \quad . \quad (7.110)$$

For a square section, $b = a$, $r = \dfrac{\pi}{2}$, then

$$Q = \frac{16}{3} a^4 n \tau \left\{ 1 - \frac{192}{\pi^5} \times 0{\cdot}9217 \right\}$$

$$= \frac{16}{3} a^4 n \tau \times 0{\cdot}4218$$

$$= 0{\cdot}8436 I n \tau \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.111)$$

Equation (7.110) is the general result for the torque on a rectangular section provided b is not less than a, and equations (7.111), and (7.108) are particular cases of (7.110), the first being applicable when $b=a$, and the second when $b>3a$.

The shear stress S_1 is given by

$$S_1 = n\left(\tau x - \frac{\partial \psi}{\partial x}\right) = n\left(2\tau x - \frac{\partial \varphi}{\partial x}\right)$$

$$= 2n\tau x - \frac{16 n\tau a}{\pi^2}\left\{\frac{\cosh\frac{\pi y}{2a}}{\cosh\frac{\pi b}{2a}}\sin\frac{\pi x}{2a} - \frac{1}{3^2}\frac{\cosh\frac{3\pi y}{2a}}{\cosh\frac{3\pi b}{2a}}\sin\frac{3\pi x}{2a} + \ldots\right\} \quad (7.112)$$

Now

$$\frac{\partial S_1}{\partial y} = \frac{8n\tau}{\pi}\left\{\frac{\sinh\frac{\pi y}{2a}}{\cosh\frac{\pi b}{2a}}\sin\frac{\pi x}{2a} - \frac{1}{3}\frac{\sinh\frac{3\pi y}{2a}}{\cosh\frac{3\pi b}{2a}}\sin\frac{3\pi x}{2a} + \ldots\right\} \quad (7.113)$$

which is clearly zero when $y=0$. Then if we allow y to vary along the side $x=a$ it follows from the last equation that S_1 is either a maximum or a minimum where $y=0$, and since S_1 is zero at the corners we may infer that the value of S_1 at the middle of the side is the greatest value along that side. Also S_2 is zero everywhere along the side, so that S_1 is the resultant shear stress at any point of the side $x=a$. This resultant shear stress at the middle of the side is

$$S'_1 = 2n\tau a - \frac{16 n\tau a}{\pi^2}\left\{\text{sech}\frac{\pi b}{2a} + \frac{1}{3^2}\text{sech}\frac{3\pi b}{2a} + \ldots\right\}$$

$$= 2n\tau a - \frac{16 n\tau a}{\pi^2}\left\{\text{sech}\, r + \frac{1}{3^2}\text{sech}\, 3r + \ldots\right\} \quad . \quad . \quad (7.114)$$

The greater the ratio of b to a is the more quickly does the series in the brackets converge. Even for the case of the square, where $r=\frac{\pi}{2}$, we may neglect every term beyond the second in the brackets because $\frac{1}{5^2}\text{sech}\frac{5\pi}{2}$ is less than 0 00005. Moreover

$$\text{sech}\frac{3\pi}{2} = \frac{2}{e^{\frac{3\pi}{2}} + e^{\frac{-3\pi}{2}}} = 2e^{\frac{-3\pi}{2}} \text{ approximately.}$$

Therefore the shear stress at the middle of a side of a square section, with sides of length $2a$, is

$$S = 2n\tau a - \frac{32 n\tau a}{\pi^2}\left\{\frac{1}{e^{\frac{\pi}{2}} + e^{\frac{-\pi}{2}}} + \frac{1}{3^2}e^{\frac{-3\pi}{2}}\right\}$$

$$= 2n\tau a - \frac{32\,n\tau a}{\pi^2} \times 0\cdot 2003$$

$$= 1\cdot 351\,n\tau a \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7\cdot 115)$$

For a rectangular section, whose shorter sides have a length $2a$, the shear stress at the middle of the longer sides is greater than 1.351 $n\tau a$ because the series in the brackets in (7.114) decreases as b increases.

Again

$$S_2 = n\left\{\frac{\partial\psi}{\partial y} - \tau y\right\} = n\frac{\partial\varphi}{\partial y}$$

$$= -\frac{16\,n\tau a}{\pi^2}\left\{\frac{\sinh\frac{\pi y}{2a}}{\cosh\frac{\pi b}{2a}}\cos\frac{\pi x}{2a} - \frac{1}{3^2}\frac{\sinh\frac{3\pi y}{2a}}{\cosh\frac{3\pi b}{2a}}\cos\frac{3\pi x}{2a} + \ldots\right\} \quad (7\cdot 116)$$

At the point $x = 0$, $y = -b$, which is the middle of one of the shorter sides, this last stress becomes

$$S'_2 = +\frac{16\,n\tau a}{\pi^2}\left\{\tanh\frac{\pi b}{2a} - \frac{1}{3^2}\tanh\frac{3\pi b}{2a} + \ldots\right\} \quad . \quad . \quad (7\cdot 117)$$

Writing, as before, r for $\frac{\pi b}{2a}$, we get

$$S'_2 = \frac{16\,n\tau a}{\pi^2}\left\{\tanh r - \frac{1}{3^2}\tanh 3r + \ldots\right\} \quad . \quad . \quad . \quad (7\cdot 118)$$

When $b = a$ the stresses given by (7.114) and (7.118), being the stresses at the middle points of the sides of a square section, must be equal. Also each of these stresses increases as b increases, that is, as r increases while a remains constant. It follows then that, when $b > a$, that stress will be the greater which increases at the greater rate. Now from (7.114) and (7.118)

$$\frac{dS'_1}{dr} = \frac{16\,n\tau a}{\pi^2}\left\{\operatorname{sech} r \tanh r + \frac{1}{3}\operatorname{sech} 3r \tanh 3r + \ldots\right\}$$

$$\frac{dS'_2}{dr} = \frac{16\,n\tau a}{\pi^2}\left\{\operatorname{sech}^2 r - \frac{1}{3}\operatorname{sech}^2 3r + \ldots\right\}$$

Since $\tanh r > \operatorname{sech} r$ it follows that

$$\frac{dS'_1}{dr} > \frac{dS'_2}{dr}$$

and therefore $S'_1 > S'_2$ when $b > a$.

It may be inferred from the preceding proof, which cannot be regarded as very rigorous, that the greatest shear stress in a twisted rod of rectangular section occurs at the middle points of the longer sides of the cross sections, and its value is approximately

$$S = 2n\tau a - \frac{16n\tau a}{\pi^2}\operatorname{sech}\frac{\pi b}{2a} \quad . \quad . \quad . \quad . \quad . \quad (7.119)$$

$2a$ being the length of one of the shorter sides. If b is large compared with a then the absolute maximum shear stress is

$$S'_1 = 2n\tau a, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.120)$$

and the shear stress at the middle points of the shorter sides is

$$S'_2 = \frac{16n\tau a}{\pi^2}\left\{1 - \frac{1}{3^2} + \frac{1}{5^2}\cdots\right\}$$

$$= \frac{16n\tau a}{\pi^2} \times 0{\cdot}9160$$

$$= 1{\cdot}485n\tau a \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.121)$$

which is very little greater than at the middle points of the sides of a square section whose sides have a length $2a$.

113. The component shear stress in any direction.

It follows from the reasoning that led to equation (7.14) that the component shear stress at any point of a section, in the direction perpendicular to an element ds of a curve in the section, is

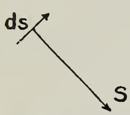

Fig. 66

$$S = n\left\{\frac{\partial\psi}{\partial s} - \tfrac{1}{2}\tau\frac{\partial}{\partial s}(x^2+y^2)\right\}, \quad . \quad . \quad . \quad . \quad (7.122)$$

the direction of the component stress being such that it makes a negative right angle with the vector ds if the outward normal to the section is parallel to the positive direction along the z-axis. If we put $ds = dx$ and $ds = dy$ in turn in (7.122) we get the expressions we already know for $-S_1$ and $+S_2$ respectively.

114. Shear stress at a sharp angle of a boundary.

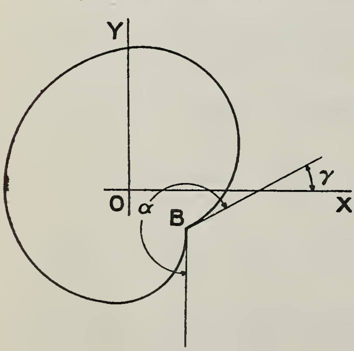

Fig. 67

If the boundary of the cross-section of a twisted prism has a reëntrant angle, such as B shown in fig. 67, the theory makes the shear stress infinite at such a point.

To deal with this problem it is convenient to shift the origin to the point B. It has to be remembered that the origin in all the preceding part of this chapter has been on the untwisted axis of the rod. Let us now write x', y', for the coordinates of an element of area referred to axes through B as origin, and let the coordinates of B itself be (x_1, y_1) referred to the old origin. Then

$$x = x_1 + x'$$
$$y = y_1 + y'$$

It follows then that

$$dx = dx'$$
$$dy = dy'$$

and therefore equations (7.12) become

$$\left.\begin{aligned}\frac{\partial\psi}{\partial y'} &= \frac{\partial w}{\partial x'},\\ \frac{\partial\psi}{\partial x'} &= -\frac{\partial w}{\partial y'},\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.123)$$

the solution of which can be written thus

$$w + i\psi = f(x' + iy'), \quad . \quad . \quad . \quad . \quad . \quad (7.124)$$

an equation exactly similar to (7.20).

Now let z be written for $(x' + iy')$. Then

$$w + i\psi = f(z) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.125)$$

If we restrict ourselves to points in the neighbourhood of B, where both x' and y' are small, and therefore also z small, we can expand $f(z)$ in ascending powers of z, and while z is very small only the lowest powers of z need be considered, since the higher powers will be negligible in comparison with the lower powers. If B were any point inside the section $f(z)$ could be expanded in integral powers of z, but, at an exceptional point such as the present position of B, the function does not consist of integral powers alone. Then we assume that

$$w + i\psi = w_0 + i\psi_0 + cz^m + c_1 z \quad . \quad . \quad . \quad . \quad (7.126)$$

where w_0 and ψ_0 are the values of w and ψ at the point B. The reason for introducing the term $c_1 z$ will be evident later.

Now let

$$\left.\begin{aligned}x' &= r\cos\theta\\ y' &= r\sin\theta\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.127)$$

so that r and θ are polar coordinates with B as pole. Then

$$\left.\begin{aligned}z &= r(\cos\theta + i\sin\theta) = re^{i\theta}\\ z^m &= r^m e^{im\theta}\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (7.128)$$

Therefore

$$w + i\psi = w_0 + i\psi_0 + cr^m e^{im\theta} + c_1 r e^{i\theta} \quad . \quad . \quad . \quad (7.129)$$

The constants c and c_1 are not necessarily real constants. They may be written in the forms $ke^{i\beta}$ and $k_1 e^{i\beta_1}$. Then

$$w + i\psi = w_0 + i\psi_0 + kr^m e^{i(m\theta+\beta)} + k_1 r e^{i(\theta+\beta_1)} \quad . \quad (7.130)$$

Differentiating both sides of this equation with respect to r and keeping θ constant we get

$$\frac{\partial w}{\partial r} + i\frac{\partial \psi}{\partial r} = mkr^{m-1}e^{i(m\theta+\beta)} + k_1 e^{i(\theta+\beta_1)}. \quad . \quad (7.131)$$

By equating the imaginary parts of both sides of this last equation we find that

$$\frac{\partial \psi}{\partial r} = mkr^{m-1}\sin(m\theta+\beta) + k_1\sin(\theta+\beta_1).$$

Now equation (7.122) shows that the shear stress perpendicular to dr is

$$S = n\left\{\frac{\partial \psi}{\partial r} - \tfrac{1}{2}\tau\frac{\partial}{\partial r}(x^2+y^2)\right\} \quad . \quad . \quad . \quad . \quad . \quad (7.132)$$

But

$$x^2+y^2 = (x_1 + r\cos\theta)^2 + (y_1 + r\sin\theta)^2$$
$$= x_1{}^2 + y_1{}^2 + 2r(x_1\cos\theta + y_1\sin\theta)$$

neglecting r^2. Hence the shear stress perpendicular to dr is

$$S = n\left\{\frac{\partial \psi}{\partial r} - \tau(x_1\cos\theta + y_1\sin\theta)\right\}$$
$$= n\left\{mkr^{m-1}\sin(m\theta+\beta) + k_1\sin(\theta+\beta_1)\right\}$$
$$- n\tau(x_1\cos\theta + y_1\sin\theta)$$
$$= nmkr^{m-1}\sin(m\theta+\beta) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.133)$$

provided we choose k_1 and β_1 so that

$$k_1\sin(\theta+\beta_1) = \tau(x_1\cos\theta + y_1\sin\theta)$$

Now let the two tangents at B make angles γ and $(\alpha+\gamma)$ with OX. Then S must be zero when θ is equal to either of these angles and when r is small but not zero. Thus we get

$$\sin(m\gamma+\beta) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.134)$$
$$\sin(m\gamma + m\alpha + \beta) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.135)$$

If we now assume that the shear stress does not vanish across any radius vector drawn from B inside the angle α then the solutions of the two preceding equations are

$$m\gamma + \beta = 0$$
$$m\gamma + m\alpha + \beta = \pi$$

whence

$$m = \frac{\pi}{\alpha} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.136)$$

For a reëntrant angle $\alpha > \pi$, and therefore $m < 1$. Consequently r^{m-1} is a negative power of r, which is very great when r is very small. It follows that the shear stress in the material near B is very great, and at B it is theoretically infinite. Of course infinite stresses are impossible; the hypotheses of the theory of elasticity fail when the stresses become very great. We can, however, fairly conclude that the elastic limit is exceeded at B.

The flaw in the preceding proof is the assumption that the shear stress does not vanish across any line drawn from B inside the angle α. That this is true however can be seen from physical considerations. A line across which the shear stress is zero is in the direction of the resultant shear stress. A series of curves can be drawn in the section such that the tangent to any curve at any point is in the direction of the resultant shear stress at that point. The boundary of the section itself is one of these curves, and it follows from considerations of continuity that the next curve of the system must be a closed curve running very near the boundary and approximately parallel to it. But this curve would have to meet the boundary at B if the shear stress vanished across any line through B except along the boundary. Then we infer that the shear stress does not vanish across any line through B inside the angle α, and therefore that the result in equation (7.136) is a correct deduction from the equations of elasticity.

115. The position of the axis of twist.

All the equations of this chapter as far as (7.38) remain true whatever be the point at which the axis of z, which is the axis about which the sections are twisted, meets a section of the rod. In all the particular cases that we have so far worked out, however, the axis of twist has passed through the centre of gravity of the sections. We shall now show that, for the same value of τ, the shear stresses, and therefore the torque, are unaltered if the axis of twist passes through any other point of a section.

Let us suppose that we have found ψ, S_1, S_2, to suit one position of the axis of twist, and let us now suppose that we require the shear stresses for a new axis of twist which meets the sections at (x_1, y_1) relative to the old axes. The new displacements at (x, y) are therefore

$$\left.\begin{aligned} u &= -\tau(y-y_1)z \\ v &= \tau(x-x_1)z \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.137)$$

Equations (7.3), (7.4), (7.5), remain true for there new displacements as for the old ones. We shall use ψ', S'_1, S'_2, for the function ψ and the shear stresses for the new strains. Then

$$\begin{aligned} S'_1 &= n\left\{\frac{\partial w'}{\partial y} + \tau(x-x_1)\right\} \\ &= n\left\{-\frac{\partial \psi'}{\partial x} + \tau(x-x_1)\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.138) \end{aligned}$$

$$S'_2 = n\left\{\frac{\partial \psi'}{\partial y} - \tau(y-y_1)\right\} \quad . \quad . \quad . \quad . \quad . \quad (7.139)$$

The function ψ' has to be determined to satisfy the equation

$$\frac{\partial^2 \psi'}{\partial x^2} + \frac{\partial^2 \psi'}{\partial y^2} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.140)$$

at every point of the section, and

$$\begin{aligned}\psi' &= \tfrac{1}{2}\tau\{(x-x_1)^2+(y-y_1)^2\}+C' \\ &= \tfrac{1}{2}\tau(x^2+y^2)-\tau(xx_1+yy_1)+K \quad . \quad . \quad . \quad (7.141)\end{aligned}$$

over the boundary.

Now the function ψ satisfies the equation

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.142)$$

at all points of a section, and

$$\psi = \tfrac{1}{2}\tau(x^2+y^2)+C \quad . \quad . \quad . \quad . \quad . \quad (7.143)$$

over the boundary. If we write

$$\psi' = \psi - \tau(xx_1+yy_1)+H \quad . \quad . \quad . \quad . \quad (7.144)$$

it will be seen that (7.140) follows from (7.142), and (7.141) follows from (7.144). Then equation (7.144) gives the new value of ψ. Also we now get

$$\begin{aligned}S'_1 &= n\left\{-\frac{\partial\psi}{\partial x}+\tau x_1+\tau(x-x_1)\right\} \\ &= n\left\{-\frac{\partial\psi}{\partial x}+\tau x\right\}=S_1;\end{aligned}$$

and likewise

$$S'_2 = n\left\{\frac{\partial\psi}{\partial y}-\tau y\right\}=S_2.$$

Thus the shear stresses are unaltered by the shift of the axis of twist. It follows, as before, that the stresses are equivalent to a pure couple of the same magnitude as with the old axis of twist.

If the axis of twist does not pass through the centres of gravity of the section of the rod then the line joining these centres is bent into a helix which has a curvature at every point. This curvature can only be maintained by a bending moment, the plane of which, at any point of the curve, is the osculating plane, that is, the plane of a small element of the curve in that neighbourhood. The central line could not, in fact, be bent into a helix without other strains than those we have assumed. We have dealt with only a small element of the rod, and we have really assumed that there was no tension parallel to the z-axis.

116. Position of greatest shear stress.

It has been remarked that, in the special torsion problems that we have worked out, the shear stress had its greatest magnitude at some point of the boundary. We can prove that this is true in every case.

Let S denote the resultant shear stress. Then

$$S^2 = S_1^2 + S_2^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.145)$$

The first conditions that S should be a maximum for variations in x and y are that

$$\frac{\partial S}{\partial x} = 0 \text{ and } \frac{\partial S}{\partial y} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.146)$$

A further condition that S should be a maximum for variations in x alone is that

$$\frac{\partial^2 S}{\partial x^2} = \text{a negative quantity} \quad . \quad . \quad . \quad . \quad (7.147)$$

And the condition when y alone is varied is

$$\frac{\partial^2 S}{\partial y^2} = \text{a negative quantity} \quad . \quad . \quad . \quad . \quad (7.148)$$

Now from (7.145), by differentiating twice with respect to x, we get

$$S\frac{\partial S}{\partial x} = S_1 \frac{\partial S_1}{\partial x} + S_2 \frac{\partial S_2}{\partial x} \quad . \quad . \quad . \quad . \quad . \quad (7.149)$$

$$S\frac{\partial^2 S}{\partial x^2} + \left(\frac{\partial S}{\partial x}\right)^2 = S_1\frac{\partial^2 S_1}{\partial x^2} + \left(\frac{\partial S_1}{\partial x}\right)^2 + S_2 \frac{\partial^2 S_2}{\partial x^2} + \left(\frac{\partial S_2}{\partial x}\right)^2 . \quad . \quad . \quad (7.150)$$

A similar equation can be obtained by differentiating all through equation (7.145) twice with respect to y. Therefore, by addition,

$$S\left\{\frac{\partial^2 S}{\partial x^2}+\frac{\partial^2 S}{\partial y^2}\right\}+\left(\frac{\partial S}{\partial x}\right)^2+\left(\frac{\partial S}{\partial y}\right)^2=S_1\left\{\frac{\partial^2 S_1}{\partial x^2}+\frac{\partial^2 S_1}{\partial y^2}\right\}+\left(\frac{\partial S_1}{\partial x}\right)^2+\left(\frac{\partial S_1}{\partial y}\right)^2 + S_2\left\{\frac{\partial^2 S_2}{\partial x^2}+\frac{\partial^2 S_2}{\partial y^2}\right\}+\left(\frac{\partial S_2}{\partial x}\right)^2+\left(\frac{\partial S_2}{\partial y}\right)^2 . \quad (7.151)$$

Now from the value of S_1 in terms of ψ we find that

$$\frac{\partial^2 S_1}{\partial x^2} + \frac{\partial^2 S_1}{\partial y^2} = -\frac{\partial}{\partial x}\left\{\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}\right\} = 0.$$

Another similar equation for S_2 can be written down. Then the expression on the right of equation (7.150) is certainly positive since it reduces to the sum of four squares. Now let us suppose that we have found a point where the first conditions for a maximum value of S, namely, those in (7.146), are satisfied. Then equation (7.150) tells us that, when S is positive,

$$\frac{\partial^2 S}{\partial x^2} + \frac{\partial^2 S}{\partial y^2} = \text{a positive quantity} \quad . \quad . \quad . \quad . \quad (7.152)$$

It follows therefore that it is impossible to satisfy both the conditions in (7.147) and (7.148) at the same time. Consequently S cannot have a maximum value for variations in x and a maximum value for variations in y at the same time; that is, S cannot have an absolute maximum value anywhere inside the section. Its greatest value must therefore be found on the boundary of the section.

We can always stipulate that S, given by equation (7.145), denotes the positive root of the right hand side. There is thus no necessity to consider negative values of S.

117. The lines of shear stress.

If we start at any point in the cross section of a twisted rod and draw a curve in this section such that the resultant shear stress at any point of the curve is in the direction of the tangent to the curve, such a curve may be called a *line of shear stress.* If we have solved the torsion problem for the particular rod we are dealing with, such a curve can be drawn from whatever point of the section we start. Any number of such curves can therefore be drawn, like the contour lines on a map. The component shear stress at any point, in the direction perpendicular to the line of shear stress through that point, is zero. If, therefore, the element of length ds in equation (7.122) be drawn along a line of shear stress it follows that the expression on the right of that equation must be zero. Let us put again

$$\xi = \psi - \tfrac{1}{2}\tau(x^2 + y^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.153)$$

Then equation (7.122) becomes

$$S = n\frac{\partial \xi}{\partial s}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.154)$$

and if ds is drawn along a line of shear stress then

$$\frac{\partial \xi}{\partial s} = 0.$$

Integrating this equation along the line of stress we find that

$$\xi = \text{constant} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.155)$$

along that line.

This must be the equation to a line of shear stress. One such line is the boundary of the section itself. If the rod is tubular, so that the section has an inner and outer boundary, ξ is constant over each boundary, but the constant is different for the two curves.

A line of shear stress cannot meet the boundary and it cannot end at any point in the cross section, for there is always a direction of resultant stress. Each line of stress must therefore be a closed curve. Moreover, two shear lines cannot intersect, nor can two branches of the same line intersect, for this would give two different directions for the resultant stress at the same point. The one exception to the last statement occurs at a point where the shear stress vanishes. At such a point two branches of the same shear line may touch, or a shear line may reduce to a closed curve of infinitesimal dimensions. In general, however, the shear lines form a system of non-intersecting closed curves, starting from the outer boundary as one of the lines, and ending with the inner boundary for a tubular rod, or ending with one or more infinitesimal closed curves at some point inside the section in case the boundary consists of only one closed curve. Even a tubular section may contain one or more sets of closed curves, the limit in each case being an infinitesimal closed curve. For the circular section the lines of stress are concentric circles; for the elliptic section they are a set of similar concentric ellipses; for the rectangular section they are a set of curves which may be described as rectangles with rounded corners, with the boundary itself at one extreme, and an infinitesimal ellipse at the other extreme.

118. Torque on an area bounded by two closed shear lines.

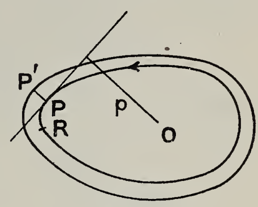

Fig. 68

In fig. 68 the two closed curves are supposed to represent two shear lines very close together in a section of some particular twisted rod. Let $\delta\xi$ be the difference of ξ for these two curves, and let δp be the perpendicular distance PP′ between the curves at some point P. At different points of the curves δp may be different, but $\delta\xi$ is constant since ξ is constant along each curve. Then if S denotes the resultant shear stress at P, equation (7.154) gives

$$S = n\frac{\delta\xi}{\delta p} \text{ approximately; } \quad . \quad . \quad . \quad . \quad . \quad (7.156)$$

that is, at different points of the inner shear line

$$S \propto \frac{1}{\delta p} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.157)$$

Thus the closeness of the shear lines indicates the intensity of the shear stress—the closer the lines the greater the stress.

Again, let p denote the perpendicular distance from the axis of twist O on to the line of action of S at P, and let ds denote the length PR of an element of the shear line itself. Then the shear force on the area contained between the two shear lines and between the two normals at P and R is $S\delta p ds$ and the moment of this about O is dQ given by the equation

$$\begin{aligned} dQ &= pS\delta p ds \\ &= np\delta\xi ds \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.158) \end{aligned}$$

Now

$$\begin{aligned} \tfrac{1}{2}pds &= \text{area of triangle OPR} \\ &= dA \text{ say} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.159) \end{aligned}$$

Therefore

$$dQ = 2n\delta\xi dA \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.160)$$

Then the torque due to the stress on the whole of the area between the two shear curves is

$$\int dQ = 2n\delta\xi \int dA$$
$$= 2nA\delta\xi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.161)$$

where A is the total area enclosed by the inner shear curve.

Equations (7.161) and (7.38) would be identical if the double integral in the latter equation were dropped. The term containing the double integral represents $2nA'\xi'$, where A′ is the area of the section, and ξ' is a mean value of ξ over the area of section. For a narrow section A′ is small compared with A, and ξ' is less than C.

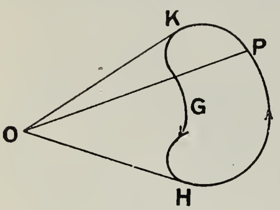

Fig. 69

It should be noticed that the result in (7.161) is independent of the position of the axis O; and it can be shown that it remains the same if O is outside the area, as in fig. 69. The area described along HPK which is the area OHPK, must be regarded as positive because the torque due to the stress along HPK is positive; and the area described along KGH is to be regarded as negative because the torque due to stress along KGH is negative. Thus the total torque for this case is

$$\int dQ = 2n\delta\xi\{\text{area OHPK} - \text{area OHGK}\}$$
$$= 2nA\delta\xi$$

where A denotes, as before, the area enclosed by the shear curve.

119. Torsion of thin tubes.

When a thin tube is subjected to torsion all the shear lines are closed curves completely encircling the inner boundary of the tube, and the variation of stress across the section is very small. We may therefore use the results in (7.156) and (7.161) for such a thin tube. The shear stress S in the former equation is really the mean stress across the thickness, not the stress at the inner or outer boundary. The two shear lines shown in fig. 68 may be regarded as the boundaries of the section of the tube.

Writing Q for the torque in a thin tube equation (7.161) gives

$$Q = 2nA\delta\xi, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.162)$$

where A had best be regarded as the mean of the areas enclosed by the inner and outer boundaries of the section, and $\delta\xi$ is the difference of ξ at the two boundaries. Moreover, if we write t for the thickness of the tube at any point, the mean shear stress across the tube at that point is, by equation (7.156),

$$S = n\frac{\delta\xi}{t},$$

whence $$n\delta\xi = tS.$$

Therefore we find that

$$Q = 2tAS, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.163)$$

and consequently

$$S = \frac{Q}{2tA} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.164)$$

We have thus got an expression for the stress in terms of the torque and the thickness t of the tube. This thickness may vary from point to point of the tube, but the stress varies with it in such a way that tS remains constant at all points of the tube. If t becomes very small at any part of the tube the stress S becomes very great. A closed tube under torsion will fail at the thinnest part of the tube.

120. Torque on a tube in terms of twist.

The result contained in (7.164) is all that is needed for many questions on the torsion of tubes, but there is nothing in the result which shows the relation between the shear stress and the twist τ. We shall now find this relation.

Let S denote the resultant shear stress at any point P of the shear line PRT (fig. 70), which is supposed to be the central shear line of the tube. Take axes OX, OY, through the axis of twist O so that OX is parallel to the shear stress S at P. Then, by equation (7.7),

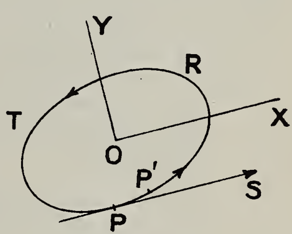

Fig. 70

$$S = S_2 = n\left(\frac{\partial w}{\partial x} - \tau y\right)$$

whence

$$n\frac{\partial w}{\partial x} = S + n\tau y.$$

If ds denotes an element of length PP′ of the shear curve at P we may write ds for dx in the last equation and get

$$n\frac{\partial w}{\partial s} = S + n\tau y \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.165)$$

Now what we know about w is that it must return to the same value if we travel once round the shear curve PRT. Then integrating both sides of (7.165) once round the curve we get

$$n\int\frac{\partial w}{\partial s}\,ds = \int S\,ds + n\tau\int y\,ds$$

that is,

$$0 = \frac{Q}{2A}\int\frac{ds}{t} + n\tau\int y\,ds \quad . \quad . \quad . \quad . \quad (7.166)$$

But

$$-\tfrac{1}{2}y\,ds = \text{area of triangle OPP}' = dA \text{ say,}$$

the negative sign being necessary because y is negative in the figure. Finally then, equation (7.166) gives

$$\frac{Q}{A}\int\frac{ds}{t} = 4n\tau A \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.167)$$

Since t is known at every point of the tube the integral involved can be worked out either analytically or graphically. If t is constant and if l is the total length of the circuit midway between the two boundaries of the tube, the result is

$$Q = 4n\tau\frac{t}{l}A^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.168)$$

The results in (7.167) and (7.168) are only approximate, and the accuracy increases as the thickness t diminishes. The result is very good so long as t is small compared with the radius of curvature of the inner boundary of the section of the tube. To be sure of getting good results a curve should be drawn midway between the two boundaries of the tube and then A is the area enclosed by this curve and ds is an element of length measured along the curve. Some examples will now be given.

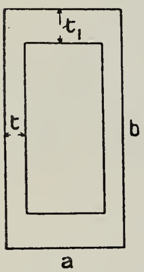
Fig. 71

121. Section in the form of a Hollow Rectangle.

Let a, b, denote the dimensions of the outer boundary; t, t_1, the thicknesses of the sides, as shown in fig. 71. Then, for the mean rectangle, the sides have lengths $(a-t)$ and $(b-t_1)$. Therefore

$$A = (a-t)(b-t_1)$$

Also

$$\int\frac{ds}{t} = 2\frac{a-t}{t_1} + 2\frac{b-t_1}{t}$$

$$= 2\frac{at + bt_1 - t^2 - t_1^2}{tt_1}$$

Therefore

$$Q = 2n\tau\frac{tt_1(a-t)^2(b-t_1)^2}{at + bt_1 - t^2 - t_1^2} \quad . \quad . \quad . \quad . \quad . \quad (7.169)$$

If we make $a=b$ and $t_1=t$, thus making the inner and outer boundaries concentric squares, we get

$$Q = n\tau t(a-t)^3 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.170)$$

If, further, we make $t=\frac{1}{2}a$, so that the central hole is infinitesimal, the result becomes

$$Q = \tfrac{1}{8}n\tau a^4 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.171)$$

Such a rod as the one to which the last equation applies can hardly be regarded as a tube, and yet the result contained in that equation differs by only about 12 % from the exact result for the square given by (7.111), where, it must be remembered, $2a$ denotes the length of the side.

Returning to equation (7.169) and making the thicknesses t and t_1 equal, and also making them negligible compared with a or b, we find

$$Q = 2n\tau \frac{a^2 b^2 t}{a + b} \qquad (7.172)$$

Also the shear stress is, by (7.164),

$$S = \frac{Q}{2tab} = n\tau \frac{ab}{a + b} \qquad (7.173)$$

This last result for the shear stress is correct where the shear lines are parallel curves, that is, along the sides of the rectangle not near a corner. It must not be forgotten that the shear stress is very great in the neighbourhood of a reëntrant angle, such as one of the corners of the inner boundary. To avoid these very great stresses the inner corners should be rounded, and not sharp angles.

122. Uniform circular tube.

The exact torque for a circular tube, of mean radius r and thickness t, is

$$\begin{aligned} Q &= n\tau I \\ &= \frac{\pi}{2} n\tau \left\{ (r + \tfrac{1}{2}t)^4 - (r - \tfrac{1}{2}t)^4 \right\} \\ &= 2\pi n\tau r t (r^2 + \tfrac{1}{4}t^2). \qquad (7.174) \end{aligned}$$

This result is obtained by the the same method as for the complete circle; ψ is constant for the tube as for the complete circle.

The preceding approximate method, applied to the circular tube, gives

$$\begin{aligned} Q &= 4n\tau \frac{t}{l} A^2 \\ &= 4n\tau \frac{t}{2\pi r} \pi^2 r^4 \\ &= 2\pi n\tau r^3 t. \qquad (7.175) \end{aligned}$$

the error in which is $\left(\frac{t}{2r}\right)^2$ of the whole. If t is as much as a quarter of the mean radius the percentage error is only 1·5.

The result given by (7.167) can be verified directly for the tube whose inner and outer boundaries are the similar ellipses

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

and

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = (1 + k)^2.$$

If the square of k be neglected, (7.167) gives

$$Q = \frac{4\pi kn\tau a^3 b^3}{a^2+b^2}.$$

Now when the section is a complete ellipse the shear lines are similar ellipses. Therefore the tube between similar ellipses is under the same stresses as when it forms part of a complete ellipse. Consequently, the torque in the tube is precisely the difference of the torques in the complete ellipses extending to the inner and outer boundaries. Thus the precise torque is, by (7.50),

$$Q = \pi n\tau\left\{\frac{a^3(1+k)^3 b^3(1+k)^3}{a^2(1+k)^2+b^2(1+k)^2} - \frac{a^3b^3}{a^2+b^2}\right\}$$

$$= \pi n\tau\frac{a^3b^3}{a^2+b^2}\left\{(1+k)^4 - 1\right\},$$

which $$= \frac{4\pi kn\tau a^3b^3}{a^2+b^2}$$

when powers of k beyond the first are neglected. Thus the approximate formula gives the same result as the exact process when k^2 is neglected.

123. Torsion of unclosed thin tubes.

The reasoning that has just been applied to closed thin tubes cannot be used for unclosed tubes, or for any rod whose cross-section consists of one closed curve. The behaviour under torsion of a closed circular tube and the same tube with a split parallel to the axis of the tube is vastly different. The essential difference is that, in the case of a closed tube, the shear lines are curves enclosing the inner boundary, whereas in the case of an unclosed tube the shear lines, while they are still closed curves, must turn back before they arrive at the split, and consequently there are shear lines running very close together in opposite directions, as, for example, in the case of a very thin rectangular section.

Let fig. 72 represent the boundary of what we shall call an open tube even when the ends do not come near together.

We shall assume that the thickness is very small compared with the radius of curvature of either the inner or outer part of the boundary, and also that the thickness either does not vary at all, or varies in such a way that the normal to the outer part of the boundary at any point is very nearly normal to the inner part also, except near the ends of the strip. Then any small portion between two normals such as PP′ and QQ′ is stressed in much the same way as a piece of a long rectangle. We shall be obliged to use the results for the long

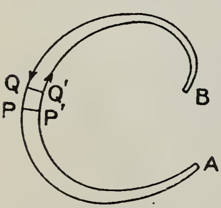

Fig. 72

rectangle to guide us in getting an approximate solution in the present case. We shall first show what the state of stress must be at points not near the ends.

Consider a small piece of the section PP′Q′Q and let us suppose that its sides are straight and parallel. Let the y-axis be taken along the middle of the strip. Now the shear lines must be parallel to the y-axis. Therefore

$$S_2 = n\left(\frac{\partial\psi}{\partial y} - \tau y\right) = 0.$$

Integrating this and dividing by n we get

$$\psi - \tfrac{1}{2}\tau y^2 = f(x)$$

$f(x)$ being any function of x. In consequence of the equation

$$\frac{\partial^2\psi}{x^2} + \frac{\partial^2\psi}{\partial y^2} = 0$$

$f(x)$ must satisfy the equation

$$\frac{d^2 f(x)}{dx^2} + \tau = 0,$$

whence we get

$$f(x) = -\tfrac{1}{2}\tau x^2 + Ax + B,$$

and therefore

$$\psi = \tfrac{1}{2}\tau(y^2 - x^2) + Ax + B.$$

The boundary condition that ψ has the same constant value when $x = \pm\frac{1}{2}t$, where t is the width of the strip, makes $A = 0$. Thus

$$\psi = \tfrac{1}{2}\tau(y^2 - x^2) + B$$

Thus the resultant shear stress is

$$\begin{aligned} S &= n\left(\tau x - \frac{\partial\psi}{\partial x}\right) \\ &= 2n\tau x \quad \ldots\ldots\ldots\ldots \quad (7.176) \end{aligned}$$

If we imagine the section to contain two rectangular pieces of different widths the shear stress is the same at the same distance from

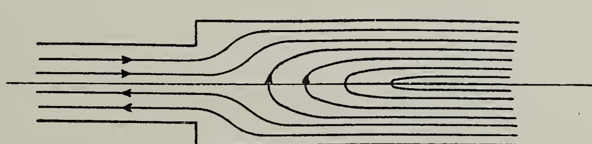

Fig. 73

the central line in the two parts. The wider part must therefore contain some extra shear lines not contained in the narrower part. Fig. 73 indicates the behaviour of the shear lines. The boundary itself must be a continuous shear line, and the extra lines in the wider part are found inside the lines coming from the narrower part. This generation of new lines must occur wherever the section broadens out, and some of these lines must close up when the section narrows again.

It would need a lot of tedious and difficult analysis to find the exact action at a point where the section suddenly broadens. We shall therefore assume that the change of width is so gradual that we may regard the shear lines at every point as parallel to the middle line of the section, except just where a line turns back. We shall assume, in consequence, that the shear stress is given by (7.176), where x means the distance from the middle line of the section.

Let t denote the width of the section at any point. Now applying (7.161) to a shear line passing at distance x from any point of the central line, the area A is

$$A=\int 2x ds \quad \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.177)$$

ds being an element of the middle line of the section. In this integral x is a function of s if the width of the section varies, for the breadth of the shear curve increases with the breadth of the section. Thus the total torque on the whole section is

$$\begin{aligned} Q &= \int 2n A d\xi \\ &= \iint 4x ds n d\xi \\ &= \iint 4x ds S dx \\ &= 8n\tau \iint x^2 ds dx \\ &= 8n\tau \int \left\{ \int_0^{\frac{1}{2}t} x^2 dx \right\} ds \\ &= \tfrac{1}{3} n\tau \int t^3 ds \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.178) \end{aligned}$$

The reason why the limits for x are 0 to $\frac{1}{2}t$ and not $-\frac{1}{2}t$ to $+\frac{1}{2}t$ is because we are not integrating across the section, but from the centre to the outside of the shear curves. The limits for s have to be taken from one end to the other of the central line. If the central line of the section is straight the result in (7.178) can be written

$$Q=4n\tau I_y \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.179)$$

the central axis being the y-axis. It should be observed that, according to (7.178), the curvature of the central line does not alter the torque.

The result given by (7.179) is correct both for an infinitely long rectangle and for an infinitely long ellipse, which are the only long sections for which we have worked out the torque accurately. The result expressed by (7.178) asserts that the torque in a body whose section is a long rectangle or a long ellipse remains the same for the same twist τ when the body is bent into tubular shape.

The approximate result given in (7.178) is usually a little above the mark for it really involves the assumption that the area over which the shear lines are not parallel to the sides of the strip is infinitesimal compared with the rest of the area. Its accuracy is, however, very high for long thin sections with nearly parallel sides. If the thickness varies rapidly at many points, so that the shear lines are not parallel curves over most of the section, the accuracy is low. As an extreme case, suppose the section were made up of a number of small circles strung together, with an infinitely thin strip connecting consecutive circles, then each circle would act independently of the rest and equation (7.179) gives, for one circle of radius r,

$$Q = \pi n \tau r^4$$

whereas the correct result is

$$Q = \tfrac{1}{2}\pi n \tau r^4$$

This, however, is a very bad case, and one to which the formula is not intended to apply; but it gives some idea of the sort of error to be expected.

124. Thin circular tube split longitudinally.

The torque in a circular tube of mean radius r and uniform thickness t (the same tube as equation (7.174) applies to, with the difference, however, that there is a split parallel to the axis) is

$$Q = \tfrac{1}{3} n\tau t^3 \times 2\pi r$$
$$= \tfrac{2}{3}\pi n \tau t^3 r \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.180)$$

The ratio of the torque in the split tube to that in the complete tube is $\frac{1}{3}\frac{t^2}{r^2}$ for the same τ. The maximum shear stresses in the two cases are, $n\tau t$ for the split tube, and $n\tau(r+\frac{1}{2}t)$ for the complete tube. If the twist is adjusted in the two cases so as to make the maximum stresses equal the ratio of the torques, is

$$\frac{1}{3}\frac{t(r+\frac{1}{2}t)}{r^2} = \frac{1}{3}\frac{t}{r} \text{ nearly} \quad . \quad . \quad . \quad . \quad (7.181)$$

The split tube is therefore much weaker than the complete tube under torsion, and very much less rigid.

125. The state of stress in a split circular tube.

We can find the state of stress in a split circular tube of uniform thickness at all points except near the ends. The result we get is just as accurate as the result for the stress in a long thin rectangle at points not near the ends. This investigation of a particular case will give confidence in the formula obtained for open tubes.

Let the inner and outer radii of the boundary be r_0 and r_1, and let the polar coordinates of any element of area of the section be (r, θ) with the common centre of the boundary curves as pole.

Now it is quite clear that the shear lines must be almost precisely circles concentric with the boundary at points not near the open ends of the tube. Then at all such points ξ must be constant when r is constant, and therefore ψ must also be constant when r is constant. Therefore ψ must be a function of r only. Now

$$w + i\psi = f(x + iy) = f(re^{i\theta})$$

The only function of $re^{i\theta}$ that contains a term not involving θ is $A \log (re^{i\theta})$, and since we want ψ to involve r only we must take $iA \log (re^{i\theta})$. A constant may also be considered as a function of $re^{i\theta}$. Therefore the most general solution giving ψ as a function fo r only is

$$\begin{aligned} w + i\psi &= iA\log_e(re^{i\theta}) + B + iC \\ &= iA\log_e r - A\theta + B + iC, \end{aligned}$$

whence

$$\psi = A\log_e r + C$$

and

$$\begin{aligned} \xi &= \psi - \tfrac{1}{2}\tau r^2 \\ &= A\log_e r - \tfrac{1}{2}\tau r^2 + C. \end{aligned}$$

The boundary condition is that

$$\xi = 0 \text{ when } r = r_0 \text{ and } r = r_1.$$

Therefore

$$\begin{aligned} 0 &= A\log_e r_0 - \tfrac{1}{2}\tau r_0{}^2 + C, \\ 0 &= A\log_e r_1 - \tfrac{1}{2}\tau r_1{}^2 + C; \end{aligned}$$

from which

$$A = \tfrac{1}{2}\tau \frac{r_1{}^2 - r_0{}^2}{\log r_1 - \log r_0}$$

and

$$C = \tfrac{1}{2}\tau \frac{r_1{}^2 \log r_0 - r_0{}^2 \log r_1}{\log r_1 - \log r_0}.$$

The resultant shear stress is

$$\begin{aligned} S &= -n\frac{d\xi}{dr} \\ &= n\left(\tau r - \frac{A}{r}\right). \end{aligned}$$

So far these are accurate results deduced from St. Venant's theory on the assumption that the shear lines are circles. This must be very nearly true for a thin split tube everywhere except over a very small range near the open ends.

To reduce the expression for the stress to the form we previously obtained let a denote the arithmetic mean of r_0 and r_1, and let t denote $(r_1 - r_0)$. Then

$$\begin{aligned} r_1{}^2 - r_0{}^2 &= (r_1 - r_0)(r_1 + r_0) \\ &= 2\,at \end{aligned}$$

and

$$\log_e \frac{r_1}{r_0} = \log_e \frac{1 + \dfrac{t}{2\,a}}{1 - \dfrac{t}{2\,a}}$$

$$=2\left\{\frac{t}{2a}+\frac{1}{3}\left(\frac{t}{2a}\right)^3+\ldots\right\}$$

If we neglect all powers of $\frac{t}{a}$ beyond the first we find that

$$A=\frac{\tau a t}{\frac{t}{a}}=\tau a^2.$$

Therefore

$$S=n\left\{\tau r-\frac{\tau a^2}{r}\right\}.$$

Now let $\quad r=a+x,$

x being thus measured radially from the middle line of the section. Then, neglecting $\left(\frac{x}{a}\right)^2$,

$$S=n\tau(a+x)-n\tau\frac{a^2}{a+x}$$

$$=n\tau(a+x)-n\tau a\left(1-\frac{x}{a}\right)$$

$$=2n\tau x,$$

just the same expression as for a strip with no curvature.

126. Torque in a rod with an I-section.

The shear lines in such a section as is shown in fig. 74 are nearly parallel to the neighbouring boundary except just in the region of the ends of the upright strip, and every shear line that runs along any one strip must return along the same strip. Then each portion of the section is in the same state of stress as a portion of an unclosed tube. If, then, the strips are fairly narrow, the formula for torque given in (7.178) can be applied to this case. Let the section be divided into three strips, as indicated by the dotted lines. Then the torque in each strip is approximately

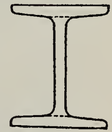

Fig. 74

$$Q=\tfrac{1}{3}n\tau\int t^3 ds,$$

t and ds having the same meanings as in (7.178). The total torque is the sum of the torques due to each strip. The middle line of the vertical strip is quite straight, and consequently the coefficient of $n\tau$ in the expression for the torque in that strip is four times the smallest moment of inertia of the strip.

127. Section shown in fig. 75.

For this section a number of shear lines follow the outer boundary into the projecting pieces such as AB. Others of these lines cling very closely to the inner boundary. The shear lines in each projecting strip go and return down the same strip, and therefore the torque in

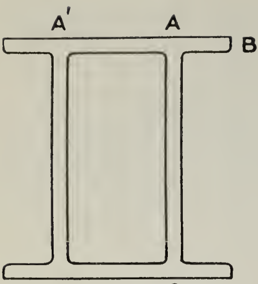

Fig. 75

each such strip is given by (7.178). The torque in the part forming a hollow rectangle can be calculated by the methods used for a closed tube, and is given by equation (7.167). As a particular case let us suppose that the thickness t is constant, and that $A'A = a$, $AB = \frac{1}{2}a$, $AC = (2a + t)$. Then the total torque in the four projecting pieces is

$$Q_1 = \tfrac{2}{3} n\tau t^3 a.$$

The torque in the hollow rectangle is, by (7.168),

$$Q_2 = 4n\tau \frac{t}{6a} \times (2a^2)^2$$
$$= \tfrac{8}{3} n\tau t a^3$$

Thus the total torque is

$$Q = Q_1 + Q_2$$
$$= Q_2\left(1 + \frac{t^2}{4a^2}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.182)$$

If t is as much as $\frac{1}{3}a$ then Q_2 is $36Q_1$, so that the torque due to the flanges is negligible in comparison with that due to the hollow rectangle. At the same time the greatest stress in a flange is

$$S_1 = 2n\tau \times \tfrac{1}{2}t = n\tau t$$

and the mean stress in the hollow rectangle, except near the corners, is, by (7.164) and (7.168),

$$S_2 = 2n\tau \frac{A}{l} = \tfrac{2}{3} n\tau a,$$

A being the area enclosed by the middle line of the hollow rectangle.

128. Distribution of shear lines.

When shear lines are being used for the purpose of reasoning about shear stresses it is useful to remember that there is the same difference in the value of ξ between any pair of consecutive lines. Consequently, since

$$S = n\frac{\delta\xi}{\delta p},$$

and since $\delta\xi$ is the same from any one line to the next line, it follows that the shear stress at any point of the section is inversely proportional to the normal distance δp from one shear line to the next in the neighbourhood of that point.

If a straight line AB of any length be drawn in a section of a twisted rod, and another parallel straight line A′B′ in another section at a distance δz from the first section, and such that AA′ and BB′ are parallel to the axis of twist, then the mean shear stress on the plane ABB′A′ is $\frac{n(\xi_2 - \xi_1)}{AB}$ where ξ_2 and ξ_1 are the values of ξ at A and B respectively. The proof follows.

It has been shown in Art 2 that a shear stress on one plane at any point requires an equal shear stress on a perpendicular plane. In fact, each shear line from the cross-section in which AB lies meets, on the line AB, another shear line running along the face ABB′A′. The shear lines in this latter plane are parallel to the axis of twist because the shear stress S_3 is zero. Then, if we take AB as x-axis, the total shear-force on the plane ABB′A′ is

$$\int S_1 dx\, \delta z = \delta z \int S_1 dx$$

$$= -\, n\delta z \int \frac{\partial \xi}{\partial x} dx$$

$$= n\delta z(\xi_2 - \xi_1) \quad . \quad . \quad . \quad . \quad . \quad (7.183)$$

Since the area on which this acts is AB $\times\, \delta z$ it follows that the mean stress is

$$\text{Mean S} = \frac{n(\xi_2 - \xi_1)}{AB} \cdot \quad . \quad . \quad . \quad . \quad . \quad (7.184)$$

When shear lines are drawn over any section of a twisted rod the number of lines crossing any line AB in the section is supposed to be proportional to the difference of ξ at A and B; that is, the number of lines is proportional to $(\xi_2 - \xi_1)$. If shear lines cross AB in opposite directions the number of lines which is proportional to $(\xi_2 - \xi_1)$ is understood to be the excess of those crossing in one direction over those crossing in the opposite direction. By this rule the total shear force on a strip such as ABB′A′ is proportional to the number of lines crossing AB.

129. Torsion of rod with section shown in fig. 76.

In such a section as this a certain number of lines enclose the area (1) only, others enclose the area (2) only, and still others enclose both areas. Let the numbers of lines enclosing these three circuits be denoted by N_1, N_2, N, respectively. We shall assume that each of the three vertical strips has the same thickness t, and each of the two horizontal strips the same thickness t', although the following method would be valid if these thicknesses were unequal.

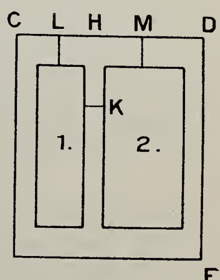

Fig. 76

Let $CD = a + t$, $DF = b + t$.

The numbers of shear lines running across the normals to the boundaries at L, M, and K, are $N + N_1$, $N + N_2$, and $N_1 - N_2$ respectively. Then the mean stresses across these three normals are

$$c\frac{N+N_1}{t'},\quad c\frac{N+N_2}{t'},\quad c\frac{N_1-N_2}{t}$$

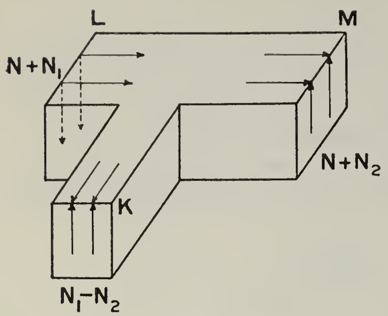

Fig. 77

where c is some constant.

Now let us regard the section in fig 76 as a horizontal plane, merely for the sake of naming directions. Then let us consider the equilibrium of the T piece whose section is LMK and which has unit length in the vertical direction. This T piece is shown in fig. 77 and arrows are drawn to show the directions of the *mean* shear stresses. Since the vertical forces must balance we get

$$c\frac{N+N_1}{t'}-c\frac{N+N_2}{t'}-c\frac{N_1-N_2}{t}=0,$$

that is,

$$c(N_1-N_2)\left(\frac{1}{t'}-\frac{1}{t}\right)=0$$

whence

$$N_1=N_2$$

Thus the number of lines crossing the normal at K (fig. 77) in one direction is equal to the number crossing in the other direction; also the total number of lines running round the four outer strips is everywhere the same, namely $(N+N_1)$. Then these four outer strips can be treated as a closed tube, and the middle strip as an independent rectangle, to which the method of the open tube can be applied. The fact that the rectangle is attached to the closed tube makes the torque rather greater than it would be if they were detached. Then the total torque is given approximately by

$$Q=Q_1+Q_2$$

where

$$Q_1\left\{\frac{2a}{t'}+\frac{2b}{t}\right\}=4n\tau a^2b^2$$

and

$$Q_2=\tfrac{1}{3}n\tau t^3(b-t')$$
$$=\tfrac{1}{3}n\tau t^3b \text{ nearly}$$

Therefore

$$Q=2n\tau\frac{a^2b^2tt'}{at+bt'}+\tfrac{1}{3}n\tau t^3b \quad . \quad . \quad . \quad . \quad . \quad (7.185)$$

The accuracy of this result will not be appreciably affected if Q_2 is neglected altogether, because we know, from the result for the uniform circular tube, that the error in Q_1 is of the same order as Q_2 when t and t' do not differ much.

130. St. Venant's approximate formula for the torque in rods with compact sections.

The torque in the rod with an elliptic section may be written in the form

$$Q = \frac{1}{4\pi^2}\frac{n\tau A^4}{I}$$

$$= 0{\cdot}0253\frac{n\tau A^4}{I} \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.186)$$

where A denotes the area of the section.

For the rod whose section is an equilateral triangle

$$Q = \frac{2}{90}\frac{n\tau A^4}{I}$$

$$= 0{\cdot}0222\frac{n\tau A^4}{I} \cdot \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.187)$$

For the square section

$$Q = 0{\cdot}0234\frac{n\tau A^4}{I} \cdot \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.188)$$

For a rectangle with sides a_1, b_1, when $b_1 > 3a_1$,

$$Q = \frac{1}{36}\frac{n\tau A^4}{I}\left(1 + \frac{a_1^2}{b_1^2}\right)\left(1 - 0{\cdot}630\frac{a_1}{b_1}\right) \quad . \quad . \quad . \quad (7.189)$$

gives a very good approximate value of the torque. If we put $b_1 = 3a_1$ in this we get

$$Q = 0{\cdot}0244\frac{n\tau A^4}{I}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.190)$$

and if we put $\frac{b_1}{a_1} = \infty$ we get

$$Q = 0{\cdot}0278\frac{n\tau A^4}{I} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.191)$$

All these results are written in the form

$$Q = k\frac{n\tau A^4}{I}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.192)$$

and it is remarkable what a small difference there is between the several values of k. Saint Venant found that, for sections that are fairly compact, that is, sections that have no projecting arms in any direction and no reëntrant angles, the coefficient k in (7.192) is remarkably near its value for an ellipse, and he gave as an approximate formula for all such sections

$$Q = \frac{1}{40}\frac{n\tau A^4}{I} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.193)$$

The formula is useless for open or closed thin tubes. Take, for example, the case of the closed circular tube with inner and outer radii r_0 and r_1. Here

$$\begin{aligned} Q &= In\tau \\ &= \tfrac{1}{2}\pi(r_1^4 - r_0^4)n\tau \end{aligned}$$

and therefore the coefficient k is

$$\begin{aligned} k &= \frac{\frac{1}{4}\pi^2(r_1^4 - r_0^4)^2}{\pi^4(r_1^2 - r_0^2)^4} \\ &= \frac{1}{4\pi^2}\left(\frac{r_1^2 + r_0^2}{r_1^2 - r_0^2}\right)^2 \end{aligned}$$

which, when $(r_1 - r_0)$ is small compared with r_0, differs greatly from $\frac{1}{40}$.

There are few sections in actual practice that will not come under either Saint Venant's approximate formula or under our rules for closed or open tubes.

CHAPTER VIII

THE ENERGY IN A STRAINED BODY

131. Strain-energy.

The forces that strain an elastic body do work on that body during the process of straining, since the body yields in the direction of the straining forces. We can find the work done on each element of the body by treating the element as a separate body under the action of forces at its surface. For an internal element the forces at the surface are the actions of the other parts of the body in contact with the surface of the element, that is, the stresses at the surface of the element. For an element at the surface of the body the straining forces are the actions of the contiguous parts of the body together with the action of the forces applied at the external boundary. Then the total work done on all the elements of the body is the work done in straining the body.

The total work obtained by dealing with each element as explained above is exactly the same as the work done by the external forces on the body, the external and internal forces on each element being assumed to be in equilibrium at every instant. For example, if a beam, clamped at one end and free at the other, is bent by the application of a force at the free end the total work done on the beam is the work done by the force at the end. If, during the process of bending, y is the deflexion at the end and Q is the force, we know that Q is proportional to y, and therefore the total work done by Q up to the time when it becomes Q_1 and produces a deflexion y_1 is $\frac{1}{2}Q_1y_1$, the factor $\frac{1}{2}$ being due to the fact that the mean force during the operation is $\frac{1}{2}Q_1$. It will be found, by the processes to be given later in this chapter, that the total work put into the elements of the body is the same as $\frac{1}{2}Q_1y_1$.

If the strained body is allowed to recover slowly its unstrained state it can, if it is perfectly elastic, react on the body or bodies maintaining the strain at any instant with exactly the same forces as when the strain was increasing; that is, the strained body can do the same amount of work on external bodies in recovering its natural state as external bodies have done on the strained body in producing the strain. Thus a perfectly elastic body is capable of giving back all the work that has been put into it, and for this reason the work done in straining such a body is regarded as energy stored in the body, and is called the *elastic energy* in the body, or sometimes the *strain-energy* in the body.

We shall begin by finding the energy in a number of simpler cases before finding the general expression for the energy in a strained body. These simpler cases contain the most useful examples, and at the same time serve the purpose of making clear the general result.

132. A uniform rod or string under a pair of opposing pulls at its ends.

Let us assume that one end remains fixed while the other end moves due to the extension of the rod by the pulls.

Let l denote the natural length of the rod and A the area of the cross-section; and let α denote the extensional strain due to a tensional stress P. The extension of the rod is $l\alpha$ and the work done by P at the free end while the extension increases by $\delta(l\alpha)$ is

$$\begin{aligned}\delta W &= P \times A \times \delta(l\alpha) \\ &= PAl\delta\alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8.1)\end{aligned}$$

The total work done by P while the strain increases from α_0 to α is

$$\begin{aligned}W &= \int_{\alpha_0}^{\alpha} PAld\alpha \\ &= \int_{\alpha_0}^{\alpha} E\alpha Ald\alpha \\ &= \tfrac{1}{2} EAl(\alpha^2 - \alpha_0{}^2) \\ &= \tfrac{1}{2} Al(E\alpha + E\alpha_0)(\alpha - \alpha_0) \\ &= \tfrac{1}{2} Al(P + P_0)(\alpha - \alpha_0) \quad . \quad . \quad . \quad . \quad . \quad (8.2)\end{aligned}$$

where P_0 and P are the tensions at the beginning and end of the operation.

By putting $\alpha_0 = 0$ and $P_0 = 0$ in (8.2) we get the work done in producing the whole strain of the rod. This work is

$$\begin{aligned}W &= \tfrac{1}{2} AlP\alpha \\ &= \tfrac{1}{2} Ts \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8.3)\end{aligned}$$

where T is the total tension across a section and s the extension of the rod.

133. Rod under variable tension.

The result in (8.3) applies to a rod under a constant stress P, and therefore also a constant strain α, all along its length. If the tension is variable, as, for example in the case of a rod hanging vertically under its own weight, we can still use the result in (8.3) for infinitesimal bits of the rod. The work dW that is done in stretching an element dx of the rod till its strain is α and its stress P is

$$dW = \tfrac{1}{2} AP\alpha dx, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8.4)$$

and therefore the whole work done in stretching a rod of length l is

$$W = \tfrac{1}{2}\int_0^l PA\alpha dx$$

$$= \tfrac{1}{2}\int_0^l \frac{P^2}{E} A dx \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8.5)$$

If we write T for the total tension across the section, namely PA, then (8.5) gives

$$W = \tfrac{1}{2}\int_0^l \frac{T^2}{EA} dx \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8.6)$$

As a particular example suppose a uniform rod hangs vertically under its own weight, the lower end being free. If w is the weight of unit volume of the material then the tension across the section at distance x from the lower end is wAx, the weight of the portion of rod below the section. Hence the tensional stress is wx, and therefore the work done in stretching the rod is

$$W = \tfrac{1}{2}\int_0^l \frac{w^2x^2}{E} A dx$$

$$= \tfrac{1}{6}\frac{w^2l^3}{E} A \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8.7)$$

134. The energy per unit volume in an element of rod.

If dV be written for the volume of the element of rod in equation (8.4) that equation becomes

$$dW = \tfrac{1}{2} P\alpha dV$$

and therefore

$$\frac{dW}{dV} = \tfrac{1}{2} P\alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8.8)$$

Either of the equations (8.7) or (8.8) may be interpreted to mean that the energy in each unit volume, at any point of a rod under a tension only, is $\tfrac{1}{2} P\alpha$, where P is the stress and α the strain at that point. The rod need not be straight. Moreover, the result is still true if the rod forms part of a larger body, provided that there are no stresses in the rod except the tension P.

135. The energy in a bent beam.

We shall find the energy that has been put into a piece of a bent beam between two cross sections at a distance δx apart, the natural state of the piece being straight.

Let dA denote an element of area of one of the sections which is at distance z from the neutral axis. The element of length δx and of area dA may be regarded as an element of rod to which (8.7) applies. Thus the energy in this piece is

$$dW = \tfrac{1}{2} P a \, dA \, \delta x$$

But $$a = \frac{z}{R}$$

and $$P = \frac{Ez}{R}$$

where R is the radius of curvature of the beam. Therefore

$$dW = \tfrac{1}{2} E \frac{z^2}{R^2} dA \, \delta x$$

Then integrating over the whole area, we find, for the energy in the piece of beam between the two cross sections

$$\delta W = \delta x \int \tfrac{1}{2} E \frac{z^2}{R^2} dA, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8.9)$$

δW being written for the integral of dW over the area because this integral is still an infinitesimal quantity of the order δx. Thus we find

$$\delta W = \delta x \times \frac{E}{2R^2} \int z^2 dA$$
$$= \delta x \times \frac{EI_y}{2R^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8.10)$$

I_y being the moment of inertia of the cross section about the neutral axis.

Now let us assume that the action across every section is a pure couple, and that the neutral axis is a principal axis of inertia of the section. With these assumptions the neutral axis passes through the centre of gravity of each section and the bending moment is

$$M = \frac{EI}{R}.$$

I being the moment of inertia about the principal axis.

It follows therefore that

$$\delta W = \frac{M}{2R} \delta x = \frac{M^2}{2EI} \delta x$$

Hence the total energy in the whole beam of length l is

$$W = \int_0^l \frac{M^2}{2EI} dx, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8.11)$$

x being measured from one end of the beam. It is worth while to notice that, if $\delta\theta$ denotes the angle subtended by δx at the centre of the circle of curvature, then

$$\frac{\delta x}{R} = \delta\theta$$

and therefore

$$\delta W = \tfrac{1}{2} M\delta\theta$$

To make our result more general let us now assume that the neutral axis is not a principal axis of inertia of the section. Let OY′, OZ′ be the principal axes of the section and let the neutral axis make an angle φ with OY′, as in fig. 78. Then the energy in δx is still

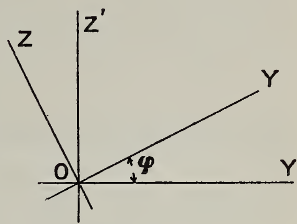

Fig. 78

$$\delta W = \delta x \times \frac{E}{2R^2}\int z^2 dA$$

But, denoting by z', y', the coordinates of dA relative to the principal axes, we get

$$z = z' \cos\varphi - y' \sin\varphi$$

Therefore

$$\delta W = \frac{E\delta x}{2R^2}\int(z'^2 \cos^2\varphi - 2y'z' \sin\varphi \cos\varphi + y'^2 \sin^2\varphi)\, dA$$

$$= \frac{E\delta x}{2R^2}\int(z'^2 \cos^2\varphi + y'^2 \sin^2\varphi)\, dA$$

$$= \frac{E\delta x}{2R^2}\left\{I_{y'} \cos^2\varphi + I_{z'} \sin^2\varphi\right\}$$

But, by (3.64) and (3.67),

$$M_{y'} = \frac{EI_{y'}}{R}\cos\varphi, \quad \ldots \ldots \ldots (8.12)$$

$$M_{z'} = \frac{EI_{z'}}{R}\sin\varphi \quad \ldots \ldots \ldots (8.13)$$

Therefore

$$\delta W = \delta x\left\{\frac{M_{y'}^2}{2\,EI_{y'}} + \frac{M_{z'}^2}{2\,EI_{z'}}\right\} \quad \ldots \ldots (8.14)$$

whence

$$W = \int\left\{\frac{M_{y'}^2}{2EI_{y'}} + \frac{M_{z'}^2}{2\,EI_{z'}}\right\}dx \quad \ldots \ldots (8.15)$$

The question of the energy in a naturally curved rod will be dealt with in chapter 11.

136. Illustrative examples.

In the case of a uniform beam, clamped at one end and free at the other, and carrying a load w per unit length, the bending moment at distance x from the free end is

$$M = \tfrac{1}{2} wx^2$$

Therefore the energy stored up in the beam is

$$W = \int_0^l \frac{w^2 x^4}{8EI} dx$$

$$= \frac{w^2 l^5}{40EI}$$

Again, for the beam investigated in Art 55,

$$M = \tfrac{1}{12} w(6x^2 - 6lx + l^2),$$

and therefore the energy in the beam is

$$W = \frac{w^2}{288EI}\int_0^l (6x^2 - 6lx + l^2)^2 dx$$

$$= \frac{w^2}{8EI}\int_0^l \left\{(x - \tfrac{1}{2}l)^2 - \tfrac{1}{12}l^2\right\}^2 dx$$

$$= \frac{w^2}{8EI}\int_{-\frac{1}{2}l}^{\frac{1}{2}l} (u^2 - \tfrac{1}{12}l^2)^2 du$$

where $u = (x - \tfrac{1}{2}l)$.

Therefore

$$W = \frac{w^2}{8EI}\int_{-\frac{1}{2}l}^{\frac{1}{2}l} (u^4 - \tfrac{1}{6}u^2 l^2 + \tfrac{1}{144}l^4) du$$

$$= \frac{2w^2 l^5}{8EI}\left\{\frac{1}{5}\cdot\frac{1}{2^5} - \frac{1}{18}\cdot\frac{1}{2^3} + \frac{1}{144}\cdot\frac{1}{2}\right\}$$

$$= \frac{w^2 l^5}{1440EI}$$

For a beam clamped at one end and carrying a load Q at the other end, which is free, the bending moment at distance x from the free end is

$$M = Qx$$

and therefore

$$W = \frac{1}{2}\int_0^l \frac{Q^2 x^2}{EI} dx$$

$$= \frac{1}{6}\frac{Q^2 l^3}{EI}.$$

If y_1 denotes the deflection at the free end it will be found that

$$y_1 = \frac{1}{3}\frac{Ql^3}{EI}$$

and therefore

$$W = \frac{1}{2}Qy_1$$

which verifies in this particular case what was stated in Art. 131.

137. Work done by the shear forces in a beam negligible

It will be observed that we have calculated the work done in bending a beam on the assumption that the stress across each cross section is purely normal, whereas we know that there is a shear stress as well when M is variable. Nevertheless, as we have previously pointed out (Art. 54), the shear stress is negligible compared with the tensional stresses in beams, and consequently the work done by the shear stresses is negligible in comparison with the work done by the tensional stresses which form the bending couple.

138. Rod under tension and bending moment.

It has been shown (Art. 74) that, when the neutral axis does not pass through the centre of gravity of a section, the total action across the section is equivalent to a tension acting at the centre of gravity, together with a couple whose moment is the actual moment of the stresses about an axis which passes through the centre of gravity of the section and is parallel to the neutral axis. With the same notation as in Art. 74, r being neglected in comparison with R, equation (8.10) remains true for this case.

Now by the theorem of parallel axes in moments of inertia

$$I_y = I + r^2A$$

I being the moment of inertia of the section about the axis through the centre of gravity parallel to the neutral axis. Therefore

$$\delta W = \delta x \frac{E}{2R^2}(I + r^2A).$$

Also, the moment of the stresses about the axis for which I is the moment of inertia is, by (6.13),

$$M = \frac{EI}{R}$$

Therefore

$$\delta W = \frac{1}{2}\delta x\left\{\frac{M^2}{EI} + EA\frac{r^2}{R^2}\right\}$$

$$= \frac{1}{2}\delta x\left\{\frac{M^2}{EI} + \frac{T^2}{EA}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8.16)$$

where T is the tension at the centre of gravity of the section, the expression for which is given in (6.12).

Thus the total energy in the rod in this case is

$$W = \frac{1}{2}\int_0^l \left\{\frac{M^2}{EI} + \frac{T^2}{EA}\right\} dx, \quad . \quad . \quad . \quad . \quad . \quad . \quad (8.17)$$

which is precisely the sum of the energies due to the couple M and the tension T separately, that is, the sum of the energies given by (8.6) and (8.11).

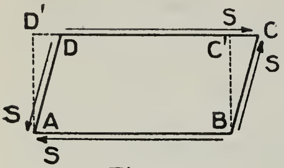

Fig. 79

139. Energy in a pure shear strain.

Suppose a naturally rectangular block, whose dimensions are p, q, r, is subjected to a pure shear stress over the faces perpendicular to the edges of length r. Thus in fig. 79

$$AB = p, \ AD' = q, \ D\hat{A}D = \theta.$$

The shear strain is θ corresponding to the shear stress S. Let us assume that the face represented by AB remains fixed while the face D′C′ is moved to DC. Then the work done by the forces on the four faces is just the work done by the force on DC because AB does not move, and the work done by the forces on AD and BC balance. While θ increases by $d\theta$ the length D′D increases by $qd\theta$ and the work done is

$$dW = (Spr) \times qd\theta$$

$$= Spqrd\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8.18)$$

But

$$S = n\theta$$

whence

$$dW = n\theta pqrd\theta,$$

and the total work done in producing the whole strain θ is therefore

$$W = \int_0^\theta npqr\theta d\theta$$

$$= \tfrac{1}{2} npqr\theta^2$$

$$= \tfrac{1}{2} Spqr\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8.19)$$

Thus the energy per unit volume is

$$\frac{W}{pqr} = \tfrac{1}{2} S\theta = \tfrac{1}{2}\frac{S^2}{n}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8.20)$$

140. Energy in a rod under torsion.

If a torque Q is applied in opposite directions at the ends of a rod of length l, and if the twist per unit length of the rod is τ, then the whole rod is twisted through an angle $l\tau$. The work done by the couple Q in increasing the angle $l\tau$, by $ld\tau$ while one end is kept fixed is

$$dW = Qld\tau \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8.21)$$

Now it can be seen from equations (7.30), (7.31), (7.32) that

$$Q = Hn\tau \qquad (8.22)$$

where H is a constant which depends only on the distribution of the area of the cross-section about the axis of twist. Therefore

$$dW = Hnl\tau d\tau,$$

and the whole energy in the rod is

$$\begin{aligned} W &= \int_0^{\tau} Hnl\tau d\tau \\ &= \tfrac{1}{2} Hnl\tau^2 \\ &= \tfrac{1}{2}\frac{Q^2}{Hn} l \\ &= \tfrac{1}{2} Ql\tau \\ &= \tfrac{1}{2} Q\varphi, \qquad (8.23) \end{aligned}$$

where φ is the whole angle through which one end of the rod is twisted relative to the other.

If we are dealing with a rod in thich the torque varies along the rod then the energy in an element of length δx in which the twist is τ is

$$\begin{aligned} \delta W &= \tfrac{1}{2} Q\tau\delta x \\ &= \tfrac{1}{2}\frac{Q^2}{Hn}\delta x, \end{aligned}$$

whence the total energy is

$$W = \tfrac{1}{2}\int \frac{Q^2}{Hn} dx \qquad (8.24)$$

141. The general expression for elastic energy.

Let us consider the work done by the stresses acting on the surface of an element of volume of dimensions δx, δy, δz, inside any strained body. Let the six stresses and corresponding strains be, as in chapter I,

$$P_1,\ P_2,\ P_3,\ S_1,\ S_2,\ S_3;$$
$$\alpha,\ \beta,\ \gamma,\ a,\ b,\ c.$$

It is easy to see that P_1 does no work except when α varies. Likewise the stress S_1, for instance, does no work on the faces on which it acts except when the strain a varies.

When α increases by $d\alpha$ the length δx increases by $d\alpha\delta x$, and therefore the work done by the stresses P_1 on the pair of faces on which they act is

$$(P_1\delta y\delta z) \times d\alpha\delta x = (P_1 d\alpha)\delta x\delta y\delta z.$$

Thus the work done per unit volume by P_1 is $P_1\, d\alpha$. Likewise, when a increases by da the work done by S_1 is, by (8.18),

$$(S_1 da)\delta x\delta y\delta z$$

It is clear then that the total work done on unit volume of the body when all the strains vary is

$$dW' = P_1 d\alpha + P_2 d\beta + P_3 d\gamma + S_1 da + S_2 db + S_3 dc \quad . \quad . \quad . \quad . \quad (8.25)$$

Now let us suppose that all the stresses started simultaneously from zero and increased up to their final values all the while maintaining the same ratios among themselves. Under these circumstances each stress is proportional to its own strain. Let us then put

$$P_1 = k_1\alpha, \; P_2 = k_2\beta, \; P_3 = k_3\gamma \quad . \quad . \quad . \quad . \quad . \quad (8.26)$$

Moreover, in all cases,

$$S_1 = na, \; S_2 = nb, \; S_3 = nc \quad . \quad . \quad . \quad . \quad . \quad . \quad (8.27)$$

Therefore the total energy in unit volume at the point where the stresses are P_1, P_2, etc., is

$$\begin{aligned} W' &= \int_0^\alpha k_1\alpha d\alpha + \ldots\ldots + \int_0^a na\,da + \ldots . \\ &= \tfrac{1}{2}k_1\alpha^2 + \ldots\ldots\ldots + \tfrac{1}{2}na^2 + \ldots . \\ &= \tfrac{1}{2}P_1\alpha + \tfrac{1}{2}P_2\beta + \tfrac{1}{2}P_3\gamma \\ &\quad + \tfrac{1}{2}S_1 a + \tfrac{1}{2}S_2 b + \tfrac{1}{2}S_3 c \quad . \quad . \quad . \quad . \quad . \quad (8.28) \end{aligned}$$

The energy in the volume $\delta x \delta y \delta z$ is $W'\delta x \delta y \delta z$, and therefore the energy in the whole body is

$$W = \iiint W' dx dy dz, \quad . \quad . \quad . \quad . \quad . \quad . \quad (8.29)$$

the triple integral extending throughout the volume of the body.

142. Energy in terms of strains, and in terms of stresses. By means of equations (2.22) and (2.17), that is, such equations as

$$P_1 = (m - n)\triangle + 2n\alpha,$$
$$S_1 = na,$$

the energy per unit volume may be expressed in terms of the strains without the stresses. The terms containing the normal stresses are

$$\begin{aligned} \tfrac{1}{2}\{P_1\alpha + P_2\beta + P_3\gamma\} &= \tfrac{1}{2}(m-n)(\alpha+\beta+\gamma)\triangle + n(\alpha^2+\beta^2+\gamma^2) \\ &= \tfrac{1}{2}(m-n)\triangle^2 + n(\alpha^2+\beta^2+\gamma^2) \end{aligned}$$

Hence

$$W' = \tfrac{1}{2}(m-n)\triangle^2 + \tfrac{1}{2}n(2\alpha^2 + 2\beta^2 + 2\gamma^2 + a^2 + b^2 + c^2) \quad (8.30)$$

Again, by means of such equations as (2.14) and (2.17), W′ can be expressed in terms of the stresses without the strains. The result is

$$W' = \frac{1}{2E}\left\{(1+\sigma)(P_1^2 + P_2^2 + P_3^2) - \sigma(P_1 + P_2 + P_3)^2\right\}$$
$$+ \frac{1}{2n}(S_1^2 + S_2^2 + S_3^2). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8.31)$$

If $P_2 = P_3 = 0$ then

$$W' = \frac{P_1^2}{2E} + \frac{1}{2n}(S_1^2 + S_2^2 + S_3^2) \quad . \quad . \quad . \quad . \quad (8.32)$$

143. Load suddenly applied.

When a load P is applied to any point of an elastic body the displacement u of that point in the direction of P is usually proportional to P when the stresses are in equilibrium with the force P. Then the work done by a variable force P while it increases slowly from zero up to its final value is, on the assumption that $P = ku$ while the work is being done,

$$V = \int_0^u P du = \int_0^u k u du = \tfrac{1}{2} k u^2$$
$$= \tfrac{1}{2} P u.$$

This, therefore, is the elastic energy put into the body by the variable force P.

Let us now suppose that a constant load R is applied at the same point of the body while the body is at rest and unstrained. When the body next comes to rest after the force R is applied the total work done on the body by the force R is in the form of elastic energy. But the work done by a constant force R in a displacement u is Ru. If P is the force straining the body the energy in the body is $\tfrac{1}{2}Pu$. Hence

$$\tfrac{1}{2} P u = R u,$$

whence

$$P = 2R$$

This is the maximum force applied to the body by the load R.

It might seem as if the force applied to the body is always R throughout the displacement. But this, it should be remarked, is not possible. The force R is really applied to the load, and the elastic body resists with a force P which must be zero when the displacement is zero, and increases as the displacement increases. The difference of R and P is generating kinetic energy in the load, which kinetic energy has to be annihilated before the load can come to rest. In order to annihilate this kinetic energy a resistance must be applied which is greater than R, and we have found above that the maximum value of this resistance is 2 R. The load R performs, in fact, one half of a simple harmonic oscillation in moving from rest to rest; the force in the direction of motion at the beginning of this half oscillation is R—P, which is R because P is zero; and the force contrary to the motion at the end of the half oscillation is P—R, which is again R since P is 2 R.

The above is what is understood in engineering by a load suddenly applied. It means that a load is applied to an unstrained body and allowed to produce its full effect. In this way oscillations arise which finally die out owing to frictional resistances. The greatest stress occurs at one end of the oscillation, and this stress is twice as great as it would be if the load were in equilibrium at the same point of the elastic body. Thus if a load R is put on an unstrained beam and left to itself the load oscillates, taking the beam with it, and the maximum stress is approximately twice as great as when the load has come to rest.

The preceding reasoning is all based on the assumption that the mass of the applied load is much greater than the mass of the part of the elastic body that is carried with the load. If this latter mass is comparable with the mass of the load the greatest stress may be very much less than twice the stress due to the load R in equilibrium.

144. The conditions of equilibrium as a consequence of minimum or maximum energy.

It is a general rule in statics that, when a body or system of bodies is in equilibrium, the positions of the parts of the system are such that the potential energy is a minimum or a maximum. This means that any *infinitesimal displacements* of the parts of the system from an equilibrium state, which are consistent with the constraints of the system, can make no change in the energy which is of the same order as the displacements. Below is given a proof of this theorem, which is clearly valid for an elastic body. Afterwards the method is applied in a particular case to show how the conditions of equilibrium can be deduced from the principle when the strain energy of the body is known. The method here used is the method of variations and comes within the scope of the calculus of variations. Let the body or system of bodies be regarded as made up of particles m_1, m_2, m_3, etc. The forces on m_1 are made up of the external forces (including what we have called body forces as well as the forces at the boundary of the body if m_1 is a particle at the boundary) and the forces applied to m_1 by each of the other particles. Thus the forces on m_1 are

X_1, Y_1, Z_1, the external forces;
X_{12}, Y_{12}, Z_{12}, due to m_2;
X_{13}, Y_{13}, Z_{13}, due to m_3;

and so on.

Let the component displacements, in the equilibrium position, of m_1, m_2, etc. be (u_1, v_1, w_1), (u_2, v_2, w_2), etc., each of these displacements being measured from any convenient reference point; in the case of an elastic body the most convenient reference point for any particle would be the unstrained position of the particle.

Let us next get an expression for the total work done on all the particles during infinitesimal displacements δu_1, δu_2, δu_3, etc. The work done on m_1 is clearly

$$\begin{aligned}\delta W_1 = &(X_1 + X_{12} + X_{13} + \ldots)\delta u_1 \\ &+ (Y_1 + Y_{12} + Y_{13} + \ldots)\delta v_1 \\ &+ (Z_1 + Z_{12} + Z_{13} + \ldots)\delta w_1 \quad . \quad . \quad . \quad . \quad (8.33)\end{aligned}$$

The increase in the potential energy, which is the negative of the total work on all particles, is

$$\delta V = -\{\delta W_1 + \delta W_2 + \ldots\} \quad . \quad . \quad . \quad . \quad . \quad (8.34)$$

Now the conditions that V is a minimum (or maximum) in the equilibrium position for all possible variations of δu_1, δu_2, etc. must be that the coefficients of every one of the small increments of the displacements must be zero except for those displacements which the constraints of the bodies make impossible.

Thus, one of these conditions is

$$(X_1 + X_{12} + X_{13} + \ldots)\delta u_1 = 0 \quad . \quad . \quad . \quad . \quad . \quad (8.35)$$

This is satisfied identically if the particle m_1 is so constrained that no motion parallel to the x-axis is possible. But if no constraints are applied to m_1 then three of the conditions for minimum V are

$$\left.\begin{aligned} X_1 + X_{12} + X_{13} + \ldots &= 0 \\ Y_1 + Y_{12} + Y_{13} + \ldots &= 0 \\ Z_1 + Z_{12} + Z_{13} + \ldots &= 0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (8.36)$$

These are the equations of equilibrium of the particle m_1.

In general, for the particle m_r, we get

$$\left.\begin{aligned} (X_r + X_{r1} + X_{r2} + \ldots)\delta u_r &= 0 \\ (Y_r + Y_{r1} + Y_{r2} + \ldots)\delta v_r &= 0 \\ (Z_r + Z_{r1} + Z_{r2} + \ldots)\delta w_r &= 0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (8.37)$$

and the same reasoning can be applied to these as we have already applied to the corresponding terms for m_1.

If a particular particle, let us say m_1, is forced by the constraints to move on a given surface, this gives one relation between δu_1, δv_1, δw_1. For, if l, m, n, are the direction-cosines of the normal to the surface at the position of m_1, the particle is free to move in a small element of the plane

$$lx + my + nz = k, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8.38)$$

and therefore the relation between the increments of its displacements is

$$l\delta u_1 + m\delta v_1 + n\delta w_1 = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (8.39)$$

By means of this relation one of the increments δu_1, δv_1, δw_1, should be eliminated from the expression for δV and then the terms should be regrouped. If δu_1 is eliminated the terms due to m_1 in δV will have the form $P\delta v_1 + Q\delta w_1$, and the conditions for a minimum or a maximum value of V will be $P = 0$, $Q = 0$, two equations now instead of three; but we have (8.39) as an extra equation now.

If the particle m_1 were constrained to move along a curve instead of a surface there would be two such equations as (8.39). Then two of the quantities δu_1, δv_1, δw_1, could be eliminated from the expression for δV. If m_1 is absolutely fixed, like a point on a fixed axis, or a point at a pinned or clamped end of a beam, then δu_1, δv_1, δw_1, are all zero, and consequently δW_1 is zero identically. Then the conditions for minimum V would not require that the coefficients of δu_1, δv_1, δw_1, should be zero. Although these coefficients are zero they are not necessary in the conditions of equilibrium, since all that the equations do is to tell us the previously unknown forces exerted by the constraining or supporting body.

Thus it will be seen that the conditions of minimum or maximum energy, subject to the actual constraints of the body, lead to all the necessary conditions of equilibrium and to no others.

145. Application to a beam.

Let the beam have a length l and let the load per unit length at distance x from one end be w. Let the forces and couples acting at

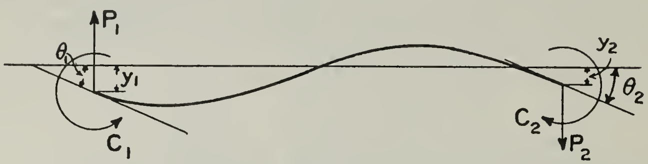

Fig. 80

the ends of the beam be P_1, P_2, C_1, C_2, as shown in fig. 80. Let the deflections at $x=0$ and $x=l$ be y_1, y_2; and let the slopes at the same points be θ_1, θ_2.

By equation (8.11) the strain energy in the beam is

$$\tfrac{1}{2}\int_0^l EI(D^2y)^2 dx$$

Since the load wdx on dx is at distance y below a fixed level the potential energy of the whole load is

$$-\int_0^l wydx.$$

The negative of the work done by the end forces is

$$\int_0^{y_1} P_1 dy_1 - \int_0^{y_2} P_2 dy_2 + \int_0^{\theta_1} C_1 d\theta_1 - \int_0^{\theta_2} C_2 d\theta_2.$$

Thus the total potential energy or available work of all the forces is

$$V = \tfrac{1}{2}\int_0^l EI(D^2y)^2 dx - \int_0^l wydx$$

$$+\int_0^{y_1} P_1 dy_1 - \int_0^{y_2} P_2 dy_2 + \int_0^{\theta_1} C_1 d\theta_1 - \int_0^{\theta_2} C_2 d\theta_2 \quad . \quad (8.40)$$

Now let us suppose that the beam is in equilibrium when

$$y = f(x) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8.41)$$

In order to use the variational method we have to assume that the form of the curve of the beam changes slightly. Let the change in y be indicated by the equation

$$\delta y = hF(x), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8.42)$$

where h is an infinitesimal constant, and $F(x)$ any function of x which is finite within the range from $x=0$ to $x=l$. Then, due to this change in y, the change in V is

$$\delta V = \tfrac{1}{2}\int_0^l \delta\Big\{ EI(D^2y)^2 \Big\} dx - \int_0^l w\delta y dx$$
$$+ P_1\delta y_1 - P_2\delta y_2 + C_1\delta\theta_1 - C_2\delta\theta_2 \quad . \quad . \quad (8.43)$$

the terms on the last line arising from such differences as the following:—

$$\int_0^{y_1+\delta y_1} P_1 dy_1 - \int_0^{y_1} P_1 dy_1 = \int_{y_1}^{y_1+\delta y_1} P_1 dy_1$$
$$= P_1\delta y_1 \quad . \quad . \quad . \quad . \quad . \quad (8.44)$$

Now

$$\delta(D^2y)^2 = \{D^2(y+\delta y)\}^2 - (D^2y)^2$$
$$= \{D^2y + D^2(\delta y)\}^2 - (D^2y)^2$$
$$= 2D^2yD^2(\delta y) \text{ to first order} \quad . \quad . \quad . \quad (8.45)$$

Then, by integration by parts,

$$\int_0^l EID^2yD^2(\delta y)dx = \Big[EID^2yD(\delta y) \Big]_0^l$$
$$- \int_0^l ED(ID^2y)D(\delta y)dx$$
$$= \Big[EID^2yD(\delta y) - ED(ID^2y)\delta y \Big]_0^l$$
$$+ \int_0^l ED^2(ID^2y)\delta y dx \quad . \quad . \quad . \quad . \quad (8.46)$$

But

$$D(\delta y) = \frac{d(y+\delta y)}{dx} - \frac{dy}{dx}$$
$$= \delta\left(\frac{dy}{dx}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8.47)$$

by the meaning of the symbol δ.

Finally we can write δV in the form

$$\delta V = \int_0^l \Big\{ ED^2(ID^2y) - w \Big\} \delta y dx$$
$$+ H_1\,\delta y_1 - H_2\delta y_2 + K_1\delta\theta_1 - K_2\delta\theta_2 \quad . \quad (8.48)$$

where

$$\left.\begin{aligned} H_1 &= P_1 + E[D(ID^2y)]_{x=0} \\ H_2 &= P_2 + E[D(ID^2y)]_{x=l} \\ K_1 &= C_1 - EI[D^2y]_{x=0} \\ K_2 &= C_2 - EI[D^2y]_{x=l} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (8.49)$$

Now in order that δV should be of smaller order than δy, δy_1, $\delta\theta_1$, etc., it is necessary that

$$\left.\begin{aligned} H_1\delta y_1 &= 0 \\ H_2\delta y_2 &= 0 \\ K_1\delta\theta_1 &= 0 \\ K_2\delta\theta_2 &= 0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8.50)$$

and
$$\int_0^l \left\{ \mathrm{ED}^2(\mathrm{ID}^2 y) - w \right\} \delta y dx = 0. \quad . \quad . \quad . \quad . \quad . \quad (8.51)$$

From the first of the conditions (8.50) it follows that, if y_1 is not given, that is, if the corresponding end is not either supported or clamped, then

$$\mathrm{H}_1 = 0,$$

that is,

$$-\mathrm{E}\frac{d}{dx}\left(\mathrm{I}\frac{d^2y}{dx^2}\right) = \mathrm{P}_1 \text{ where } x = 0 \quad . \quad . \quad . \quad . \quad (8.52)$$

This merely means that the shearing force at the end is equal to the applied force at that end, which agrees with what we learnt in the chapter on beams. Similar conclusions can be drawn for the other end.

Again the third of the conditions (8.50) is satisfied if θ_1 is given; but if it is not given then it follows that

$$\mathrm{EID}^2 y = \mathrm{C}_1 \text{ where } x = 0; \quad . \quad . \quad . \quad . \quad . \quad (8.53)$$

that is, at an end where the slope is not fixed, the applied couple is equal to the quantity we have previously called the bending moment.

Thus the four conditions (8.50) give precisely the end conditions of the beam.

Next, in order that (8.51) should be true for all possible values of δy, it is necessary that

$$\mathrm{ED}^2(\mathrm{ID}^2 y) - w = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8.54)$$

If this were not zero at every point of the beam but were equal to some function $\varphi(x)$, then, by taking $\mathrm{F}(x)$ in (8.42) identical with $\varphi(x)$, that is, by taking

$$\delta y = h\varphi(x) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8.55)$$

we should get

$$\int_0^l \left\{ \mathrm{ED}^2(\mathrm{ID}^2 y) - w \right\} \delta y\, dx = \int_0^l h \left\{ \varphi(x) \right\}^2 dx \quad . \quad . \quad . \quad (8.56)$$

which could not possibly be zero if $\varphi(x)$ were not everywhere zero. Then it follows that equation (8.54) must be true at every point of the beam. This then is the differential equation for y.

Thus we see that, from an assumed expression for the energy in an elastic body, it is possible, by the variational method, to deduce the differential equations and the boundary conditions that are consistent with that energy expression. Moreover, since there is a connection between the differential equation and the boundary conditions it is clear that these boundary conditions are consistent with the differential equation. It follows then that, even if there is an error in the original energy expression, nevertheless the boundary conditions deduced therefrom are consistent with the differential equation. There is one particular problem in this subject, namely the bending of thin plates, where the variational method gave boundary conditions consistent with the differential equation when intuitional methods had failed. It was Kirchhoff who put the finishing touch to the work of the brilliant French mathematicians who had preceded him in the treatment of the subject of thin plates.

146. Extension of the minimum energy principle.

Suppose V_1 is the potential energy of the internal and external forces of a body in an equilibrium position, and suppose V is the potential energy in any other position of the body under the same or different external forces. Then let

$$V = V_1 + V' \quad \ldots \ldots \ldots \quad (8.57)$$

Now, in any equilibrium position, V is a minimum or maximum for all possible displacements. But V_1, being the potential energy in a given equilibrium position, is invariable. Consequently V' must be a minimum or maximum in any new equilibrium position. If, then, there are other equilibrium positions besides the one where $V = V_1$ these positions can be found from the conditions that V' is a maximum or minimum.

The preceding theorem is very useful in dealing with stability questions.

One of the simplest examples in elasticity to which we can apply this rule is the case of an elastic string carrying a weight at its lower end. If l is its natural length, A the area of its cross section, x_1 its extension due to a load W_1, we get, by (8.6),

$$V_1 = \tfrac{1}{2}\frac{EA}{l}x_1^2 - W_1 x_1 \quad \ldots \ldots \quad (8.58)$$

Now suppose a further load W_2 is attached to the end. Then the new potential energy in the second state is

$$V_1 + V' = \tfrac{1}{2}\frac{EA}{l}(x_1 + x_2)^2 - (W_1 + W_2)(x_1 + x_2),$$

whence

$$V' = \tfrac{1}{2}\frac{EA}{l}(x_2^2 + 2x_1x_2) - (W_1 + W_2)x_2 - W_2x_1 \quad \ldots \quad (8.59)$$

Now x_1 is supposed to be given and therefore invariable. The condition that V' should be a minimum when x_2 varies is

$$\frac{dV'}{dx_2} = 0, \quad \ldots \ldots \ldots \quad (8.60)$$

that is,

$$\frac{EA}{l}(x_2 + x_1) - (W_1 + W_2) = 0,$$

whence

$$W_1 + W_2 = EA\frac{x_1 + x_2}{l}, \quad \ldots \ldots \quad (8.61)$$

which is clearly right.

We shall now apply the theorem to an example of a different kind, the sort of example for which the theorem is most useful. We take the case of a strut pinned at both ends under a thrust T. Let l be the natural length, $(l-u_1)$ the length in the straight state just when instability begins, $(l-u_1-u)$ the distance between the ends in the bent state.

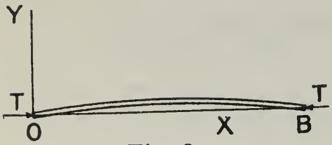

Fig. 81

The potential energy in the straight state is

$$V_1 = \tfrac{1}{2}\frac{EA}{l}u_1^2 - Tu_1; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8.62)$$

and in the bent state

$$V_1 + V' = \tfrac{1}{2}\frac{EA}{l}u_1^2 - Tu_1 - Tu + \tfrac{1}{2}\int_0^l \frac{M^2}{EI}\,dx \quad . \quad . \quad (8.63)$$

Thus

$$V' = \tfrac{1}{2}\int_0^l \frac{M^2}{EI}dx - Tu \quad . \quad . \quad . \quad . \quad . \quad . \quad (8.64)$$

Now since the thrust in the rod is appreciably the same in the bent state as in the straight state it follows that the length of the rod is not altered by bending; that is, the difference between the lengths of the straight line OB and the central axis of the rod is u, the displacement of B since bending started. Thus if ds is the length of an element PQ of the central axis of the rod and dx its projection P′Q′ on the axis of x, then

$$u = \int(ds - dx)$$

$$= \int_0^{l-u-u_1}\left(\frac{ds}{dx} - 1\right)dx$$

$$= \int_0^l \left(\frac{ds}{dx} - 1\right)dx \text{ approximately} \quad . \quad . \quad (8.65)$$

But

$$(ds)^2 = (dx)^2 + (dy)^2,$$

whence

$$\frac{ds}{dx} = \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}} = 1 + \tfrac{1}{2}\left(\frac{dy}{dx}\right)^2 \text{ nearly} \quad . \quad . \quad . \quad (8.66)$$

Therefore

$$u = \int_0^l \tfrac{1}{2}\left(\frac{dy}{dx}\right)^2 dx \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8.67)$$

Then finally

$$V' = \tfrac{1}{2}\int_0^l \frac{M^2}{EI}dx - \tfrac{1}{2}\int_0^l T\left(\frac{dy}{dx}\right)^2 dx$$

$$= \tfrac{1}{2}\int_0^l \left\{EI\left(\frac{d^2y}{dx^2}\right)^2 - T\left(\frac{dy}{dx}\right)^2\right\}dx \quad . \quad . \quad . \quad . \quad (8.68)$$

By using the conditions that V′ should be a minimum for all possible variations in the form of the curve, that is, for all possible variations in y, we can get the differential equation for the strut. Thus suppose y changes to $(y+\delta y)$, δy being equal to a function of x multiplied by a small coefficient.

Then the new value of V′ is

$$V'+\delta V'=\tfrac{1}{2}\int_0^l\left[EI\left\{\frac{d^2(y+\delta y)}{dx^2}\right\}^2-T\left\{\frac{d(y+\delta y)}{dx}\right\}^2\right]dx,$$

whence, neglecting squares of δy and its differential coefficients,

$$\delta V'=\tfrac{1}{2}\int_0^l\left\{2EI\frac{d^2y}{dx^2}\frac{d^2(\delta y)}{dx^2}-2T\frac{dy}{dx}\frac{d(\delta y)}{dx}\right\}dx$$

Now by integration by parts

$$\int_0^l\frac{dy}{dx}\frac{d(\delta y)}{dx}dx=\left[y\frac{d(\delta y)}{dx}\right]_0^l-\int_0^l y\frac{d^2(\delta y)}{dx^2}dx \quad . \quad . \quad . \quad (8.69)$$

The integrated term is zero at both limits because $y=0$ at both ends. Therefore

$$\delta V'=\int_0^l\left\{EI\frac{d^2y}{dx^2}+Ty\right\}\frac{d^2\delta y}{dx^2}dx \quad . \quad . \quad . \quad . \quad (8.70)$$

Now if the quantity in brackets under the integral sign is not zero at every point of the rod it is possible to make δy such a function of x that the quantity to be integrated is positive at all points of the rod, and consequently $\delta V'$ will not be zero for this particular variation of y. But if

$$EI\frac{d^2y}{dx^2}+Ty=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8.71)$$

then $\delta V'$ is certainly zero for any values of δy. This then is the condition that V′ should be a maximum or minimum. But we know that equation (8.74) is the differential equation for the strut. Thus the minimum condition for V′ has given the correct differential equation.

147. Approximate solutions by the minimum energy principle.

There is another way in which an equation such as (8.68) can be made to serve a useful purpose. There are many stability problems of the same type as the strut problem which lead tc differential equations the solutions of which are either not known or so cumbersome as to be useless. In such cases, if we can write down the potential energy, it is usually possible to get quite good approximate solutions to the problems by assuming a reasonable type of strain in which constants are left to be determined by the minimum energy principle. The reasonableness of the assumed type has to be decided by intuition. Of course the type will not be reasonable if it does not satisfy the boundary conditions.

Application to the strut.

The boundary conditions for the strut being that $y = 0$ where $x = 0$ and $x = l$ we must assume that

$$y = x(l - x)f(x), \quad \ldots \quad (8.72)$$

$f(x)$ being a function of x left be chosen by intuition.

It will simplify our problem if we move the origin to the middle of the rod. Then let $l = 2a$ and let

$$s = \frac{x}{a} - 1,$$

so that s is a variable varying from -1 to $+1$ along the rod and proportional to the distance from the middle. Thus equation (8.68) becomes

$$V' = \tfrac{1}{2}\int_{-1}^{1}\left\{\frac{EI}{a^4}\left(\frac{d^2y}{ds^2}\right)^2 - \frac{T}{a^2}\left(\frac{dy}{ds}\right)^2\right\}a\,ds \quad \ldots \quad (8.73)$$

Now the form taken by equation (8.72) is

$$y = (1 - s^2)\,F(s),$$

and clearly, if T is the smallest buckling thrust, y is an even function of s. Therefore

$$y = (1 - s^2)f(s^2) \quad \ldots \quad (8.74)$$

We may take

$$f(s^2) = c + b_1 s^2 + b_2 s^4 + \ldots \quad \ldots \quad (8.75)$$

and then determine the constants c, b_1, etc., from the conditions.

$$\frac{\partial V'}{\partial c} = 0, \quad \frac{\partial V'}{\partial b_1} = 0, \text{ etc.} \quad \ldots \quad (8.76)$$

By this means we should, if we took an infinite series in (8.75), get the absolutely correct solution to the problem. We should, in fact, find that y is a cosine function of a multiple of s. But, of course, the method loses all its virtue if we do not get our result easily, and we are not likely to find an infinite series very easy to handle. Intuition tells us that we ought not to be very far wrong if we take only

$$y = c(1 - s^2) \quad \ldots \quad (8.77)$$

The substitution of this in (8.73) gives

$$V' = \frac{1}{2a^3}\int_{-1}^{1}\left\{4c^2EI - 4c^2a^2s^2T\right\}ds$$

$$= \frac{4c^2}{a^3}\left\{EI - \frac{1}{3}a^2T\right\} \quad \ldots \quad (8.78)$$

The condition

$$\frac{\partial V'}{\partial c} = 0$$

gives

$$\frac{8c}{a^3}\left\{EI-\frac{1}{3}a^2T\right\}=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8.79)$$

whence either

$$c=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8.80)$$

or

$$T=\frac{3EI}{a^2}=\frac{12EI}{l^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8.81)$$

The alternatives that we get in the last two equations are due to the fact that the rod has two possible states of equilibrium, a straight state and a curved state The last equation gives the approximate value of T in the curved state. The factor 12 should, as we well know, be π^2.

To get a better result put

$$\begin{aligned} y&=k(1-s^2)(b+s^2)\\ &=k\{b+(1-b)s^2-s^4\} \quad . \quad . \quad . \quad . \quad . \quad (8.82)\end{aligned}$$

Since it is slightly more convenient to have a single letter for the coefficient of s^2 inside the brackets we shall put

$$h=1-b$$

Then

$$y=k(1-h+hs^2-s^4) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8.83)$$

Now equation (8.76) becomes

$$\begin{aligned} V'&=\frac{k^2}{2a^3}\int_{-1}^{1}\left\{EI(2h-12s^2)^2-a^2T(2hs-4s^3)^2\right\}ds\\ &=\frac{k^2}{a^3}\left\{EI\left(4h^2-16h+\frac{144}{5}\right)-a^2T\left(\frac{4}{3}h^2-\frac{16}{5}h+\frac{16}{7}\right)\right\} \quad (8.84)\end{aligned}$$

The equations

$$\frac{\partial V'}{\partial k}=0, \quad \frac{\partial V'}{\partial h}=0,$$

become

$$EI\left(h^2-4h+\frac{36}{5}\right)=a^2T\left(\frac{1}{3}h^2-\frac{4}{5}h+\frac{4}{7}\right) \quad . \quad . \quad . \quad (8.85)$$

and

$$EI(2h-4)=a^2T\left(\tfrac{2}{3}h-\tfrac{4}{5}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (8.86)$$

By eliminating h from these we get an equation giving T in terms of EI. It seems easier, however, to determine h first.

By dividing the sides of (8.85) by the corresponding sides of (8.86) we get

$$\frac{h^2-4h+\frac{36}{5}}{2(h-2)}=\frac{\frac{1}{3}h^2-\frac{4}{5}h+\frac{4}{7}}{2\left(\frac{1}{3}h-\frac{2}{5}\right)},$$

the solution of which is

$$h \doteq 5{\cdot}718.$$

Then (8.86) gives

$$\frac{a^2T}{EI} = \frac{2h - 4}{\frac{2}{3}h - \frac{4}{5}} = 2{\cdot}469,$$

whence

$$\frac{l^2T}{EI} = 9{\cdot}875, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8.87)$$

a result which is remarkably near the true value $\pi^2 = 9{\cdot}8696$.

Thus an expression with only two variable coefficients in it gives a result agreeing nearly perfectly with the true result which requires an infinite series to express it completely. It should be noticed that this method is, in effect, precisely the same as the one given in Art 101.

CHAPTER IX

TRANSVERSE OSCILLATIONS OF THIN RODS

148. The equation of motion.

Suppose a naturally straight uniform rod, fixed in any way at the ends, is oscillating in one plane. Let the x-axis be taken along the line of centres of gravity of the sections of the rod in its unstrained state, and the y-axis perpendicular to the x-axis in the plane of motion. When the rod is in motion let y denote the displacement of a particle on the central line which was at (x, o), the component displacement in the direction of the x-axis being assumed to be negligible if it is not zero. The origin of coordinates may be taken at any convenient point on the x-axis. Let P, P′ be two points on the central line of the rod, whose abscissae in the undisturbed state were x and $(x + dx)$. Let the inclination to the x-axis, at any instant, of the tangents to the central line at P and P′ be φ and $(\varphi + d\varphi)$, as shown in fig. 82. Now the position of the particular point P varies with the time t; that is, y is a function of t for a point which has a given abscissa x. Also, at any particular instant, y is a function of x, which function is shown by the curve of the central line at that instant. If we imagine that a series of instantaneous photographs of the central line of the rod are taken at different instants these show y as a function of x at each instant, but a different function at different instants. In fact, y is a function of the two independent variables x and t.

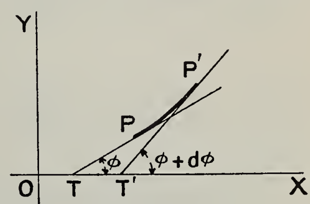

Fig. 82

The acceleration of P is $\dfrac{\partial^2 y}{\partial t^2}$, the symbol for partial differentiation being used because, when dealing with the motion of a particular particle of the rod, x is constant while t varies. Again the slope of the curve at P at a *given instant* is

$$\tan\varphi = \frac{\partial y}{\partial x} \qquad \text{. . . . . (9.1)}$$

In this differentiation t is constant while x varies. The slope expressed by $\tan\varphi$ is the slope of the curve on one of the instantaneous photographs mentioned above.

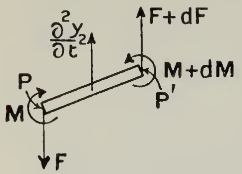

Fig. 83

Let F and M denote the shearing force and bending moment at P (fig. 83), $(F+dF)$ and $(M+dM)$ their values at P′. Let a denote the area of the cross section and w the weight per unit volume of the material of the rod. Resolving in the direction of the y-axis for the motion of the element PP′, and assuming that there are no forces on this element except the shearing force and bending moments at its ends, we get

$$(F+dF)-F=\frac{wadx}{g}\frac{\partial^2 y}{\partial t^2};$$

that is,

$$\frac{\partial F}{\partial x}=\frac{wa}{g}\frac{\partial^2 y}{\partial t^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9.2)$$

Next we have to get the relation between M and F by taking moments. To do this we really need the moment of inertia of the piece PP′. Regarding this as a straight piece of rod, its moment of inertia about an axis through its centre of gravity perpendicular to the rod is

$$\begin{aligned} I_1 &= (\text{mass})\left\{\tfrac{1}{12}(dx)^2+k^2\right\} \\ &= \frac{wa}{g}dx\left\{\tfrac{1}{12}(dx)^2+k^2\right\} \end{aligned}$$

where k denotes the radius of gyration of a section of the rod about the axis through its centre of gravity perpendicular to the plane of motion.

Now the angular velocity of PP′ is $\frac{\partial\varphi}{\partial t}$, and the angular acceleration is $\frac{\partial^2\varphi}{\partial t^2}$. Then, taking moments about the centre of gravity of the piece PP′, we get, for the motion of this piece,

$$\begin{aligned} I_1\frac{\partial^2\varphi}{\partial t^2} &= \tfrac{1}{2}dx\cos\varphi\{F+(F+dF)\}+dM \\ &= Fdx+dM, \end{aligned}$$

neglecting small quantities of higher order than the first.

Dividing the last equation all through by dx the result becomes

$$\frac{wa}{g}\left\{\tfrac{1}{12}(dx)^2+k^2\right\}\frac{\partial^2\varphi}{\partial t^2}=F+\frac{\partial M}{\partial x}$$

which, when dx is infinitely small, reduces to

$$\frac{\partial M}{\partial x}=-F+\frac{wa}{g}k^2\frac{\partial^2\varphi}{\partial t^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9.3)$$

Differentiating this with respect to x we get

$$\frac{\partial^2 M}{\partial x^2} = -\frac{\partial F}{\partial x} + \frac{wa}{g} k^2 \frac{\partial^3 \varphi}{\partial x \partial t^2} \quad . \quad . \quad . \quad . \quad . \qquad (9.4)$$

Since φ is small we may write φ for $\tan\varphi$ in equation (9.1). Then, differentiating both sides of that equation with respect to x, we get

$$\frac{\partial \varphi}{\partial x} = \frac{\partial^2 y}{\partial x^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (9.5)$$

Now making use of equations (9.2) and (9.5) the equation (9.4) becomes

$$\frac{\partial^2 M}{\partial x^2} = -\frac{wa}{g}\frac{\partial^2 y}{\partial t^2} + \frac{wa}{g} k^2 \frac{\partial^4 y}{\partial x^2 \partial t^2} \quad . \quad . \quad . \quad . \qquad (9.6)$$

The relation between the bending moment and the curvature is just the same as for rods in equilibrium. That is,

$$M = EI\frac{\partial^2 y}{\partial x^2} = Eak^2 \frac{\partial^2 y}{\partial x^2}, \quad . \quad . \quad . \quad . \quad . \qquad (9.7)$$

where I denotes the moment of inertia of the cross section of the rod, as in the bending of beams. In the present problem we have taken y positive upwards in our figure, and M is reckoned positive in the direction contrary to that used in beam equations in Chapter VI. This double change of signs makes no change of sign in equation (9.7).

Substituting in (9.6) the value of M from (9.7) we find, on dividing by a,

$$Ek^2\frac{\partial^4 y}{\partial x^4} = -\frac{w}{g}\frac{\partial^2 y}{\partial t^2} + \frac{w}{g} k^2 \frac{\partial^4 y}{\partial x^2 \partial t^2} . \quad . \quad . \quad . \quad . \qquad (9.8)$$

The last term in (9.8) is due to the *rotary inertia* of the rod. If the rod is thin k is small but E is large, so that Ek^2 is not small, whereas wk^2 is small. The term due to the rotary inertia is, then, small compared with the other two terms in the equation for any ordinary rod or beam. It is usual to neglect this term in dealing with the transverse oscillations of rods since it makes no appreciable difference to the results. Then the final differential equation for the transverse oscillations of thin uniform rods is

$$Ek^2\frac{\partial^4 y}{\partial x^4} = -\frac{w}{g}\frac{\partial^2 y}{\partial t^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (9.9)$$

If the rod were not uniform the equation would be

$$\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 y}{\partial x^2}\right) = -\frac{wa}{g}\frac{\partial^2 y}{\partial t^2}$$

The true proof that the term due to the rotary inertia is negligible is not quite so simple as it appears above, but a more rigorous proof would lead to the same conclusion.

149. Normal modes of oscillation.

A rod can oscillate transversely in an infinite variety of ways, in every one of which the displacement satisfies equation (9.9). But there are certain simple types or modes of oscillation in each one of which every particle of the rod executes simple harmonic motion in the same period and the same phase, but with different amplitudes. That is, the period is independent of x but the amplitude is a function of x. These modes are called the normal modes of oscillation of the rod. A normal mode is expressed by

$$y = u \sin (pt + a) \quad \ldots \ldots \quad (9.10)$$

where u is a function of x and not of t, and is called a *normal function* for the rod. Each normal mode of oscillation has its own normal function.

With the value of y given by (9.10) we find that

$$\frac{\partial^2 y}{\partial t^2} = -p^2 u \sin (pt + a)$$

and

$$\frac{\partial^4 y}{\partial x^4} = \frac{d^4 u}{dx^4} \sin (pt + a)$$

The substitution of these values in (9.9) gives

$$Ek^2 \frac{d^4 u}{dx^4} \sin (pt + a) = + \frac{w}{g} p^2 u \sin (pt + a), \quad \ldots \quad (9.11)$$

whence

$$\frac{d^4 u}{dx^4} = m^4 u, \quad \ldots \ldots \quad (9.12)$$

where

$$m^4 = \frac{wp^2}{gEk^2} \quad \ldots \ldots \quad (9.13)$$

Our assumption that y could be expressed in the form given in (9.10), where u was assumed to be a function of x alone, is verified by (9.12) since this equation does not involve t; that is, it determines u as a function of x alone.

To solve (9.12), which is a linear equation with constant coefficients, assume

$$u = Ae^{nx},$$

where n is a constant which must be determined. Then

$$\frac{d^4 u}{dx^4} = n^4 Ae^{nx}, \quad \ldots \ldots \quad (9.14)$$

and therefore equation (9.12) gives

$$n^4 Ae^{nx} = m^4 Ae^{nx}$$

or

$$n^4 = m^4,$$

whence

$$n^2 = \pm m^2,$$

and therefore $n = \pm m$ or $\pm im$, ... (9.15)

i being written for $\sqrt{-1}$.

The following are therefore separate solutions of (9.12)

$$\left.\begin{aligned} u &= A_1 e^{mx} \\ u &= A_2 e^{-mx} \\ u &= A_3 e^{imx} \\ u &= A_4 e^{-imx} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9.16)$$

and it is easy to verify that a solution is obtained by equating u to the sum of all the quantities on the right of equations (9.16). That is

$$u = A_1 e^{mx} + A_2 e^{-mx} + A_3 e^{imx} + A_4 e^{-imx} \quad . \quad . \quad (9.17)$$

Now since

$$\begin{aligned} e^{mx} &= \cosh mx + \sinh mx \\ e^{-mx} &= \cosh mx - \sinh mx \\ e^{imx} &= \cos mx + i \sin mx \\ e^{-imx} &= \cos mx - i \sin mx \end{aligned}$$

equation (9.17) can be written in the form

$$u = A \cos mx + B \sin mx + H \cosh mx + K \sinh mx \quad . \quad (9.18)$$

the new constants being connected with the old constants by the equations

$$\begin{aligned} H &= A_1 + A_2 \\ K &= A_1 - A_2 \\ A &= A_3 + A_4 \\ B &= i(A_3 - A_4) \end{aligned}$$

It can be verified directly that the value of u given by (9.18) is a solution of (9.12), for the fourth differential coefficient of every term on the right of (9.18) is the product of m^4 and the term itself. Also, since (9.12) is a differential equation of the fourth order and equation (9.18) contains four arbitrary constants, or constants of integration—the requisite number for the complete solution of a differential equation of the fourth order—it follows that (9.18) gives the complete solution of the equation (9.12). The value of y corresponding to this value of u is given by (9.10).

Just as for a beam in equilibrium under given loads there are four conditions to be satisfied by y and its differential coefficients with respect to x, which conditions depend on the forces applied at the ends of the oscillating rod. These conditions, which are exactly the same as for a loaded beam whose ends are fixed in the same way as those of the oscillating beam, are given in equations (5.16), (5.17), and (5.18).

150. Rod clamped at one end and free at the other.

If the origin is taken at the clamped end the end-conditions for a rod of length l are

$$y = 0 \text{ and } \frac{\partial y}{\partial x} = 0 \text{ where } x = 0;$$

$$\frac{\partial^2 y}{\partial x^2} = 0 \text{ and } \frac{\partial^3 y}{\partial x^3} = 0 \text{ where } x = l.$$

Since y involves x only so far as it is contained in u these conditions are equivalent to

$$u = 0 \text{ and } \frac{du}{dx} = 0 \text{ where } x = 0 \quad . \quad . \quad . \quad . \quad (9.19)$$

$$\frac{d^2u}{dx^2} = 0 \text{ and } \frac{d^3u}{dx^3} = 0 \text{ where } x = l \quad . \quad . \quad . \quad . \quad (9.20)$$

Applying these conditions to the value of u in (9.18) we get

$$A + H = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9.21)$$

$$m(B + K) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9.22)$$

$$m^2\{-A\cos ml - B\sin ml + H\cosh ml + K\sinh ml\} = 0 \quad (9.23)$$

$$m^3\{A\sin ml - B\cos ml + H\sinh ml + K\cosh ml\} = 0 \quad . \quad (9.24)$$

When H and K are eliminated from the last two equations by means of (9.21) and (9.22) the equations take the forms

$$-A(\cos ml + \cosh ml) = B(\sin ml + \sinh ml) \quad . \quad . \quad . \quad (9.25)$$

$$A(\sin ml - \sinh ml) = B(\cos ml + \cosh ml) \quad . \quad . \quad . \quad (9.26)$$

Each of these equations gives a value of the ratio of A to B. The equation obtained by eliminating this ratio is

$$\sinh^2 ml - \sin^2 ml = (\cosh ml + \cos ml)^2$$

or $$2\cosh ml \cos ml = -(\cosh^2 ml - \sinh^2 ml) - (\cos^2 ml + \sin^2 ml)$$
$$= -2$$

whence $$\cos ml = -\frac{1}{\cosh ml} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9.27)$$

This equation determines ml, and therefore determines p since all the other quantities involved in m are known. There are an infinite number of roots of (9.27), and the period of oscillation corresponding to each value of p is $\frac{2\pi}{p}$. Then this last equation determines all the possible periods of what we have called the normal modes of oscillation.

Equation (9.27) can be solved by graphs and then by successive approximations. Let $z = ml$ and then plot the curves

$$y_1 = \cos z$$

$$y_2 = -\frac{1}{\cosh z} = -\operatorname{sech} z$$

For quite moderate values of z the value $\cosh z$ differs very little from $\frac{1}{2}e^z$, and the value of $\operatorname{sech} z$ therefore differs little from $2e^{-z}$. For example

$$\operatorname{sech} \pi = \frac{2}{e^{\pi} + e^{-\pi}} = \frac{2e^{-\pi}}{1 + e^{-2\pi}}$$
$$= \frac{2e^{-\pi}}{1 + 0.00187}$$

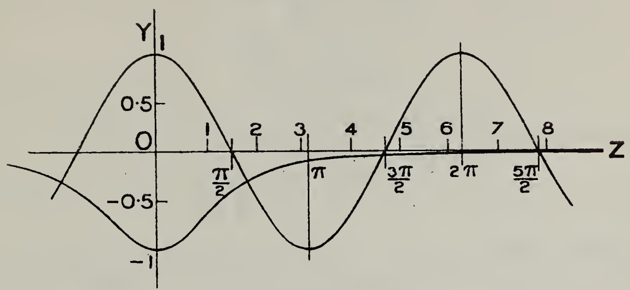

Fig. 84

The curves for y_1 and y_2 are shown in fig. 84. It can easily be seen from the figure that the roots after the first are approximately

$$z = \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \ldots \qquad (9.28)$$

The first root is approximately

$$z = 1{\cdot}87.$$

Let the first root be

$$z = 1{\cdot}87 + v,$$

where v is a number that we know is small. Then the equation for v is

$$\cosh(1{\cdot}87 + v) \times \cos(1{\cdot}87 + v) = -1$$

But, when we neglect v^2, Taylor's theorem gives

$$f(a + v) = f(a) + vf'(a)$$

Therefore

$$\cosh(1{\cdot}87 + v) = \cosh 1{\cdot}87 + v \sinh 1{\cdot}87$$
$$\cos(1{\cdot}87 + v) = \cos 1{\cdot}87 - v \sin 1{\cdot}87$$

Hence our equation for v becomes

$$(\cosh 1{\cdot}87 + v \sinh 1{\cdot}87)(\cos 1{\cdot}87 - v \sin 1{\cdot}87) = -1;$$

or, again neglecting v^2,

$$v\{\cosh 1{\cdot}87 \sin 1{\cdot}87 - \sinh 1{\cdot}87 \cos 1{\cdot}87\} = 1 + \cosh 1{\cdot}87 \cos 1{\cdot}87,$$

which gives

$$v = \frac{0{\cdot}0206}{4{\cdot}11} = 0{\cdot}0050.$$

Then a better approximation to the first root is

$$z = 1{\cdot}8750 \qquad (9.29)$$

In a similar way better approximations to the other roots of equation (9.27) can be found than those given in (9.28).

Suppose z_r denotes the rth positive root of (9.27). Then the corresponding value of p is, by (9.13),

$$p_r = \left(\frac{z_r}{l}\right)^2 \sqrt{\frac{g\mathrm{E}k^2}{w}}$$

$$= \frac{kz_r^2}{l^2}\sqrt{\frac{g\mathrm{E}}{w}}, \qquad (9.30)$$

and the corresponding period of oscillation is

$$t_r = \frac{2\pi}{p_r} = \frac{2\pi l^2}{kz_r^2}\sqrt{\frac{w}{g\mathrm{E}}} \qquad (9.31)$$

The different periods of the normal modes of oscillation for a rod clamped at one end are therefore approximately proportional to

$$\frac{1}{1{\cdot}875^2}, \quad \frac{2^2}{3^2\pi^2}, \quad \frac{2^2}{5^2\pi^2}, \text{ etc.}$$

For rods of the same material w and E are the same, and therefore the periods of corresponding modes for such rods are proportional to $\frac{l^2}{k}$. If the rods have equal and similar sections these periods of corresponding modes are proportional to l^2. Thus, if two steel bars with the same cross-section, one of which is twice as long as the other, oscillate in their slowest modes, the period of the longer rod is four times as great as the period of the shorter.

Again, if two rods of the same material have the same length and similar but unequal sections, the periods of corresponding modes are proportional to the inverse of the linear dimensions of the sections. If one section has twice the linear dimensions of the other the periods of its normal modes are half as great as the corresponding periods of the thinner rod. The thicker the rod, other things being equal, the quicker the oscillations.

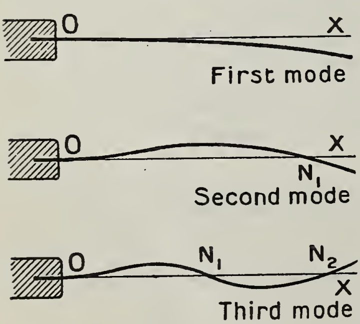

Fig. 85

151. Shape of the curve for the different modes.

For any mode of oscillation of the rod clamped at one end and free at the other the equation to the curve is

$$y = \{\mathrm{A}(\cos mx - \cosh mx) + \mathrm{B}(\sin mx - \sinh mx)\}\sin(pt + \alpha)$$

where m is one of the roots of (9.27), p is related to m by equation (9.13), and B is related to A by either of equations (9.25) or (9.26).

Let R be written for the common value of each side of equation (9.25). Then

$$A = -\frac{R}{\cosh ml + \cos ml},$$

$$B = \frac{R}{\sinh ml + \sin ml},$$

and therefore, in terms of the new constant R,

$$y = R\left\{\frac{\cosh mx - \cos mx}{\cosh ml + \cos ml} - \frac{\sinh mx - \sin mx}{\sinh ml + \sin ml}\right\}\sin(pt + \alpha) \qquad (9.32)$$

The magnitude of the constant R depends on the way in which the rod is started. If R sin $(pt + \alpha)$ is treated as a small constant at any instant, the curve of the central line of the rod at that instant can be plotted for any particular mode. For all modes except the first the rod has *nodes*, that is, points which remain at rest on the x-axis while the rest of the rod oscillates. The second mode has one node, the third mode has two, and so on. These are marked N_1, N_2, in fig. 85.

152. Positions of the nodes.

The nodes are at the points where

$$\frac{\cosh mx - \cos mx}{\cosh ml + \cos ml} = \frac{\sinh mx - \sin mx}{\sinh ml + \sin ml} \quad . \quad . \quad . \quad (9.33)$$

Now for all the modes of oscillation except the first, in which case there is no node, cosh ml and sinh ml are nearly equal and each is large in comparison with cos ml or sin ml. Then the nodes are very near the points where

$$\cosh mx - \cos mx = \sinh mx - \sin mx$$

or

$$\cosh mx - \sinh mx = \cos mx - \sin mx$$

or

$$e^{-mx} = \cos mx - \sin mx$$

$$= \sqrt{2}\cos\left(\frac{\pi}{4} + mx\right) \quad . \quad . \quad (9.34)$$

It is easy to see from the curves

$$y_1 = e^{-mx}$$

$$y_2 = \sqrt{2}\cos\left(\frac{\pi}{4} + mx\right)$$

that the values of mx that satisfy equation (9.34) are approximately

$$\left.\begin{aligned} mx_1 &= \frac{5\pi}{4}, \\ mx_2 &= \frac{9\pi}{4}, \\ mx_3 &= \frac{13\pi}{4}, \\ &\text{etc.} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9.35)$$

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For the second mode of oscillation the only node is at x_1, and since $ml = \frac{3\pi}{2}$ nearly for this mode it follows that

$$x_1 = \tfrac{5}{6}\, l \text{ nearly.}$$

For the third mode, for which $ml = \frac{5\pi}{2}$ nearly, the two nodes are near

$$x_1 = \tfrac{1}{2}\, l, \quad x_2 = \tfrac{9}{10}\, l.$$

153. Rod pinned at both ends.

The assumption in this case is that the rod is fixed by smooth parallel pins at both ends. The end condition are therefore

$$y = 0 \text{ and } \frac{\partial^2 y}{\partial x^2} = 0$$

at both ends. That is,

$$\left.\begin{aligned} u = 0 \text{ and } \frac{d^2 u}{dx^2} = 0 \\ \text{both where } \quad x = 0 \text{ and where } x = l \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (9.36)$$

Applying these conditions to (9.18) we get

$$A + H = 0$$
$$-A + H = 0$$
$$A \cos ml + B \sin ml + H \cosh ml + K \sinh ml = 0$$
$$-A \cos ml - B \sin ml + H \cosh ml + K \sinh ml = 0$$

The first two of these conditions give

$$A = H = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9.37)$$

Then the last two give

$$B \sin ml = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9.38)$$

and

$$K \sinh ml = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9.39)$$

Since sinh ml cannot be zero the last equation requires that K should be zero. Also (9.38) is satisfied provided

either $\qquad B = 0$

or $\qquad \sin ml = 0$

If the former is true then y is always zero and therefore the rod is at rest. The other alternative gives

$$ml = \pi, \text{ or } 2\pi, \text{ or } 3\pi, \text{ etc.,} \quad . \quad . \quad . \quad . \quad (9.40)$$

whence

$$p \frac{l^2}{k} \left(\frac{w}{gE} \right) = \pi^2, \text{ or } 2^2\pi^2, \text{ or } 3^2\pi^2, \quad . \quad . \quad . \quad (9.41)$$

Thus the curve of the rod for the nth mode is

$$y = B_n \sin \frac{n\pi x}{l} \sin (p_n t + a) \quad . \quad . \quad . \quad . \quad . \quad (9.42)$$

which, at any given instant, is a pure sine curve having n half-wave lengths; and the period of oscillation of this mode is

$$t_n = \frac{2\pi}{p_n} = \frac{2}{n^2\pi}\frac{l^2}{k}\left(\frac{w}{gE}\right)^{\frac{1}{2}}. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (9.43)$$

The *frequencies* of the normal modes—the frequency being the number of oscillations per second— are proportional to 1^2, 2^2, 3^2, etc.

154. Rod clamped at both ends.

Taking the origin at one end, as usual, the conditions at the ends in this case are

$$\left.\begin{array}{l} u = 0 \\ \dfrac{du}{dx} = 0 \end{array}\right\} \begin{array}{l} \text{both where} \\ \quad x = 0 \\ \text{and } x = l \end{array} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (9.44)$$

Applying these conditions to u given in (9.18) we get

$$A + H = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (9.45)$$

$$B + K = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (9.46)$$

$$A \cos ml + B \sin ml + H \cosh ml + K \sinh ml = 0 \quad . \quad . \qquad (9.47)$$

$$-A \sin ml + B \cos ml + H \sinh ml + K \cosh ml = 0 \quad . \quad . \qquad (9.48)$$

The elimination of H and K from the last two equations by means of the preceding two gives

$$A(\cosh ml - \cos ml) = -B(\sinh ml - \sin ml) \quad . \quad . \quad . \qquad (9.49)$$

$$A(\sinh ml + \sin ml) = -B(\cosh ml - \cos ml) \quad . \quad . \quad . \qquad (9.50)$$

Now eliminating the ratio A : B from the last two equations we get

$$(\cosh ml - \cos ml)^2 = \sinh^2 ml - \sin^2 ml$$

or

$$2 \cosh ml \cos ml = \cosh^2 ml - \sinh^2 ml + \cos^2 ml + \sin^2 ml$$
$$= 2$$

whence

$$\cos ml = \frac{1}{\cosh ml} = \operatorname{sech} ml \quad . \quad . \quad . \quad . \quad . \qquad (9.51)$$

Since sech ml is very small except when $ml < 3$ it follows that the larger values of ml satisfying (9.51) are near roots of the equation

$$\cos ml = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (9.52)$$

By plotting the curves

$$y_1 = \cos z$$
$$y_2 = \operatorname{sech} z$$

it will be seen that the smallest root of (9.51) is the one near $\frac{3\pi}{2}$. Then we know that roots of (9.51) are approximately

$$ml = \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \text{ etc.,} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (9.53)$$

and therefore the frequencies of the normal modes are approximately proportional to

$$3^2,\ 5^2,\ 7^2, \text{ etc.}$$

The first root of (9.51) is actually

$$ml = 4.730, \text{ whereas } \frac{3\pi}{2} \text{ is } 4.712 \quad . \quad . \quad . \quad . \qquad (9.54)$$

If the origin be taken at the middle instead of at the ends of the rod clamped at both ends the values of u take the forms

$$u = \mathrm{R}\left\{\frac{\sinh mx}{\sinh \frac{1}{2} ml} - \frac{\sin mx}{\sin \frac{1}{2} ml}\right\} \quad . \quad . \quad . \quad . \quad . \quad (9.55\mathrm{a})$$

or

$$u = \mathrm{R}\left\{\frac{\cosh mx}{\cosh \frac{1}{2} ml} - \frac{\cos mx}{\cos \frac{1}{2} ml}\right\} \quad . \quad . \quad . \quad . \quad . \quad (9.55\mathrm{b})$$

according as the middle of the rod is, or is not, a node. In the even modes, the second, fourth, etc., the middle is a node, but in the odd modes it is not a node. Consequently (9.55a) is correct for the even modes, and (9.55b) for the odd modes. The forms of the first and second modes are shown in fig. 86.

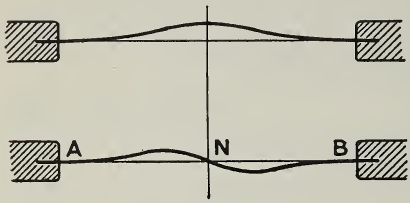

Fig. 86

155. Rod clamped at one end and pinned at the other. We are assuming that the pin is smooth and is in the line of the tangent at the clamped end. That is, the end-conditions, taking the origin at the pin, are

$$\left.\begin{aligned} u &= 0, \\ \mathrm{M} = 0 \text{ or } \frac{d^2u}{dx^2} &= 0, \end{aligned}\right\} \text{where } x = 0; \quad . \quad . \quad . \quad (9.56)$$

and

$$\left.\begin{aligned} u &= 0, \\ \frac{du}{dx} &= 0, \end{aligned}\right\} \text{where } x = l \quad . \quad . \quad . \quad . \quad (9.57)$$

Now the portion NB of the rod in fig. 86 above satisfies the four end-conditions of the present problem, except that the length NB is called $\frac{1}{2} l$ in that problem and l in the present one. Putting therefore l for $\frac{1}{2} l$ in (9.55a) we get

$$u = \mathrm{R}\left\{\frac{\sinh mx}{\sinh ml} - \frac{\sin mx}{\sin ml}\right\} \quad . \quad . \quad . \quad . \quad . \quad (9.58)$$

which satisfies all the four conditions in (9.56) and (9.57). We are certain that it satisfies those in (9.57) since they are the same conditions as in the last problem. We must show that the two conditions in (9.56) are also satisfied.

The first of the equations (9.56) is obviously satisfied because $\sin 0 = 0$ and $\sinh 0 = 0$.

Also

$$\frac{d^2u}{dx^2} = m^2\mathrm{R}\left\{\frac{\sinh mx}{\sinh ml} + \frac{\sin mx}{\sin ml}\right\}, \quad . \quad . \quad . \quad (9.59)$$

which is also obviously zero where $x=0$. Thus (9.58) is the complete value of u for this problem. It should have been obvious, without any calculation, that the bending moment is zero at N in fig. 86, for this point is clearly a point of inflection on the curve, that is, a point where the curvature changes sign, and therefore where the curvature is zero.

The periods of oscillation in the present case can be obtained from the periods of the even modes of the clamped-clamped rod by putting $2l$ for l; or we may get m directly, in another way, from the second of the conditions in (9.57). Thus

$$mR\left\{\frac{\cosh ml}{\sinh ml}-\frac{\cos ml}{\sin ml}\right\}=0$$

or

$$\tan ml=\tanh ml \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9.60)$$

It can be shown that (9.51) is equivalent to

$$\tan\frac{ml}{2}=\pm\tanh\frac{ml}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9.61)$$

which, if l be put for $\frac{1}{2}l$, contains the equation (9.60).

In short, a clamped-pinned rod oscillates exactly like one half of a clamped-clamped rod of twice the length when the latter is oscillating in even modes, that is, in the second, fourth, etc., modes.

156. Rod clamped at one end and free at the other, and carrying a finite load W at the free end.

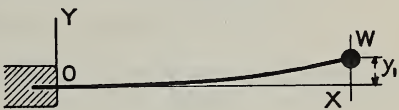

Fig. 87

With the origin at the fixed end the conditions at this end are

$$\left.\begin{aligned}u&=0\\ \frac{du}{dx}&=0\end{aligned}\right\}\text{ where } x=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (9.62)$$

At the other end the bending moment is zero; that is,

$$\frac{d^2u}{dx^2}=0 \text{ where } x=l \quad . \quad . \quad . \quad . \quad . \quad . \quad (9.63)$$

The other condition at the load is that the shearing force is the force due to the inertia of W. If y_1 denotes the value of y at the load the acceleration of the load is $\frac{d^2y_1}{dt^2}$. Then the force on W in the direction in which y_1 is measured is $\frac{W}{g}\frac{d^2y_1}{dt^2}$, which force is applied by the rod. There is an equal and opposite reaction on the rod. If y_1 is positive this reaction is in the direction in which the shearing force is negative. Thus the condition is

$$F=-\frac{W}{g}\frac{d^2y_1}{dt^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9.64)$$

Now since

$$y = u \sin (pt + \alpha)$$

this last condition becomes, after division by $\sin (pt + \alpha)$,

$$\mathrm{EI}\frac{d^3u}{dx^3} = -\frac{\mathrm{W}}{g}p^2u;$$

that is, when p^2 is expressed in terms of m by means of (9.13),

$$\frac{d^3u}{dx^3} = -\frac{m^4\mathrm{W}}{aw}u \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9.65)$$

where a is the area of the cross section of the rod. With the value of u given by (9.18) the two conditions (9.62) give

$$\mathrm{A} + \mathrm{H} = 0$$

and

$$\mathrm{B} + \mathrm{K} = 0$$

whence

$$u = \mathrm{A}(\cos mx - \cosh mx) + \mathrm{B}(\sin mx - \sinh mx) \quad . \quad . \quad (9.66)$$

Now (9.63) gives

$$-\mathrm{A}(\cos ml + \cosh ml) - \mathrm{B}(\sin ml + \sinh ml) = 0 \quad . \quad (9.67)$$

Also (9.65) gives

$$-m^3\{\mathrm{A}(\sinh ml - \sin ml) + \mathrm{B}(\cosh ml + \cos ml)\}$$
$$= \frac{m^4\mathrm{W}}{aw}\{\mathrm{A}(\cosh ml - \cos ml) + \mathrm{B}(\sinh ml - \sin ml)\} \quad (9.68)$$

On eliminating the ratio of A to B from the last two equations and clearing the resulting equation of fractions we get

$$-(\sinh^2 ml - \sin^2 ml) + (\cosh ml + \cos ml)^2$$
$$= \frac{m\mathrm{W}}{aw}\left\{\begin{matrix} +(\cosh ml - \cos ml)(\sinh ml + \sin ml) \\ -(\sinh ml - \sin ml)(\cosh ml + \cos ml)\end{matrix}\right\};$$

that is,

$$2 + 2\cosh ml \cos ml = 2\frac{m\mathrm{W}}{aw}\{\cosh ml \sin ml - \sinh ml \cos ml\}$$

If we write z for ml, and c for $\frac{\mathrm{W}}{alw}$, which is the ratio of the weight W to the weight of the whole rod, the equation for z is

$$1 + \cosh z \cos z = cz(\cosh z \sin z - \sinh z \cos z)$$

or

$$\frac{1 + \cosh z \cos z}{\cosh z \sin z - \sinh z \cos z} = cz \quad . \quad . \quad . \quad . \quad (9.69)$$

This equation can be solved by plotting the curve

$$y_1 = \frac{1 + \cosh z \cos z}{\cosh z \sin z - \sinh z \cos z}$$
$$= \frac{\cos z + \operatorname{sech} z}{\sin z - \cos z \tanh z} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9.70)$$

and the straight line

$$y_2 = cz \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9.71)$$

and finding the values of z at the intersections. For all except small values of z we may use the approximate values

$$\tanh z = 1, \ \operatorname{sech} z = 0$$

and then (9.70) becomes

$$y_1 = \frac{\cos z}{\sin z - \cos z}$$

$$= \frac{1}{\sqrt{2}} \frac{\cos z}{\sin\left(z - \frac{\pi}{4}\right)} \quad . \quad . \quad . \quad . \quad . \quad . \quad (9.72)$$

The true value of y_1 vanishes when

$$\cos z = -\operatorname{sech} z \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9.73)$$

and is infinite when

$$\tan z = \tanh z \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9.74)$$

whereas the approximate value of y_1, given by (9.72), vanishes when z is an odd multiple of $\frac{\pi}{2}$, and is infinite when $\left(z - \frac{\pi}{4}\right)$ is a multiple of π or zero.

The curve given by (9.70) is shown in fig. 88.

Two straight lines are drawn satisfying (9.71), one when c is very small and the other when $c = 1$. The graph shows that, for a small value of c, the first few roots of (9.69) nearly coincide with the roots of (9.73), which is the same equation as (9.27). This tells us what we might easily have guessed without calculation, namely, that the possible periods of vibration when the rod carries a load which is small compared with the weight of the rod are nearly the same as when there is no load on the end.

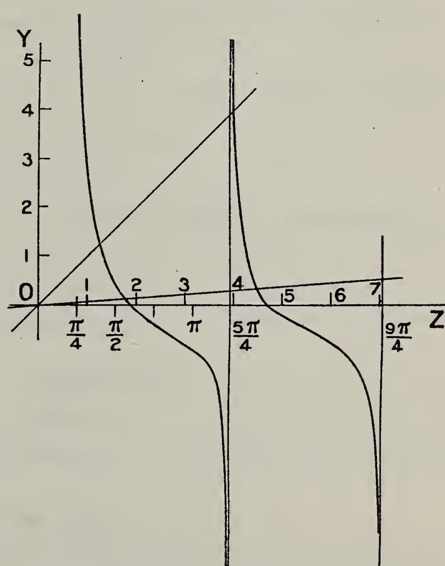

Fig. 88

Agaim when $c = 1$ the graph gives roughly the following values of z at the intersections of the line and the curves

$$z = 1{\cdot}23,\ 4{\cdot}04,\ \frac{9\pi}{4},\ \frac{13\pi}{4} \quad . \quad . \quad . \quad . \quad . \quad . \quad (9{\cdot}75)$$

In getting a second approximation to the first root the calculations are simplified if the first approximation be taken as 1·25. The second approximations to the first two roots are

$$z = 1{\cdot}238,\ 4{\cdot}045 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9{\cdot}76)$$

For the other roots it is sufficiently accurate to solve the equation

$$z = \frac{1}{\sqrt{2}} \frac{\cos z}{\sin\left(z - \frac{\pi}{4}\right)}.$$

Writing θ for $\left(z - \frac{\pi}{4}\right)$ in this we get

$$\left(\theta + \frac{\pi}{4}\right) \sin\theta = \frac{1}{\sqrt{2}} \cos\left(\theta + \frac{\pi}{4}\right)$$

$$= \tfrac{1}{2}(\cos\theta - \sin\theta)$$

whence

$$\tan\theta = \frac{1}{2\theta + \frac{\pi}{2} + 1}$$

We know that the approximate solutions are $\theta = n\pi$. Then let

$$\theta = n\pi + v$$

where v is small. Thus

$$\tan v = \frac{1}{2n\pi + \frac{\pi}{2} + 1}$$

approximately; or, since v is small,

$$v = \frac{2}{\pi(4n + 1) + 2}.$$

For the third root $n = 2$, and therefore

$$v = \frac{2}{9\pi + 2}$$

$$= 0{\cdot}0660.$$

Therefore the third root is approximately

$$z = \frac{9\pi}{4} + \frac{2}{9\pi + 2}$$

$$= 7{\cdot}135 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9{\cdot}77)$$

The periods of the first three normal modes are the values of t given by

$$\sqrt{\frac{gE}{w} \cdot \frac{k}{l^2}}\, t = \frac{2\pi}{z_1^2},\ \frac{2\pi}{z_2^2},\ \frac{2\pi}{z_3^2},$$

$$= 4{\cdot}101,\ 0{\cdot}3841,\ 0{\cdot}1234 \quad . \quad . \quad . \quad . \quad (9{\cdot}78)$$

Thus, for a solid steel rod of length 20 inches having a circular section with a diameter one inch, if $E = 32 \times 10^6$ pounds per square inch, $w = 490$ pounds per cubic foot, the period of the slowest mode is, since $k = \frac{1}{4}$ inch,

$$t = 4{\cdot}10\sqrt{\frac{w}{gE}\frac{l^2}{k}}$$

$$= 4{\cdot}10\sqrt{\frac{490}{32{\cdot}2 \times 32 \times 10^6 \times 144}}\ \frac{20^2}{\frac{1}{4} \times 12}$$

$$= 0{\cdot}0315 \text{ sec.}; \quad \ldots \quad (9.79)$$

that is, the rod makes about 1905 oscillations per minute.

If the weight on the end has any other ratio to the weight of the rod the same method will give the periods. There is, however, one interesting case that can be investigated without the graph. It is the first mode of oscillation when W is many times the weight of the rod. This first mode is a very slow mode and the corresponding value of z is small. Now when z is small

$$\cos z = 1 - \frac{z^2}{2}$$

$$\cosh z = 1 + \frac{z^2}{2}$$

$$\sin z = z - \frac{z^3}{6}$$

$$\sinh z = z + \frac{z^3}{6}$$

approximately. Therefore equation (9.69) becomes,

$$1 + \left(1 - \frac{z^4}{4}\right) = cz^2\left\{\left(1 + \tfrac{1}{3}z^2\right) - \left(1 - \tfrac{1}{3}z^2\right)\right\}$$

$$2 - \frac{z^4}{4} = \tfrac{2}{3}cz^4.$$

When $\frac{1}{4}z^4$ is neglected on the left this gives

$$z = \left(\frac{3}{c}\right)^{\frac{1}{4}}$$

The corresponding period is

$$t = \frac{l^2}{k}\sqrt{\frac{w}{gE}}\,2\pi\left(\frac{c}{3}\right)^{\frac{1}{2}}$$

$$= 2\pi\frac{l^2}{k}\sqrt{\frac{w}{gE} \times \frac{W}{3wal}}$$

$$= \frac{2\pi}{k}\sqrt{\frac{Wl^3}{3agE}} \quad \ldots \quad (9.80)$$

a result which does not involve the weight of the rod. This shows that our approximation involves the assumption that the rod has no inertia, or that the inertia of W is infinitely greater than that of the rod.

This last result could have been obtained by a much simpler process, without, in fact, the theory of the oscillation of rods at all. If the inertia of the rod is negligible then the internal forces in the rod, that is, the shearing force and bending moment, are the same as if the rod were at rest under the action of forces at the ends. At the clamped end there are the necessary forces to maintain the position and direction of the end; and at the free end there is a shearing force, which is the reaction to the force causing the acceleration of W. Let y_1 be the displacement of W at any time, and assume that

$$y_1 = u_1 \sin(pt + \alpha).$$

Then the shearing force at the end is

$$\begin{aligned} F &= -\frac{W}{g}\frac{d^2y_1}{dt^2} \\ &= \frac{W}{g}p^2u_1 \sin(pt + \alpha) \\ &= \frac{W}{g}p^2y_1 \quad \text{. . . . . (9.81)} \end{aligned}$$

This load F on the free end of a clamped-free beam causes a deflection y given by

$$EIy = F(\tfrac{1}{2}lx^2 - \tfrac{1}{6}x^3),$$

and at the free end this deflection is

$$y_1 = \frac{1}{3}\frac{Fl^3}{EI} \quad \text{. . . . . (9.82)}$$

Equating the two values of y_1 in (9.81) ane (9.82) we get

$$\frac{gF}{Wp^2} = \frac{1}{3}\frac{Fl^3}{EI}$$

whence

$$p^2 = \frac{3gEI}{Wl^3}.$$

Since the period is $\frac{2\pi}{p}$ this agrees with (9.80).

157. Free and forced oscillations.

The particular cases of rods oscillating transversely that we have so far worked out are cases of oscillation under no external forces except such as are necessary to keep the ends fixed. These are called *free* oscillations to distinguish them from the oscillations which the rods would have if periodic forces acted on them. Suppose, for example, that the ends of any one of the rods we have dealt with were forced to oscillate in any particular way, then it is clear that this motion would induce a motion of the same period in the rest of the rod. These induced oscillations are called forced oscillations. One very important distinction between free and *forced* oscillations is this; although there are an infinite number of possible free oscillations yet the difference between the frequencies of any two modes is finite; whereas the frequency of a forced oscillation is always the same as that of the disturbing force, and can therefore have any magnitude whatever.

158. Any free motion is a combination of normal modes.

When a rod is set in motion by a blow, or by being bent and then let go, it is very unlikely that it will begin to oscillate in one of the normal modes. Suppose, for example, that the clamped-free rod that we dealt with first were bent by a force acting at the free end and then let go from rest, it would not then begin to oscillate in the first normal mode. It could only oscillate in this mode if the curve into which it were bent at the start were the same curve as the rod assumes in one extreme position in the first normal mode. But clearly the curve represented by the coefficient of $\sin(pt+\alpha)$ in (9.32) is not the same as the curve due to a load on one end of a beam which is clamped at the other end. The one equation involves hyperbolic and circular functions of x, and the other is an algebraic equation involving powers of x up to the cube. It can be shown, however, that the subsequent motion is composed of a number of normal modes of which the first is by far the most important.

Suppose a rod is oscillating freely, and suppose that, at any instant (at which we shall assume that $t=0$) the curve of the rod and the velocity of each point are given by

$$y = \mathrm{F}(x) \qquad \ldots \ldots \ldots (9.83)$$

and

$$\frac{\partial y}{\partial t} = f(x), \qquad \ldots \ldots \ldots (9.84)$$

the two functions $\mathrm{F}(x)$ and $f(x)$ being any physically possible functions. We shall show how to represent the subsequent motion by means of a combination of normal modes of the rod.

At clamped, pinned, or free ends of a rod one of the following pairs of conditions is usually true.

$$y=0,\ \frac{\partial y}{\partial x}=0; \qquad \ldots \ldots \ldots (9.85)$$

$$y=0,\ \frac{\partial^2 y}{\partial x^2}=0; \qquad \ldots \ldots (9.86)$$

$$\frac{\partial^2 y}{\partial x^2}=0,\ \frac{\partial^3 y}{\partial x^3}=0; \qquad \ldots \ldots (9.87)$$

We shall assume for the present that one of these pairs of conditions is true at either end of the rod we are considering. We shall deal with exceptional cases later.

The case of the rod pinned at both ends gives rise, as we have found, to only circular functions. We shall consider this case separately because it is the simplest, and can therefore usefully lead up to the harder cases. One of the normal modes in this case is represented, as in (9.42), by

$$y = B_n \sin \frac{n\pi x}{l} \sin (p_n t + \alpha_n) \quad . \quad . \quad . \quad . \quad . \quad (9.88)$$

where n is an integer, B_n and α_n are arbitrary constants, and p_n is given by

$$p_n = \frac{n^2\pi^2 k}{l^2}\left(\frac{gE}{w}\right)^{\frac{1}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9.89)$$

Differentiating both sides of (9.88) with respect to t we get

$$\frac{\partial y}{\partial t} = p_n B_n \sin \frac{n\pi x}{l} \cos (p_n t + \alpha_n) \quad . \quad . \quad . \quad . \quad (9.90)$$

Now if we write y_n for the value of y given by (9.88) and then put

$$y = y_1 + y_2 + y_3 + \ldots \text{ to } \infty \quad . \quad . \quad . \quad . \quad . \quad (9.91)$$

we find, when $t = 0$, that

$$y = B_1 \sin \alpha_1 \sin \frac{\pi x}{l} + B_2 \sin \alpha_2 \sin \frac{2\pi x}{l} + \ldots \text{ to } \infty \quad . \quad . \quad (9.92)$$

and $$\frac{\partial y}{\partial t} = p_1 B_1 \cos \alpha_1 \sin \frac{\pi x}{l} + p_2 B_2 \cos \alpha_2 \sin \frac{2\pi x}{l} + \ldots \text{ to } \infty \quad . \quad (9.93)$$

The expressions on the right-hand sides of equations (9.92) and (9.93) are Fourier series, and it is possible, as Fourier has shown, to determine the coefficients so as to make each of these series represent any given single-valued continuous function of x such as $F(x)$ and $f(x)$ in (9.83) and (9.84) must be. That is, we may put

$$F(x) = B_1 \sin \alpha_1 \sin \frac{\pi x}{l} + B_2 \sin \alpha_2 \sin \frac{2\pi x}{l} + \quad . \quad . \quad . \quad (9.94)$$

$$f(x) = p_1 B_1 \cos \alpha_1 \frac{\sin \pi x}{l} + p_2 B_2 \cos \alpha_2 \sin \frac{2\pi x}{l} + \quad . \quad . \quad (9.95)$$

To determine the coefficient of $\sin \dfrac{n\pi x}{l}$ in equation (9.94) multiply both sides of the equation by $\sin \dfrac{n\pi x}{l}$ and integrate from 0 to l. Thus

$$\int_0^l F(x) \sin \frac{n\pi x}{l}\, dx = B_n \sin \alpha_n \int_0^l \sin^2 \frac{n\pi x}{l}\, dx \quad . \quad . \quad . \quad (9.96)$$

the other terms on the right vanishing because

$$\int_0^l \sin\frac{n\pi x}{l}\sin\frac{r\pi x}{l}dx = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (9.97)$$

if n is not equal to r.

Also $$\int_0^l \sin^2\frac{n\pi x}{l}dx = \int_0^l \frac{1}{2}\left\{1 - \cos\frac{2n\pi x}{l}\right\}dx$$

$$= \frac{1}{2}\left[x - \frac{l}{2n\pi}\sin\frac{2n\pi x}{l}\right]_0^l$$

$$= \tfrac{1}{2}l \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9.98)$$

Therefore (9.96) gives

$$B_n \sin\alpha_n = \frac{2}{l}\int_0^l F(x)\sin\frac{n\pi x}{l}dx \quad . \quad . \quad . \quad . \quad (9.99)$$

which determines $B_n \sin\alpha_n$ because $F(x)$ is known.

In the same way (9.95) gives

$$p_n B_n \cos\alpha_n = \frac{2}{l}\int_0^l f(x)\frac{\sin n\pi x}{l}dx \quad . \quad . \quad . \quad . \quad (9.100)$$

The two equations (9.99) and (9.100) determine the two arbitrary constants B_n and α_n; and by putting $n = 1, 2, 3$, etc. in turn, all the constants in (9.92) are determined. Thus y is known completely at any time t and at any place x.

As a particular case suppose the rod pinned at both ends is bent into the form of the parabola

$$y = cx(l - x) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9.101)$$

and let go from rest (a pure couple at each end would bend the rod into the initial state). Then the functions $F(x)$ and $f(x)$ are

$$F(x) = cx(l - x) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9.102)$$

and $$f(x) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9.103)$$

Therefore

$$B_n \sin\alpha_n = \frac{2}{l}\int_0^l cx(l - x)\sin\frac{n\pi x}{l}dx$$

Now integration by parts gives

$$\int x(l - x)\sin\frac{n\pi x}{l}dx = -\frac{l}{n\pi}x(l - x)\cos\frac{n\pi x}{l}$$

$$+\frac{l^2}{n^2\pi^2}(l - 2x)\sin\frac{n\pi x}{l} - \frac{2l^3}{n^3\pi^3}\cos\frac{n\pi x}{l}$$

whence it follows that

$$B_n \sin\alpha_n = \frac{2c}{l}\frac{2l^3}{n^3\pi^3}\{1 - \cos n\pi\} \quad . \quad . \quad . \quad . \quad (9.104)$$

since all the other terms disappear at the limits.

Putting $n = 1, 2, 3$, in turn we find

$$\left.\begin{aligned} B_1 \sin \alpha_1 &= \frac{8cl^2}{\pi^3} \\ B_2 \sin \alpha_2 &= 0 \\ B_3 \sin \alpha_3 &= \frac{8cl^2}{3^3\pi^3} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (9.105)$$

Again, since $f(x) = 0$, equation (9.100) gives

$$B_n \cos \alpha_n = 0.$$

Since B_n is not zero this last equation is satisfied by taking

$$\alpha_n = \frac{\pi}{2} \cdot$$

Then, since every α is $\frac{\pi}{2}$ it follows that $\sin \alpha_n$ is unity for all values of n. Therefore the B's are determined completely by the equations (9.105). Also

$$\sin(p_n t + \alpha_n) = \sin\left(p_n t + \frac{\pi}{2}\right) = \cos p_n t.$$

The value of y at any time t after the rod is released is, by (9.92),

$$y = \frac{8cl^2}{\pi^3}\left\{\sin\frac{\pi x}{l}\cos p_1 t + \frac{1}{3^3}\sin\frac{3\pi x}{l}\cos p_3 t + \frac{1}{5^3}\sin\frac{5\pi x}{l}\cos p_5 t + \right\} \quad (9.106)$$

where the p's are given by (9.89).

Thus the complete motion is composed of an infinite number of oscillations of different periods, but the eye would probably only notice the first one, which has an amplitude twenty-seven times as great as the second, and one hundred and twenty-five times as great as the third. The real reason why the first mode preponderates so much over the other modes is because the curve into which the rod was bent at the start differs very little from the curve of the first mode, namely

$$y = \frac{8cl^2}{\pi^3}\sin\frac{\pi x}{l}$$

159. Rod with ends pinned, clamped, or free.

We return now to the general problem of the analysis into normal modes of the motion of a rod whose conditions at the start, i. e. when $t = 0$, are given by (9.83) and (9.84), and the ends of which are subject to any one of the pairs of conditions (9.85), (9.86), or (9.87). There may be the same, or a different, pair of conditions at the two ends.

The normal modes are represented by equations of the type

$$y_n = R_n u_n \sin(p_n t + \alpha_n) \quad . \quad . \quad . \quad . \quad . \quad . \quad (9.107)$$

where u_n is a function of x which satisfies the equation

$$\frac{d^4 u_n}{dx^4} = m_n^4 u_n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9.108)$$

at all points of the rod, and satisfies the four end-conditions of the rod. These four end-conditions do not, as we have found, determine the four constants in u_n; they determine the three ratios among the four constants and the equation obtained by eliminating these ratios determines m_n, and therefore also p_n, since m involves p. There is thus one constant left undetermined in y_n and this is represented by R_n in (9.107).

Let u_n and u_r be a pair of the normal functions for the rod we are considering. Let us write, for shortness, D for $\frac{d}{dx}$, D^2 for $\frac{d^2}{dx^2}$ etc. Then the equations for u_n and u_r are

$$D^4 u_n = m_n^4 u_n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9.109)$$

$$D^4 u_r = m_r^4 u_r \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9.110)$$

Now multiplying both sides of (9.109) by u_r and then integrating by parts we get

$$\begin{aligned} m_n^4 \int u_n u_r dx &= \int u_r D^4 u_n dx \\ &= u_r D^3 u_n - D u_r D^2 u_n + D^2 u_r D u_n - u_n D^3 u_r + \int u_n D^4 u_r dx \end{aligned}$$

Now by means of (9.110) the last equation becomes

$$(m_n^4 - m_r^4) \int u_n u_r dx = u_r D^3 u_n - D u_r D^2 u_n + D^2 u_r D u_n - u_n D^3 u_r \quad (9.111)$$

If we take this integral over the whole extent of the rod then

$$(m_n^4 - m_r^4) \int u_n u_r dx = 0 \quad . \quad . \quad . \quad . \quad . \quad (9.112)$$

because one pair of the conditions (9.85), (9.86), (9.87), holds at each end, and those conditions are equivalent to

$$\left.\begin{aligned} u = 0, \quad D u &= 0; \\ u = 0, \quad D^2 u &= 0; \\ D^2 u = 0, \quad D^3 u &= 0; \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (9.113)$$

these equations being true whether we put u_r or u_n for u. It will be found, on examining these conditions, that one factor of each term on the right of (9.111) is zero whichever of the three pairs of conditions in (9.113) is true.

It now follows from (9.112) that, if n and r are not equal,

$$\int u_n u_r dx = 0,$$

the integral being taken from one end of the rod to the other.

It will be convenient to assume that the origin is at one end of the rod, though it makes no difference to the following argument whether it is or not. Then our last result can be written

$$\int_0^l u_n u_r dx = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9.114)$$

Now let

$$y = R_1 u_1 \sin(p_1 t + \alpha_1) + R_2 u_2 \sin(p_2 t + \alpha_2) + \text{etc.}, \quad . \quad . \quad (9.115)$$

and let us assume that (9.83) and (9.84) are true when $t = 0$. Then

$$F(x) = R_1 u_1 \sin\alpha_1 + R_2 u_2 \sin\alpha_2 + \ldots + R_n u_n \sin\alpha_n + \quad . \quad . \quad (9.116)$$

and $f(x) = p_1 R_1 u_1 \cos\alpha_1 + p_2 R_2 u_2 \cos\alpha_2 + \ldots + p_n R_n u_n \cos\alpha_n +$ (9.117)

Next multiply both sides of (9.116) by u_r and integrate from 0 to l. Then, making use of (9.114), we get

$$\int_0^l u_r F(x)\,dx = \int_0^l R_r u_r^2 \sin\alpha_r\,dx\,;$$

that is,

$$R_r \sin\alpha_r \int_0^l u_r^2\,dx = \int_0^l u_r F(x)\,dx \quad . \quad . \quad . \quad . \quad (9.118)$$

Since u_r and $F(x)$ are known functions of x this equation gives $R_r \sin\alpha_r$. In the same way

$$p_r R_r \cos\alpha_r \int_0^l u_r^2\,dx = \int_0^l u_r f(x)\,dx, \quad . \quad . \quad . \quad (9.119)$$

which determines $R_r \cos\alpha_r$. The two equations (9.118) and (9.119) together determine both the constants R_r and α_r. Moreover, we can find α_r without integrating u_r^2, since an equation for tan α_r is obtained by dividing corresponding sides of (9.118) and (9.119). In particular, if $F(x)$ is zero for all values of x between 0 and l, then $\tan\alpha_r = 0$ and therefore $\alpha_r = 0$; whereas if $f(x) = 0$ then $\tan\alpha_r = \infty$ and $\alpha_r = \frac{1}{2}\pi$.

When all the R's and all the α's are determined their values can be substituted in (9.115) and then y is completely determined.

The integral of u_n^2, which is required for the determination of the constants, can be obtained in every particular case by direct integration of the terms in u_n^2, but this process is laborious and it is easier to get a general result to cover all cases. This result can be deduced from (9.111) by an ingenious method which is given in the late Lord Rayleigh's "Theory of Sound". Although m_r and m_n in that equation are understood to represent values of m belonging to a pair of possible modes of oscillation, nevertheless the actual equation as it stands remains true whatever constant values the m's may have, because the only assumptions used in getting (9.111) were that

$$D^4 u_n = m_n^4 u_n,$$

and

$$D^4 u_r = m_r^4 u_r,$$

and, until we use the end-conditions of the rod, m_n and m_r can have any values we like. Then let

$$u_n = u$$
$$m_r = m + dm$$

where m is written for m_n. Also let us assume that the constants of integration in u_r are the same as those in u_n. Then to the first order in dm,

$$m_r^4 - m_n^4 = 4m^3 dm\,,$$

$$u_r = u + \frac{du}{dm}dm\,,$$

$$\mathrm{D}u_r = \mathrm{D}u + \frac{d}{dm}(\mathrm{D}u)dm\,,$$

etc.

Now
$$\frac{du}{dm} = x\frac{du}{d(xm)} = \frac{x}{m}\frac{du}{dx} = \frac{x}{m}\mathrm{D}u\,,$$

since u is a function of mx only.

Likewise

$$\frac{d}{dm}(\mathrm{D}u) = \frac{x}{m}\frac{d}{dx}(\mathrm{D}u) = \frac{x}{m}\mathrm{D}^2u\,,$$

and so on for the other differential coefficients. Therefore

$$u_r\mathrm{D}^3u_n - u_n\mathrm{D}^3u_r = \left(u + \frac{x}{m}\mathrm{D}u\,dm\right)\mathrm{D}^3u - u\left(\mathrm{D}^3u + \frac{x}{m}\mathrm{D}^4u\,dm\right)$$
$$= \frac{x}{m}dm(\mathrm{D}u\mathrm{D}^3u - u\mathrm{D}^4u);$$

and in the same way

$$\mathrm{D}^2u_r\mathrm{D}u_n - \mathrm{D}u_r\mathrm{D}^2u_n = \frac{x}{m}dm(\mathrm{D}^3u\mathrm{D}u - \mathrm{D}^2u\mathrm{D}^2u)$$

The substitution of the preceding results in (9.111) gives, when we divide by dm and then make dm approach zero,

$$-4m^3\int u^2dx = \frac{x}{m}\left\{\mathrm{D}u\mathrm{D}^3u - u\mathrm{D}^4u + \mathrm{D}^3u\mathrm{D}u - (\mathrm{D}^2u)^2\right\},$$

or
$$\int u^2dx = \frac{x}{4m^4}\left\{u\mathrm{D}^4u - 2\,\mathrm{D}u\mathrm{D}^3u + (\mathrm{D}^2u)^2\right\}$$

Thus we find

$$\int_0^l u^2\,dx = \frac{l}{4m^4}\left\{u\mathrm{D}^4u - 2\mathrm{D}u\mathrm{D}^3u + (\mathrm{D}^2u)^2\right\}_{x=l}\,, \quad (9.120)$$

the quantity in brackets on the right having its value at the end $x=l$. We may, of course, substitute for D^4u its value m^4u. If the end $x=l$ is held by a smooth pin then $u=0$ and $\mathrm{D}^2u=0$ at that end, so that

$$\int_0^l u^2\,dx = -\frac{l}{2m^4}(\mathrm{D}u\mathrm{D}^3u)_{x=l} \quad . \quad . \quad . \quad . \quad (9.121)$$

If the end $x = l$ is clamped then $u = 0$ and $Du = 0$, so that

$$\int_0^l u^2 dx = \frac{l}{4m^4}(D^2 u)_{x=l} \quad . \quad . \quad . \quad . \quad . \quad (9.122)$$

If the end $x = l$ is free then $D^2 u = 0$ and $D^3 u = 0$ so that

$$\int_0^l u^2 dx = \tfrac{1}{4} l (u^2)_{x=l} \quad . \quad . \quad . \quad . \quad . \quad . \quad (9.123)$$

Since the origin can be taken indifferently at either end of the rod we may take the end $x = l$ to be that end which suits our convenience, and the result in (9.120) remains the same for either end. It is a point of interest that the expression in the brackets in (9.120) must have the same value at both ends of any rod.

160. Clamped-free rod struck at the free end.

As an example to show the method of using the preceding analysis we shall find the motion resulting from a blow given at the free end of a clamped-free rod when the rod is at rest. In this case $F(x) = 0$ at all points of the rod, and $f(x)$ is zero everywhere except near the free end where it is very great. Therefore equation (9.118) gives

$$R_r \sin \alpha_r = 0$$

whence

$$\alpha_r = 0$$

Then (9.119) gives

$$p_r R_r \int_0^l u_r^2 dx = \int_0^l u_r f(x) dx$$

Let us suppose that the range within which $f(x)$ is not zero is from $x = l - \varepsilon$ to $x = l$, and let u'_r be written for the value of u_r at $x = l$. Then, by means of (9.123), the equation for R_r is

$$p_r R_r \times \tfrac{1}{4} l u'^2_r = \int_{l-\varepsilon}^l u_r f(x) dx$$

$$= u'_r \int_{l-\varepsilon}^l f(x) dx$$

$$= N u'_r ,$$

where N is written for the value of the integral of $f(x)$. The reason why u_r can be taken from under the integral sign is because u_r is practically constant in the small range from $(l - \varepsilon)$ to l.

Finally

$$R_r = \frac{4N}{l p_r u'_r}$$

Therefore the value of y when t seconds have elapsed since the blow was struck is

$$y = y_1 + y_2 + y_3 + \dots$$
$$= \frac{4N}{l}\left\{\frac{1}{p_1}\frac{u_1}{u'_1}\sin p_1 t + \frac{1}{p_2}\frac{u_2}{u'_2}\sin p_2 t + \dots\right\},$$

the series having an infinite number of terms. If we write t_1, t_2, etc., for the periods of the normal modes we can put y in the form

$$y = \frac{2N}{\pi l}\left\{t_1 \frac{u_1}{u'_1}\sin p_1 t + t_2 \frac{u_2}{u'_2}\sin p_2 t + \dots\right\}$$

which is a convergent series because the periods, after the first, are approximately proportional to $\frac{1}{3^2}, \frac{1}{5^2}, \frac{1}{7^2}$, etc., which are the terms of a convergent series, and $\frac{u_r}{u'_r}$ is not greater than unity. The position of the free end of the rod is obtained by putting $u_r = u'_r$. Thus at the free end

$$y = \frac{2N}{\pi l}\{t_1 \sin p_1 t + t_2 \sin p_2 t + \dots\}$$

It follows from equations (9.28), (9.29), (9.31), that the ratio of the amplitude of the first to that of the second mode at the free end of the rod is

$$\frac{t_1}{t_2} = \left(\frac{3\pi}{2 \times 1{\cdot}875}\right)^2 = 6{\cdot}32.$$

Also the ratio of the amplitude of the first to that of the third mode is

$$\frac{t_1}{t_3} = \left(\frac{5\pi}{2 \times 1{\cdot}875}\right)^2 = 17{\cdot}53 \quad .$$

These numerical values give an idea of the relative importance of the different terms in the expression for y.

161. The case of the rod carrying a weight at a free end. The conditions (9.113) do not cover the case of the rod clamped at one end and free at the other where it carries a load. The pair of conditions at the loaded end have been shown to be

$$D^2 u = 0 \text{ and } D^3 u = -m^4 h u,$$

where h is the same constant for all modes. For this case (9.111) gives, the load being assumed to act at $x = l$.

$$(m_n^4 - m_r^4)\int_0^l u_n u_r dx = (u_r D^3 u_n - u_n D^3 u^r)_{x=l}$$
$$= -(m_n^4 - m_r^4) h (u_n u_r)_{x=l} \quad . \quad (9.124)$$

If, as before, we used dashed letters to indicate values at the end $x = l$, then

$$\int_0^l u_n u_r dx = -h u'_n u'_r \quad . \quad . \quad . \quad . \quad . \quad (9.125)$$

Then, still using the form for y in (9.115), equation (9.118) is replaced by

$$\int_0^l u_r \mathrm{F}(x)\,dx = -hu'_r\left\{\mathrm{R}_1 u'_1 \sin\alpha_1 + \mathrm{R}_2 u'_2 \sin\alpha_2 + \ldots\right\} + \mathrm{R}_r \sin\alpha_r \int_0^l u_r^2\,dx,$$

the term involving u_r being omitted from the bracket. Then

$$\int_0^l u_r \mathrm{F}(x)\,dx = -hu'_r\left\{\mathrm{F}(l) - \mathrm{R}_r u'_r \sin\alpha_r\right\} + \mathrm{R}_r \sin\alpha_r \int_0^l u_r^2\,dx,$$

whence

$$\mathrm{R}_r \sin\alpha_r\left\{\int_0^l u_r^2\,dx + hu'^2_r\right\} = \int_0^l u_r \mathrm{F}(x)\,dx + hu'_r \mathrm{F}(l), \qquad (9.126)$$

which determines $\mathrm{R}_r \sin\alpha_r$ in terms of known functions and their integrals. In the same way we can get an equation for $\mathrm{R}_r \cos\alpha_r$ and thus find the values of all the R's and all the α's.

162. Oscillations of a beam under transverse forces.

We can prove that a beam on which transverse forces act that do not vary with time oscillates just as freely as if the forces did not act; but the mean position of the central line of the beam is the equilibrium position. For example, a horizontal beam is deflected by its own weight, and if it is set in oscillation we should find just the same motion relative to the equilibrium position as if we had ignored altogether the action of gravity while still taking account of the inertia of the beam. This follows very easily from the equations of motion as we shall now show.

Let the transverse force on a length dx of the beam be $f(x)dx$. Then it is easy to show that (9.9) is replaced by

$$\mathrm{EI}\frac{\partial^4 y}{\partial x^4} = -\frac{aw}{g}\frac{\partial^2 y}{\partial t^2} + f(x) \quad . \quad . \quad . \quad . \quad . \quad (9.127)$$

Now let

$$y = y_1 + y_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9.128)$$

where y_1 is a function of x, and not of t, which satisfies the equation

$$\mathrm{EI}\frac{d^4 y_1}{dx^4} = f(x) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9.129)$$

and the end-conditions of the beam. Thus y_1 is the deflection of the beam when there is no motion, that is, the ordinary equilibrium deflection of the beam. We can define y_2 by saying that it is $(y - y_1)$, y being the actual deflection during oscillation.

Differentiating both sides of (9.128) with respect to x four times we get

$$\frac{\partial^4 y}{\partial x^4}=\frac{d^4 y_1}{dx^4}+\frac{\partial^4 y_2}{\partial x^4} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9.130)$$

It must be clearly understood that y_2 is a function of both x and t, and therefore the symbol for partial differentiation is needed.

Again, from (9.128),

$$\frac{\partial^2 y}{\partial t^2}=\frac{\partial^2 y_2}{\partial t^2} . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9.131)$$

since y_1 is not a function of t. Therefore equation (9.127) becomes

$$\mathrm{EI}\frac{d^4 y_1}{dx^4}+\mathrm{EI}\frac{\partial^4 y_2}{\partial x^4}=-\frac{aw}{g}\frac{\partial^2 y_2}{\partial t^2}+f(x) \quad . \quad . \quad . \quad (9.132)$$

By equation (9.129) this reduces to

$$\mathrm{EI}\frac{\partial^4 y_2}{\partial x^4}=-\frac{aw}{g}\frac{\partial^2 y_2}{\partial t^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (9.133)$$

which is the same equation for y_2 as (9.9) is for y. That is, y_2, which is the displacement relative to the equilibrium position, is just the same as y would have been if the transverse force had been neglected altogether. Thus the modes of oscillation and the periods are all unaffected by gravity, or by any other force which is independent of time.

To take another example, suppose a fiddle bow is drawn transversely across a thin rod clamped at one end and free at the other. If the bow is drawn at a constant speed it may set up oscillations in which the maximum velocity at the point where the bow acts is equal to that of the bow, but does not exceed it. Then the friction is a constant force acting in the same direction throughout, and therefore comes under the class we have represented by $f(x)$ in (9.127). The friction then does not, as might be expected, reverse its direction and damp down the oscillations. It merely gives to the curve of the central line of the rod a new equilibrium position about which the oscillations take place, and it does not affect the periods of these oscillations.

163. An approximate method of finding the periods of oscillation.

The late Lord Rayleigh has shown (*Sound,* Vol. I Art. 89) that very good approximations to the periods of oscillation of a rod can be obtained by assuming any reasonable shape for the curve of the rod and using energy methods. We shall first of all deduce an accurate equation for the period, and then show that the approximate method is a good one.

Let u be one of the normal functions for a particular rod so that a possible oscillation is represented by

$$y=\mathrm{H}u \sin pt \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9.134)$$

Now from the principle of the conservation of energy it follows that the sum of the kinetic and potential energies of the rod is constant throughout the motion. But the energy is all potential at either end of the oscillation, and all kinetic at the middle of the oscillation. Then the preceding principle can be used in the following form: the kinetic energy when the velocity is a maximum is equal to the potential energy when y is a maximum Now.

$$\frac{\partial y}{\partial t}=p\mathrm{H}u\cos pt \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9.135)$$

Then the maximum value of the velocity at any point of the rod is

$$v=p\mathrm{H}u \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9.136)$$

and the maximum value of y is

$$y=\mathrm{H}u \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9.137)$$

The bending moment corresponding to the maximum y is

$$\mathrm{M}=\mathrm{EIH}\frac{d^2u}{dx^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9.138)$$

and the potential energy in this position is, by (8.11),

$$\mathrm{W}=\frac{1}{2}\int_0^l\frac{\mathrm{M}^2}{\mathrm{EI}}dx$$

$$=\frac{1}{2}\int_0^l\mathrm{EIH}^2\left(\frac{d^2u}{dx^2}\right)^2dx \quad . \quad . \quad . \quad . \quad . \quad . \quad (9.139)$$

Also the maximum kinetic energy is

$$\int_0^l\frac{1}{2}\frac{wadx}{g}v^2=\frac{1}{2}\int_0^l\frac{w}{g}ap^2\mathrm{H}^2u^2dx \quad . \quad . \quad . \quad . \quad (9.140)$$

By equating these two energies we arrive at the following equation for p

$$p^2\int_0^l\frac{w}{g}au^2dx=\int_0^l\mathrm{EI}\left(\frac{d^2u}{dx^2}\right)^2dx \quad . \quad . \quad . \quad . \quad (9.141)$$

In terms of m this equation becomes, for a uniform rod,

$$m^4l^4\int_0^lu^2dx=l^4\int_0^l\left(\frac{d^2u}{dx^2}\right)^2dx \quad . \quad . \quad . \quad . \quad (9.142)$$

When the normal function u is known this gives the period, and it is precisely the same equation for m^4l^4 as the earlier method in this chapter would give. But the really useful thing about this result is that the value of p which equation (9.141) gives is very little affected by the form of the function u provided that the curve it represents looks reasonably like the curve for the mode of oscillation that we are considering. Before offering any reason for the last statement let us try to find, by this method, the slowest period of the uniform rod clamped at both ends, and compare the result with the one obtained in Art. 154.

Let us assume that the beam clamped at both ends oscillates between the extreme positions given by

$$y = \pm Hx^2(l-x)^2$$

This is a reasonable curve since it is the curve that the rod does actually assume when it is held horizontally under a uniform load.

Then, since a factor H^2 would occur on each side of (9.142), we may drop the useless factor H and take

$$u = x^2(l-x)^2,$$

from which

$$\frac{d^2u}{dx^2} = 2l^2 - 12lx + 12x^2,$$

$$\int_0^l u^2 dx = \frac{l^9}{630}, \quad \ldots \quad (9.144)$$

and

$$\int_0^l \left(\frac{d^2u}{dx^2}\right)^2 dx = \frac{4l^5}{5} \quad \ldots \quad (9.145)$$

Therefore

$$m^4 l^4 = 504,$$

whence

$$ml = 4{\cdot}739 \quad \ldots \quad (9.146)$$

This should be compared with the result 4·730 obtained by the exact process in equation (9.54).

In order to put the method to a further test let us try the same problem with a different value of u. Let us assume that the rod, in its extreme positions, takes the form it would have if it were a strut. That is, let us take

$$u = 1 - \cos\frac{2\pi x}{l}$$

$$= 2\sin^2\frac{\pi x}{l} \quad \ldots \quad (9.147)$$

Then

$$\frac{d^2u}{dx^2} = \frac{4\pi^2}{l^2}\cos\frac{2\pi x}{l}$$

Therefore

$$\int_0^l u^2 dx = \int_0^l 4\sin^4\frac{\pi x}{l}\,dx \quad \ldots \quad (9.148)$$

To work out this integral put

$$\theta = \frac{\pi x}{l};$$

then

$$dx = \frac{l}{\pi}d\theta$$

and therefore

$$\int_0^l u^2 dx = 4\int_0^\pi \frac{l}{\pi}\sin^4\theta\,d\theta$$

Now it is proved in works on the integral calculus that

$$\int_0^{\frac{q\pi}{2}} \sin^n \theta d\theta = \frac{n-1}{n}\int_0^{\frac{q\pi}{2}} \sin^{n-2} \theta d\theta \quad . \quad . \quad . \quad . \quad (9.149)$$

if q is an integer.

Consequently, by two successive applications of this formula, we find that

$$\int_0^l u^2 dx = \frac{4l}{\pi} \times \frac{3}{4} \times \frac{1}{2}\int_0^{\pi} 1.d\theta$$

$$= \tfrac{3}{2}l \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9.150)$$

In the same way

$$\int_0^l \left(\frac{d^2u}{dx^2}\right)^2 dx = \frac{16\pi^4}{l^4}\int_0^l \cos^2 \frac{2\pi x}{l} dx$$

$$= \frac{8\pi^4}{l^3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9.151)$$

Now equation (9.142) gives

$$m^4 l^4 = \frac{16\pi^4}{3},$$

whence

$$ml = \frac{2\pi}{\sqrt{3}} = 4 \cdot 774 \quad . \quad . \quad . \quad . \quad . \quad (9.152)$$

which is still fairly near the truth, but not so good as the result obtained by assuming that u had the same form as when the rod was horizontal and deflected by its own weight.

A good result for the period of oscillation is usually obtained by assuming that u has the form it would have if the rod were a beam deflected by the actual loads that oscillate with the rod. It is not easy to give very convincing reasons why this method should be so accurate. Lord Rayleigh showed, however, that the frequency given by the approximate method must lie between the greatest and the least of the frequencies of the normal modes. If, therefore, we use the approximate method to find the frequency of the slowest mode, we are sure that the result we get will err by being too high. The proof is given below.

164. Proof of the principle for a thin rod.

Let the normal functions for the oscillating rod—which need not have a uniform section—be u_1, u_2, u_3, etc., and let the corresponding values of p be p_1, p_2, etc. Then it can be shown that any function $f(x)$ of x can be expanded in the form

$$f(x) = A_1u_1 + A_2u_2 + A_3u_3 + \ldots \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9.153)$$

When wa and EI are not constant the normal functions are such that

$$\int_0^l wa\, u_n u_m dx = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (9.154)$$

provided m and n are unequal. Also, by integration by parts,

$$\int_0^l \mathrm{EI}\frac{d^2u_n}{dx^2}\frac{d^2u_m}{dx^2}dx = \left[\mathrm{EI}\frac{d^2u_n}{dx^2}\frac{du_m}{dx} - u_m\frac{d}{dx}\left(\mathrm{EI}\frac{d^2u_n}{dx^2}\right)\right]_0^l + \int_0^l u_m\frac{d^2}{dx^2}\left(\mathrm{EI}\frac{d^2u_n}{dx^2}\right)dx$$

Now the integrated terms are zero at both limits whatever the end conditions are. Moreover, the differential equation for u_n is

$$\frac{d^2}{dx^2}\left(\mathrm{EI}\frac{d^2u_n}{dx^2}\right) = \frac{wa}{g}p_n^2 u_n \quad . \quad . \quad . \quad . \quad . \quad (9.155)$$

Therefore

$$\int_0^l \mathrm{EI}\frac{d^2u_n}{dx^2}\frac{d^2u_m}{dx^2}dx = p_n^2\int_0^l \frac{wa}{g}u_n u_m dx$$

$$= 0 \text{ by } (9.154) \quad . \quad . \quad . \quad . \quad (9.156)$$

Now the approximate method consists in replacing the single normal function u in (9.141) by $f(x)$, which is equivalent to the sum of a number of normal functions with arbitrary coefficients, as given by equation (9.153). Thus replacing u by $f(x)$ in (9.141) we get, as the approximate equation for p,

$$p^2\int_0^l \frac{wa}{g}\{\mathrm{A}_1{}^2u_1{}^2 + \mathrm{A}_2{}^2u_2{}^2 + \mathrm{A}_3{}^2u_3{}^2 + \ldots\}dx$$

$$= \int_0^l \mathrm{EI}\left\{\mathrm{A}_1{}^2\left(\frac{d^2u_1}{dx^2}\right)^2 + \mathrm{A}_2{}^2\left(\frac{d^2u_2}{dx^2}\right)^2 + \ldots\right\}dx \quad . \quad . \quad (9.157)$$

But equation (9.141) is accurate for a single mode; that is,

$$p_n^2\int_0^l \frac{wa}{g}u_n^2 dx = \int_0^l \mathrm{EI}\left(\frac{d^2u_n}{dx^2}\right)^2 dx \quad . \quad . \quad . \quad . \quad (9.158)$$

Therefore equation (9.157) becomes

$$p^2\int_0^l \frac{wa}{g}\{\mathrm{A}_1{}^2u_1{}^2 + \mathrm{A}_2{}^2u_2{}^2 + \ldots\}dx$$

$$= \int_0^l \frac{wa}{g}\{p_1{}^2\mathrm{A}_1{}^2u_1{}^2 + p_2{}^2\mathrm{A}_2{}^2u_2{}^2 + \ldots\}dx,$$

whence

$$p^2 = \frac{p_1{}^2\mathrm{A}_1{}^2\mathrm{U}_1{}^2 + p_2{}^2\mathrm{A}_2{}^2\mathrm{U}_2{}^2 + \ldots}{\mathrm{A}_1{}^2\mathrm{U}_1{}^2 + \mathrm{A}_2{}^2\mathrm{U}_2{}^2 + \ldots} \quad . \quad . \quad . \quad (9.159)$$

where

$$\mathrm{U}_n^2 = \int_0^l \frac{wa}{g}u_n^2 dx \quad . \quad . \quad . \quad . \quad . \quad . \quad (9.160)$$

It is now clear that p^2 lies between the greatest and least of the quantities p_1^2, p_2^2, etc. Then if p_1^2 is the least of these it is certain that p^2 is greater than p_1^2. Moreover, if A_1 is much greater than the other coefficients, then A_1^2 preponderates still more over A_2^2, A_3^2, etc. Consequently p^2 must be nearly equal to p_1^2. Now if $f(x)$ is a function of x obviously of the same type as u_1, then we may be confident that the coefficients A_2, A_3, etc., are much smaller than A_1. But the greatest confidence is gained by applying the method to cases where the result is known, for the accuracy of the results nearly always exceeds anything that could be expected.

165. The period of oscillation with several masses.

Lord Rayleigh's principle can be used to prove an approximate rule for finding the period of an elastic body when it carries several masses, provided the period is known for each separate mass.

Let V denote the potential energy of the elastic forces during oscillation, and T the kinetic energy of the oscillating system. Let us suppose that the system is oscillating in one of its normal modes. Let m_1, m_2, m_3, etc. be the oscillating masses, and let the displacements of these masses in the actual oscillation be y_1, y_2, y_3, etc. Then the kinetic energy is

$$T=\tfrac{1}{2}m_1\dot{y}_1{}^2+\tfrac{1}{2}m_2\dot{y}_2{}^2+\tfrac{1}{2}m_3\dot{y}_3{}^2+\ldots \quad (9.161)$$

the dots indicating differentiation with respect to time.

Now since all the particles are oscillating in the same period and same phase there must be a common factor cos pt in all the y's. Thus

$$y_1=b_1\cos pt;\quad y_2=b_2\cos pt;\quad \text{etc.}$$

Therefore

$$T=\left\{\tfrac{1}{2}m_1b_1{}^2+\tfrac{1}{2}m_2b_2{}^2+\tfrac{1}{2}m_3b_3{}^2+\ldots\right\}p^2\sin^2 pt \quad (9.162)$$

Moreover, the potential energy being proportional to the square of the displacements, which are proportional to $\cos pt$, it can be written in the form

$$V=c^2\cos^2 pt, \quad (9.163)$$

the factor c^2 depending on the mode of the oscillation; in the case of a vibrating rod c^2 depends on the form of the curve of the rod in the position of maximum displacement.

Thus the energy equation, namely

$$T+V=\text{const.}$$

becomes

$$\tfrac{1}{2}\left\{m_1b_1{}^2+m_2b_2{}^2+\ldots\right\}p^2\sin^2 pt+c^2\cos^2 pt=\text{const} \quad (9.164)$$

Since the constant cannot be a function of t the coefficients of $\cos^2 pt$ and $\sin^2 pt$ must be equal, in which case the left hand side of the last equation reduces to c^2. Therefore

$$\tfrac{1}{2}p^2\left\{m_1b_1{}^2+m_2b_2{}^2+m_3b_3{}^2+\ldots\right\}=c^2. \quad (9.165)$$

Now let us suppose that the form of the curve in the actual oscillation. is not very different from the form of the curve for one of the modes of oscillation when m_1 alone is attached to the elastic body. In that case we know from Rayleigh's rule that we can get quite a good approximation to the period for the mass m_1 alone by using the same curve as we have used in arriving at equation (9.165). But the potential energy is exactly the same whether one or several masses is attached. Therefore, if p_1 is the value of p when m_1 alone is oscillating, equation (9.165) gives

$$\tfrac{1}{2}p_1{}^2 m_1 b_1{}^2 = c^2 \text{ approximately},$$

whence

$$\tfrac{1}{2}m_1 b_1{}^2 = \frac{c^2}{p_1{}^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9.166)$$

Likewise, if the mode when m_2 alone is ocillating is not very different from the actual mode when all the masses oscillate, then we get

$$\tfrac{1}{2}m_2 b_2{}^2 = \frac{c^2}{p_2{}^2}$$

If the same is true of all the masses equation (9.165) gives, after division by p^2c^2,

$$\frac{1}{p_1{}^1} + \frac{1}{p_2{}^2} + \frac{1}{p_3{}^2} + \ldots = \frac{1}{p^2}.$$

Writing t_1, t_2, etc. for the periods of oscillation corresponding to p_1, p_2, . . . we find from the last equation

$$t_1{}^2 + t_2{}^2 + t_3{}^2 + \ldots = t^2 \quad . \quad . \quad . \quad . \quad . \quad (9,167)$$

This is the approximate equation for the period t when all the masses are attached in terms of then periods when the several masses oscillate alone.

It is as well to recall the assumptions from which equation (9.167) is deduced. It is assumed that the actual shape of the curve when all the masses are attached is nearly the same as when each one oscillates separately. Now although this assumption may be very much wrong for some of. the attached masses it must be very nearly right for the masses which have most kinetic energy during the motion, for these are the masses that have most control over the motion when all the masses oscillate together. Moreover these are the masses for which the period is greatest, and therefore they contribute the largest terms to the left hand side of (9.167). Since then the assumptions we have made are nearly true for the largest terms in the expression for t it follows that the error in t cannot be very big.

Since the approximate equation (9.166) always gives a value for $p_1{}^2$ greater than the true value it follows that, if we use the true values of t_1, t_2, etc., in (9.167), we shall get a value of t which is greater than the correct value. Equation (9.167) can also be used whether each of the periods t_1, t_2, etc., is due to a system of masses or due to a single mass. Thus, for example, if t_1 is the period of a rod under its own mass, and t_2 the period of the same rod (regarded as having no mass itself) when a mass M is attached at some point, then the period when both masses are taken into account is approximately $\sqrt{t_1^2+t_2^2}$. Let us apply this to the rod with a mass at one end, the case shown in fig. 87. Equation (9.80) gives the period of the slowest mode when the mass of the rod is neglected, and equations (9.31) and (9.29) give the corresponding period when the mass at the end is neglected. Denoting these by t_2 and t_1 respectively, our present method gives

$$t^2=t_1^2+t_2^2=t_1^2\left\{1+\frac{t_2^2}{t_1^2}\right\}$$

$$=t_1^2\left\{1+\frac{1\cdot875^4}{3}\frac{\mathrm{W}}{wal}\right\}$$

Now let z denote ml, as in the example that we are quoting, and take $\mathrm{W}=wal$; then we can write the last equation thus

$$\frac{1}{z^4}=\frac{1}{z_1^4}\left\{1+\frac{z_1^4}{3}\right\}$$

Therefore $$z=z_1\left\{1+4\cdot11\right\}^{-\frac{1}{4}}=1\cdot25 \quad . \quad . \quad . \quad (9.168)$$

A more accurate value of z for this case is given in (9.76) as 1·238, which differs by only one per cent from the result in (9.168).

166. The whirling of shafts.

Suppose a shaft, whose section is a complete or a hollow circle, rotates in bearings, either carrying no loads but its own weight or carrying loads such as pulleys. There are certain speeds of rotation at which the straight form of the shaft becomes unstable, just as a strut becomes unstable for certain values of the end-thrusts. The theory concerning this instability is not yet very satisfactorily worked out, although the conclusions from the usually accepted theory are probably correct in the main. The following theory, up to the point where the tension is taken into account, follows the conventional lines.

Let us suppose that, by some means or other, the central line takes for a moment the form of a plane curve. The problem is really to determine the subsequent behaviour of the shaft. This problem in all its completeness is never worked out. The assumption is made that the plane of the curve rotates at the same speed as the shaft, and that therefore, if the shaft oscillates, its motion is confined to a plane which rotates at the same speed as the shaft itself. This cannot, however, be strictly true, for it is not difficult to see that it is possible for the plane of the curve to remain fixed while the shaft rotates.

Consider, for example, the curve into which the weight of the shaft itself, (assuming the shaft to be horizontal) bends the central line. If the speed of the rotation is slow, it is quite certain that this curve remains very nearly in a vertical plane. In this kind of motion, however, there is a small relative motion of the particles of the shaft since each line of fibres parallel to the axis, except those along the axis itself, is alternately stretched and contracted. The viscosity due to this relative motion tends to carry the curve round with the shaft. We shall assume then that, when the speed is constant, the shaft settles into a state in which there is no relative motion of the particles; that is, the shaft either remains straight or the central line takes the form of an unchanging curve in a plane which rotates at the same speed as the shaft itself. This last statement needs qualifying if a torque is being transmitted along the shaft, for, in that case, the curve is not a plane curve.

167. Steady motion of the shaft in a plane curve.

Let the angular velocity of the shaft be ω radians per second, and let the equation to the curve of the central line be

$$y = f(x)$$

the curve being assumed to rotate with angular velocity ω also. The element of mass $\frac{wa}{g}dx$ situated at (x, y) has an acceleration yw^2 towards the x-axis. The product of the mass and this acceleration *reversed* may be treated as a load on the shaft, which product is usually called *centrifugal force.* Then the problem is the same as a beam problem with a lateral load $\frac{wa}{g}y\omega^2 dx$ on the length dx; that is $\frac{wa}{g}y\omega^2$ per unit length. Then the equation of relative equilibrium is,

$$\mathrm{EI}\frac{d^4y}{dx^4} = \frac{wa\omega^2}{g}y \quad \ldots\ldots\ldots \quad (9.169)$$

This last equation is correct on the assumption that the centre of gravity of each element of the shaft was absolutely on the x-axis when the shaft was unstrained. This, however, is not possible, for it is beyond human capacity to make an absolutely straight line or absolutely homogeneous material. Let us assume then that the centres of gravity of the different elements lay on a curve in the unstrained state, and let the projection of this curve on the plane of the rotating curve be

$$y_1 = \mathrm{F}(x) \quad \ldots\ldots\ldots \quad (9.170)$$

Then the elastic righting force depends on $(y - y_1)$ and not on y, so that the correct equation of relative equilibrium is,

$$\mathrm{EI}\frac{d^4(y - y_1)}{dx^4} = \frac{wa\omega^2}{g}y \quad \ldots\ldots\ldots \quad (9.171)$$

Now it is possible to expand $F(x)$ in terms of the normal functions for the shaft we are dealing with; that is, in terms of the functions which occur in the transverse oscillation of the shaft with the actual end-conditions of the shaft. If these normal functions are u_1, u_2, u_3, etc., then

$$y_1 = C_1 u_1 + C_2 u_2 + C_3 u_3 + \dots \quad (9.172)$$

To simplify the problem let us assume that only one of these normal functions occurs in y_1. Thus

$$y_1 = C_n u_n \quad (9.173)$$

where u_n is a function which satisfies the equation for transverse oscillations, namely,

$$EI \frac{d^4 u_n}{dx^4} = \frac{w a p_n^2}{g} u_n, \quad (9.174)$$

as well as the end-conditions of the shaft, the constant p_n being also determined by these end-conditions.

Equation (9.171) now becomes

$$EI \frac{d^4 y}{dx^4} = \frac{w a \omega^2}{g} y + \frac{w a p_n^2}{g} C_n u_n \quad (9.175)$$

We may write this, after dividing by EI, in the form

$$\frac{d^4 y}{dx^4} = s^4 y + m_n^4 C_n u_n \quad (9.176)$$

This is a linear equation the solution of which consists of the sum of several terms, one of which is due to the term containing u_n and is called the particular integral. Let us first find this particular integral. A particular integral is any value of y which makes one side of equation (9.160) identical with the other. Let us try

$$y = H u_n \quad (9.177)$$

Then equation (9.176) gives

$$m_n^4 H u_n = s^4 H u_n + m_n^4 C_n u_n \quad (9.178)$$

which is an identity provided that

$$H = \frac{m_n^4}{m_n^4 - s^4} C_n \quad (9.179)$$

Thus part of the value of y is

$$y = \frac{m_n^4}{m_n^4 - s^4} C_n u_n \quad (9.180)$$

If now $s^4 = m_n^4$, that is, if

$$\omega^2 = p_n^2 \quad (9.181)$$

then the value of y is infinite. This means that the deflection of the shaft will become very great if the period of rotation of the shaft happens to be nearly coincident with the period of oscillation in the n^{th} mode.

If we take the general expression for y_1 in terms of all the modes the complete particular integral of (9.171) is

$$y = \frac{m_1^4}{m_1^4 - s^4} C_1 u_1 + \frac{m_2^4}{m_2^4 - s^4} C_2 u_2 + \dots \quad . \quad . \quad (9.182)$$

so that y will become very great when the period of rotation nearly coincides with any of the possible periods of oscillation of the shaft when there is no rotation. Any one of these critical speeds of rotation is called a whirling speed for the shaft. If a shaft continued to rotate at or near one of these whirling speeds, and if the rigidity were the only righting force, the deflection of the shaft would go on increasing until rupture occurred. It should be observed that the term in equation (9.182) containing $C_1 u_1$ has the same sign as $C_1 u_1$ as long as s^4 is less than m_1^4, that is, as long as ω^2 is less than p_1^2; but when ω^2 is greater than p_1^2 then the term has the opposite sign from $C_1 u_1$. A similar statement is true for any other of the terms in equation (9.182). When ω^2 is very great compared with p_1^2 then the part of y containing u_1 during rotation is very small and has the opposite sign from the corresponding part when there is no rotation. Thus a high speed of rotation, provided ω does not nearly coincide with one of the values of p, tends to straighten out an originally crooked shaft, and what little deflection remains is in the direction contrary to the natural deflection.

168. The effect of tension on a rod vibrating transversely.

Suppose a uniform rod is attached by smooth pins at its ends to two absolutely rigid bodies, so that when the rod is loaded transversely or oscillates transversely the central line has to extend because the pins remain at a fixed distance l apart. Let us consider how the transverse oscillations of this rod are affected by the tension.

It has been proved in Art. 95 that the central line has its length increased by

$$\tfrac{1}{2}\int_0^l \left(\frac{dy}{dx}\right)^2 dx$$

when this line is bent into a curve. The coordinates of a point on this curve are (x, y), referred to an x-axis through the ends and a y-axis through one end of the rod. Taking a as the area of the cross-section the total tension across any section of the rod is

$$P = \frac{Ea}{2l}\int_0^l \left(\frac{dy}{dx}\right)^2 dx \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad 9.183)$$

This tension P acts at both ends of the rod and may be regarded as acting along the x-axis. The rod is therefore in the condition of a tie rod with the forces due to inertia as lateral loads. Let M denote the whole bending moment at x, and M_1 the bending moment due to the accelerations and to the lateral forces at the ends. Then, since the bending moment due to the force P at either end is Py, we get

$$M = Py + M_1$$

Therefore, differentiating twice with respect to x, we get

$$\frac{\partial^2 M}{\partial x^2} = P\frac{\partial^2 y}{\partial x^2} + \frac{\partial^2 M_1}{\partial x^2} \quad . \quad . \quad . \quad . \quad . \quad (9.184)$$

But, since the product of the acceleration and the mass of unit length is $\frac{wa}{g}\frac{\partial^2 y}{\partial t}$, and since this vector reversed may be treated as a force, it follows that

$$\frac{\partial^2 M_1}{\partial x^2} = -\frac{wa}{g}\frac{\partial^2 y}{\partial t^2}$$

Therefore equation (9.184) becomes

$$EI\frac{\partial^4 y}{\partial x^4} = P\frac{\partial^2 y}{\partial x^2} - \frac{wa}{g}\frac{\partial^2 y}{\partial t^2} \quad . \quad . \quad . \quad . \quad . \quad (9.185)$$

which agrees with the equation (9.9) except that it contains an extra term due to the tension P. It is worth while to verify equation (9.185) in quite a different way.

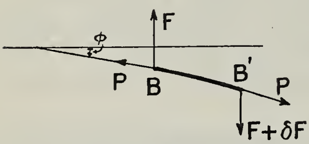

Fig. 89

Consider an element of rod BB′ of length δx. In fig. 89 the displacement y is measured downwards so that the conventions are the same as in the beam theory in Chapter V.

The component, in the direction of y, of the force P acting at B is $-P\sin\varphi$, or approximately $-P\frac{\partial y}{\partial x}$. The corresponding component at B′ is $P\frac{\partial y}{\partial x} + \frac{\partial}{\partial x}\left(P\frac{\partial y}{\partial x}\right)\delta x$. The difference of these is a force in the direction of y of amount $\frac{\partial}{\partial x}\left(P\frac{\partial y}{\partial x}\right)\delta x$. Then equating the forces in the direction of the acceleration $\ddot{y}$ to the product of mass and acceleration we get

$$\delta F + \frac{\partial}{\partial x}\left(P\frac{\partial y}{\partial x}\right)\delta x = \left(\frac{wa}{g}\delta x\right)\frac{\partial^2 y}{\partial t^2};$$

that is,

$$\frac{\partial F}{\partial x} + \frac{\partial}{\partial x}\left(P\frac{\partial y}{\partial x}\right) = \frac{wa}{g}\frac{\partial^2 y}{\partial t^2}.$$

Now P has approximately the same value at all points of the rod at the same instant since the accelerations in the x-direction are obviously of the second order. Hence P is a function of t but not of x. Therefore the last equation becomes

$$\frac{\partial F}{\partial x} + P\frac{\partial^2 y}{\partial x^2} = \frac{wa}{g}\frac{\partial^2 y}{\partial t^2}, \quad . \quad . \quad . \quad . \quad . \quad (9.186)$$

which will be identical with (9.185) when F is expressed in terms of curvature.

To solve equation (9.185) put

$$y = T \sin\frac{\pi x}{l}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (9.187)$$

where T is a function of t but not of x. This is the right type of solution for pinned ends because it makes y and M zero at the ends. Now equation (9.185) gives

$$EI\frac{\pi^4}{l^4} T \sin\frac{\pi x}{l} = -\frac{\pi^2}{l^2} PT \sin\frac{\pi x}{l} - \frac{wa}{g}\frac{d^2T}{dt^2}\sin\frac{\pi x}{l} \quad . \quad (9.188)$$

Also, from (9.183),

$$P = \frac{Ea}{2l}\frac{\pi^2}{l^2} \times \tfrac{1}{2} l T^2$$

$$= \frac{\pi^2}{4l^2} EaT^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9.189)$$

Then the equation for T becomes

$$EI\frac{\pi^4}{l^4} T = -\frac{\pi^4}{4l^4} EaT^3 - \frac{wa}{g}\frac{d^2T}{dt^2};$$

that is, when ak^2 is written for I,

$$\frac{d^2T}{dt^2} = -\frac{\pi^4}{4l^4}\frac{Eg}{w}\left\{4k^2T + T^3\right\} \quad . \quad . \quad . \quad . \quad (9.190)$$

Now let

$$\zeta = \frac{dT}{dt}$$

Then
$$\frac{d^2T}{dt^2} = \frac{d\zeta}{dt} = \frac{d\zeta}{dT}\frac{dT}{dt} = \zeta\frac{d\zeta}{dT}$$

Thus (9.190) becomes

$$\zeta\frac{d\zeta}{dT} = -\frac{\pi^4}{4l^4}\frac{Eg}{w}\left\{4k^2T + T^3\right\}$$

whence
$$\zeta^2 = -\frac{\pi^4}{2l^4}\frac{Eg}{w}\left\{2k^2T^2 + \tfrac{1}{4}T^4 - 2k^2T_0^2 - \tfrac{1}{4}T_0^4\right\} (9.191)$$

where T_0 is the maximum value of T, which occurs when ζ is zero.

Therefore

$$\frac{dt}{dT} = \frac{1}{\zeta}$$

$$= \frac{2l^2}{\pi^2}\sqrt{\frac{2w}{Eg}}\sqrt{\frac{1}{(T_0^2 - T^2)(T_0^2 + T^2 + 8k^2)}} \quad (9.192)$$

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This can be integrated by the substitution

$$T = T_0 \sin\theta \qquad dT = T_0 \cos\theta\, d\theta \qquad (9.193)$$

If we write c^2 for $\frac{T_0^2}{2T_0^2+8k^2}$ the result takes the form

$$t = \frac{2l^2}{\pi^2}\sqrt{\frac{2w}{Eg}}\frac{1}{\sqrt{2T_0^2+8k^2}}\int\frac{d\theta}{\sqrt{1-c^2\cos^2\theta}} \quad . \quad . \quad (9.194)$$

The relation between T and t can only be expressed by means of elliptic integrals, but the complete period can be obtained without much trouble. It is clear that T goes through one quarter of its cycle of values while θ varies from 0 to $\frac{\pi}{2}$, and this takes place in a quarter of the period. Then the whole period is

$$t_1 = \frac{8l^2}{\pi^2}\sqrt{\frac{2w}{Eg}}\frac{1}{\sqrt{2T_0^2+8k^2}}\int_0^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{1-c^2\cos^2\theta}}$$

Now

$$\int_0^{\frac{\pi}{2}}(1-c^2\cos^2\theta)^{-\frac{1}{2}}d\theta = \int_0^{\frac{\pi}{2}}\left(1+\frac{1}{2}c^2\cos^2\theta+\frac{1\cdot 3}{2\cdot 4}c^4\cos^4\theta+\ldots\right)d\theta$$

$$= \left\{1+\left(\frac{1}{2}\right)^2c^2+\left(\frac{1\cdot 3}{2\cdot 4}\right)^2c^4+\ldots\right\}\frac{\pi}{2},$$

the values of which are given in tables of complete elliptic integrals for different values of $\sin^{-1}c$. Then finally

$$t_1 = \frac{4l^2}{\pi}\sqrt{\frac{w}{Eg(T_0^2+4k^2)}}\left\{1+\left(\frac{1}{2}\right)^2c^2+\left(\frac{1\cdot 3}{2\cdot 4}\right)^2c^4+\right\} \quad . \quad . \quad (9.195)$$

Here T_0 is the absolute maximum amplitude of the rod, that is, the greatest value of the y at the middle of the rod. For a rod whose section is a complete circle of radius r we know that $k^2 = \frac{1}{4}r^2$. Now if T_0^2 is very small compared with $4k^2$, in which case T_0 is small compared with r, then we may neglect c^2 and T_0^2 altogether and in that case we get

$$t_1 = \frac{2l^2}{\pi k}\sqrt{\frac{w}{Eg}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9.196)$$

which is the period of the first mode when the tension is not taken into account. We might have seen from physical reasoning that the effect of the tension would be negligible if the amplitude were very small. The foregoing result tells us that the tension exercises very little control over the oscillations of the first mode when the maximum amplitude is very small compared with the radius of the rod.

Let us make the opposite assumption, namely, that T_0^2 is large compared with $4k^2$. We will take the extreme case and suppose that $c^2 = \frac{1}{2}$; this is the case of a perfectly flexible string. Then

$$t_1 = \frac{4l^2}{\pi T_0}\sqrt{\frac{w}{Eg}}\left\{1 + \frac{1}{4}\frac{1}{2} + \frac{9}{64}\frac{1}{2^2} + \ldots\right\}$$

$$= \frac{4l^2}{\pi T_0}\sqrt{\frac{w}{Eg}} \times 1\cdot1803 \text{ approximately.} \quad . \quad (9.197)$$

It should be particularly observed that when the motion is controlled entirely by the tension, that tension being zero in the equilibrium position, the period varies inversely as the amplitude T_0; that is, the greater the amplitude the less the period.

We have only dealt with the first mode of the rod pinned at both ends, but the work is just the same for any other mode. For the n^{th} mode all that is necessary is to write $\frac{l}{n}$ for l in the work for the first mode.

The equations are much more difficult for a rod whose ends are fixed in any other way. The curve does not then take the form of a simple sine curve. The problem has not enough practical interest to repay us for the labour of working out more difficult cases.

CHAPTER X

LONGITUDINAL AND TORSIONAL OSCILLATIONS OF RODS

169. Longitudinal motion of a rod.

It is possible for the particles of a straight rod to move in such a way that the only stress in the rod is a pure tension along its length. This tension will be different at different parts of the rod. The main and important part of the motion of the particles is longitudinal, although there must be, of curse, lateral motion due to the extension, which lateral motion is determined by Poisson's ratio. The strain does not differ greatly from the homogeneous strain which was investigated in Art. 32, Chap. III. The stresses S_1, S_2, S_3 are all zero.

Let the x-axis be taken along the line joining the centres of inertia of the cross sections, the origin being taken at some particle of the rod. Our assumption is that the plane of particles, which would be at x if there were no strain, are all at $(x+u)$ at any instant, u being a function of both x and the time t. If we write P for the tensional stress in the x-direction, and f for the acceleration of the particles in the cross-section at $(x+u)$, then equation (2.24) gives, since the shear stresses are zero,

$$\frac{\partial P}{\partial x} + \varrho X = \varrho f, \qquad (10.1)$$

X being the external force on unit mass of the rod.

We shall assume that the particle at which the origin is taken is either at rest or has a constant velocity. In that case the acceleration relative to the origin is the true acceleration f. That is,

$$\frac{\partial^2 u}{\partial t^2} = f \qquad (10.2)$$

Moreover, in most of the problems we shall solve, X will be zero. Then, making these two assumptions, equation (10.1) becomes

$$\begin{aligned}\frac{\partial P}{\partial x} &= \varrho \frac{\partial^2 u}{\partial t^2} \\ &= \frac{w}{g}\frac{\partial^2 u}{\partial t^2}, \qquad (10.3)\end{aligned}$$

where w is the weight of unit volume of the material of the rod.

Again, since the other two tensional stresses are zero, equation (2.14) gives

$$P = E\frac{\partial u}{\partial x}; \qquad (10.4)$$

whence

$$\frac{\partial P}{\partial x} = E\frac{\partial^2 u}{\partial x^2}, \qquad (10.5)$$

and therefore (10.3) becomes

$$E\frac{\partial^2 u}{\partial x^2} = \frac{w}{g}\frac{\partial^2 u}{\partial t^2}, \qquad (10.6)$$

or

$$c^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, \qquad (10.7)$$

where

$$c^2 = \frac{gE}{w} \qquad (10.8)$$

Equation (10.7) determines the motion of the rod when the initial state of the rod is given and the conditions at the ends are known. Before attempting to solve the equation we shall show that the equation of motion of a rod subject to torsion alone is exactly similar to (10.7).

170. Torsional motion of a rod.

Let the x-axis be taken along the centres of inertia of the cross-section of the rod just as for longitudinal motion. We are assuming that there is pure torsion in each element of the rod. Let us assume that the section at distance x from the origin is twisted through the angle θ relative to the section at the origin, and let ω denote the angular velocity of this section. The element of length δx is twisted through the angle $\delta\theta$ and consequently the twist per unit length of this element is

$$\tau = \frac{\partial\theta}{\partial x} \qquad (10.9)$$

Let Q denote the torque in this element. Then

$$Q = Kn\tau = Kn\frac{\partial\theta}{\partial x}, \qquad (10.10)$$

where K is a constant which depends on the shape and size of the section of the rod and has the dimensions of the moment of inertia of an area. The constants for several sections have been calculated in chapter VII.

Let I denote the moment of inertia of the cross-section of the rod about the x-axis. Then the moment of inertia of the element of the rod of length δx is $\frac{w}{g}I\,\delta x$, and the angular momentum of this element is $\frac{w}{g}I\omega\,\delta x$. The rate of increase of this angular momentum is equal to the total torque on the element, and this total torque is the excess of the torque at $(x+\delta x)$ over that at x, namely, $\frac{\partial Q}{\partial x}\delta x$. Therefore

$$\frac{\partial}{\partial t}\left(\frac{w}{g}\mathrm{I}\omega\delta x\right)=\frac{\partial \mathrm{Q}}{\partial x}\delta x;$$

that is,

$$\frac{w}{g}\mathrm{I}\delta x\frac{\partial\omega}{\partial t}=\frac{\partial \mathrm{Q}}{\partial x}\delta x$$

$$=\mathrm{K}n\frac{\partial^2\theta}{\partial x^2}\delta x$$

or

$$\frac{gn}{w}\frac{\mathrm{K}}{\mathrm{I}}\frac{\partial^2\theta}{\partial x^2}=\frac{\partial\omega}{\partial t} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10.11)$$

Now if the section at the origin is at rest, or is rotating with constant angular velocity, then the angular acceleration of the section at x relative to the section at the origin is the true acceleration of the section at x. That is,

$$\frac{\partial^2\theta}{\partial t^2}=\frac{\partial\omega}{\partial t}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10.12)$$

in which case (10.11) becomes

$$c_1{}^2\frac{\partial^2\theta}{\partial x^2}=\frac{\partial^2\theta}{\partial t^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10.13)$$

where

$$c_1{}^2=\frac{gn}{w}\frac{\mathrm{K}}{\mathrm{I}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10.14)$$

Equation (10.13) differs from (10.7) only in having θ for u and $c_1{}^2$ instead of c^2.

The constant K has the same dimensions as I, and for a rod whose section is a complete or a hollow circle K is equal to I. Coulomb's theory of torsion, which was the accepted theory before St. Venant published his work on torsion, made K equal to I for all rods.

171. Solution of the differential equation.

Corresponding to every problem in longitudinal oscillations there is a problem in torsional oscillations, and it is easy to see the relations between the pairs of problems. We shall therefore deal first with longitudinal oscillations and make use of our results for torsional oscillations afterwards.

The complete solution of (10.7) is

$$u=f(ct-x)+\mathrm{F}(ct+x) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10.15)$$

where f and F denote any arbitrary functions. To prove that (10.15) gives the complete solution of (10.7) put

$$\xi=ct-x, \quad \eta=ct+x, \quad . \quad . \quad . \quad . \quad . \quad (10.16)$$

and let us regard u as a function of ξ and η, which we can do since t and x can be expressed in terms of ξ and η, from which it follows that any function of t and x can be expressed in terms of ξ and η. Now

$$du = \frac{\partial u}{\partial \xi} d\xi + \frac{\partial u}{\partial \eta} d\eta,$$

where the partial differential coefficients are obtained on the assumption that η is constant in the first, and ξ constant in the second.

But
$$d\xi = c\,dt - dx,$$
$$d\eta = c\,dt + dx;$$

Therefore

$$du = \frac{\partial u}{\partial \xi}(c\,dt - dx) + \frac{\partial u}{\partial \eta}(c\,dt + dx). \quad . \quad . \quad . \quad (10.17)$$

If we keep t constant then we make dt zero, and therefore the last equation gives, when both sides are divided by dx,

$$\left(\frac{\partial u}{\partial x}\right)_t = -\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}.$$

Again, by repeating this process,

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = -\frac{\partial}{\partial \xi}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial \eta}\left(\frac{\partial u}{\partial x}\right)$$

$$= -\frac{\partial}{\partial \xi}\left(-\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}\right) + \frac{\partial}{\partial \eta}\left(-\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}\right)$$

$$= \frac{\partial^2 u}{\partial \xi^2} - \frac{\partial^2 u}{\partial \xi \partial \eta} - \frac{\partial^2 u}{\partial \eta \partial \xi} + \frac{\partial^2 u}{\partial \eta^2}$$

$$= \frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} - 2\frac{\partial^2 u}{\partial \xi \partial \eta} \quad . \quad . \quad . \quad . \quad . \quad . \quad (10.18)$$

If we next keep x constant, and consequently make dx zero in (10.17) we get

$$\frac{\partial u}{\partial t} = c\left\{\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}\right\}.$$

A repetition of this process gives

$$\frac{\partial^2 u}{\partial t^2} = c^2\left\{\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} + 2\frac{\partial^2 u}{\partial \xi \partial \eta}\right\} \quad . \quad . \quad . \quad . \quad (10.19)$$

By means of (10.18) and (10.19) equation (10.7) gives

$$4c^2\frac{\partial^2 u}{\partial \xi \partial \eta} = 0$$

whence

$$\frac{\partial}{\partial \xi}\left(\frac{\partial u}{\partial \eta}\right) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10.20)$$

Integrating this with respect to ξ on the assumption that η is constant we get

$$\frac{\partial u}{\partial \eta} = \text{constant}$$
$$= \text{any function of } \eta \quad . \quad . \quad . \quad . \quad . \quad (10.21)$$

Let us write $F'(\eta)$ for the function of η and then integrate again with respect to η. This gives, since ξ is constant in this step,

$$\begin{aligned} u &= F(\eta) + \text{any function of } \xi \\ &= F(\eta) + f(\xi) \\ &= F(ct + x) + f(ct - x), \end{aligned}$$

and this is clearly the most general solution of (10.20), and therefore of (10.7).

172. Interpretation of the solution

We see that u is the sum of two terms u_1 and u_2 where

$$u_1 = f(ct - x) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10.22)$$

$$u_2 = F(ct + x) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10.23)$$

Since the functions f and F are arbitrary it is quite possible, in particular cases, for one of the functions to be zero so that u is equal to only one of the functions. Let us suppose that u_1 is zero. Then

$$u = F(ct + x) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10.24)$$

When $t = 0$ this becomes

$$u = F(x) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10.25)$$

In order to visualise these results let us suppose that curves are plotted with u as ordinate and x as abscissae showing the relations expressed by (10.24) and (10.25). We must regard t as constant in the former equation, so that our curve gives an instantaneous graph showing u at all points for a particular value of t. Now the curve representing (10.24) differs from that representing (10.25) only in being bodily displaced a distance ct in the negative direction parallel to the x-axis. If, then, (10.24) be plotted for different values of t, a series of curves is produced each of which could be obtained by sliding the whole curve represented by (10.25) parallel to the x-axis. Then as t varies continuously the curve moves continuously, and, since it moves a distance ct in time t, it is moving with a velocity c, and this velocity is parallel to the negative direction along the x-axis.

To put the argument in another form, suppose we pick on any particular ordinate of the curve (10.25), say the ordinate at $x = x^1$ The ordinate is

$$u^1 = F(x^1) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10.26)$$

Now let t and x both vary so that

$$x + ct = x^1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10.27)$$

Then at the point x given by (10.27) the ordinate of (10.24) is

$$u = F(ct + x)$$
$$= F(x^1) = u^1$$

But from (10.27)

$$\frac{dx}{dt} + c = 0,$$

that is,
$$\frac{dx}{dt} = -c. \qquad (10.29)$$

Then in order to satisfy (10.27) the point x must travel backwards along the x-axis with velocity c, and in this way the ordinate u remains constant and equal to u^1. This means that every ordinate of the curve (10.24) travels backwards along the x-axis with velocity c and maintains its size. It follows that the whole curve travels in one direction, as a wave travels in water. For this reason we shall refer to $F(ct + x)$ as a wave.

From the fact that every value of u travels backwards with the same velocity it follows that the values of $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial t}$ also travel backwards That is, the whole state of the rod is travelling backwards so that, for example, the tension that existed at x^1 when t was zero exists at $(ct + x)$ at the instant t seconds later provided that x and t satisfy (10.27).

In the same way it can be shown that

$$u = f(ct - x)$$

represents a wave or a state of the rod travelling *forward* along the x-axis with velocity c. Then we may picture the general case, represented by (10.15), as two waves or states travelling in opposite directions with the same velocity c, each wave maintaining its form as it passes over the other. The actual state at any point is the sum of the states due to each wave. For example, the total tension is the sum of the tensions in the two waves. Likewise the velocity of any particle of the rod is the sum of the velocities due to the effect of the two waves at the point where the particle is.

The velocity c is called the velocity of sound in the material of the rod because these longitudinal oscillations in a rod generate sound waves in air, and any sound waves impinging on one end of the rod will travel to the other end with the velocity c.

173. Particular solutions.

(i) One particular solution is given by

$$u = H(ct - x) + K(ct + x)$$
$$= (H + K)ct + (K - H)x, \qquad (10.30)$$

where H and K are constants.

Then the tensional stress is

$$P = E\frac{\partial u}{\partial x}$$
$$= E(K - H) \quad . \quad . \quad . \quad . \quad . \quad . \quad (10.31)$$

Also the velocity of any particle of the rod is

$$\frac{\partial u}{\partial t} = (H + K)c \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10.32)$$

Since H and K are arbitrary constants it is clear that (H + K) and (K — H) are also arbitrary. Then this is the case of a string moving with a constant velocity and having a constant tension. There is no relative motion of the string. If we make H = — K then the velocity is zero. Thus we get the simple case of a stretched string at rest.

(ii) Another simple solution is given by

$$u = K\{(x + ct)^2 - (ct - x)^2\}$$
$$= 4Kcxt \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10.33)$$

Then

$$P = 4EKct \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10.34)$$

and the velocity is

$$\frac{\partial u}{\partial t} = 4Kcx \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10.35)$$

Thus the tension is the same at all points of the string at any instant but increases at a uniform rate. The velocity is proportional to the distance x from the origin. This is the case of a string which is fixed at one end — the origin — and the other end of which moves at a constant speed, so that the strain increases at a uniform rate.

(iii) A third solution is

$$u = K\left\{\sin\frac{\pi(ct + x)}{l} - \sin\frac{\pi(ct - x)}{l}\right\}$$
$$= 2K\sin\frac{\pi x}{l}\cos\frac{\pi ct}{l} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10.36)$$

Here

$$P = \frac{2\pi}{l}EK\cos\frac{\pi x}{l}\cos\frac{\pi ct}{l} \quad . \quad . \quad . \quad . \quad . \quad (10.37)$$

$$\frac{\partial u}{\partial t} = -\frac{2\pi c}{l}K\sin\frac{\pi x}{l}\sin\frac{\pi ct}{l} \quad . \quad . \quad . \quad . \quad . \quad (10.38)$$

The displacement u is always zero at the origin. Therefore the origin is a fixed point on the rod. The velocity of any point on the rod varies with time as indicated by the factor $\sin\frac{\pi ct}{l}$. This represents simple harmonic motion, and the amplitude of the particular particles at x varies as $\sin\frac{\pi x}{l}$. We see also that the points where $x = l$, $2l$, $3l$, etc., are all at rest. At certain particular instants, namely, when

$$\sin\frac{\pi ct}{l}=0,$$

every particle of the rod is in its natural position, but it is not at rest; it is, in fact, in the same state as a pendulum bob swinging through its equilibrium position.

174. Reflexion from a fixed point.

Suppose one point of the rod is fixed, and let the origin be taken at this point. Then

$$u=0 \text{ where } x=0$$

Therefore, from (10.15),

$$0=f(ct)+F(ct)$$

for all values of t. That is, writing z for the variable ct

$$f(z)=-F(z),$$

which expresses the function f in terms of the function F. Now the complete value for u is

$$u=F(ct+x)-F(ct-x) \quad . \quad . \quad . \quad . \quad . \quad (10.39)$$

The wave $F(ct+x)$ travels along the rod towards the fixed point while the wave $-F(ct-x)$, which is the negative of the reversed forward wave, travels in the opposite direction. The physical interpretation is this: the wave travelling towards the fixed point carries to that point a succession of displacements; the fixed point reverses each of these as it arrives and sends it back along the rod thus forming the wave which travels in the opposite direction. The total displacement at any point x_1 of the rod at any instant is the displacement due to the wave travelling towards the origin plus the displacement which has been reflected from the origin and has now arrived at x_1. This reflected displacement passed through the same point x_1 in the contrary direction at an earlier instant, the interval of time between the two passages being $\frac{2x_1}{c}$, since the displacement has travalled $2x_1$ with a velocity c in that time.

Although the total displacement at the origin is zero it is possible for each wave to cause a displacement at that point, but each of these displacements is the negative of the other. It is in this sense that every displacement on the wave travelling towards the origin reaches that point undiminished and is suddenly reversed there to form the return wave.

175. Rod fixed at both ends.

Suppose a rod is fixed at the ends $x=0$ and $x=l$. Then, because the origin is fixed, the displacement is given by

$$u=f(ct-x)-f(ct+x) \quad . \quad . \quad . \quad . \quad . \quad . \quad (10.40)$$

Again, because the end $x=l$ is fixed,

$$0 = f(ct - l) - f(ct + l) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10.41)$$

which is true for all values of t.

Now let z be written for $(ct - l)$. Then $(ct + l)$ is $(z + 2l)$, and the last equation gives

$$f(z + 2l) = f(z) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10.42)$$

Thus $f(z)$ is such a function of the variable z that the same value of the function recurs when z is increased by $2l$. That is, $f(z)$ is a periodic function which goes through its cycle while z increases by $2l$.

The periodicity we have just found could have been foreseen from the more physical point of view of reflexion. Suppose we fix our attention on the same point of the rod as time varies. During the time that ct increases by $2l$ each wave travels a distance $2l$. That is, the displacement due to either of the waves travels to one end of the rod, is then reflected to the other end with changed sign, and is again reflected with a second change of sign to the starting point. Thus two reflexions and two changes of sign bring the whole wave into exactly the same position as at the start.

176. Reflexion from a free end.

At a free end the tension is zero. Let the origin be taken at the free end. Then the condition at this end is

$$\frac{\partial u}{\partial x} = 0 \text{ where } x = 0.$$

Using the general value of u in (10.15) this gives

$$-f'(ct) + F'(ct) = 0,$$

whence

$$F'(ct) = f'(ct)$$

for all values of t. Integrating both sides with respect to ct we get

$$F(ct) = f(ct) + C.$$

Therefore

$$u = f(ct - x) + f(ct + x) + C.$$

Now the constant C is unnecessary because we can suppose $\frac{1}{2}$ C to be absorbed into each of the functions; that is, we may put

$$\varphi(z) = f(z) + \tfrac{1}{2}C;$$

then

$$u = \varphi(ct - x) + \varphi(ct + x) \quad . \quad . \quad . \quad . \quad . \quad . \quad (10.43)$$

This form shows that the free end reflects the displacements which arrive there with unchanged sign. Likewise a tension or a velocity arriving at the free end on one wave is reflected by the return wave as an equal tension or a velocity in the same direction.

177. Rod free at both ends.

If the free ends be taken at $x = 0$ and $x = l$ the displacement is given by

$$u = \varphi(ct - x) + \varphi(ct + x)$$

The condition at the other end is

$$\frac{\partial u}{\partial x} = 0 \text{ when } x = l;$$

that is,

$$0 = -\varphi'(ct - l) + \varphi'(ct + l)$$

for all values of t. Integrating with respect to ct and neglecting the constant for the reason given in the last article

$$0 = -\varphi(ct - l) + \varphi(ct + l)$$

Now writing z for $(ct - l)$ we get

$$\varphi(z + 2l) = \varphi(z) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10.44)$$

which shows that the function φ is periodic and its values are certain to be repeated when z has increased by $2l$.

178. Rod fixed at one end and free at the other.

Let the end $x = 0$ be fixed and the end $x = l$ be free. Then, because the origin is fixed, the displacement is given by

$$u = f(ct - x) - f(ct + x)$$

The condition at the free end gives

$$0 = \frac{\partial u}{\partial x}$$

$$= -f'(ct - l) - f'(ct + l)$$

for all values of t. Integrating this with respect to ct we get

$$u = f(ct - l) + f(ct + l).$$

Again writing z for $(ct - l)$ the result becomes

$$f(z + 2l) = -f(z) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10.45)$$

Thus $f(z)$ is repeated, but with its sign changed, when z increases by $2l$. Then if z be increased by $2l$ twice in succession the function will be repeated with the same sign. Thus

$$f(z + 4l) = -f(z + 2l) = +f(z) \quad . \quad . \quad . \quad . \quad . \quad (10.46)$$

Thus each wave travels a distance of $4l$, that is, travels twice in each direction along the rod, before the cycle is complete. It follows that the whole state of the rod is certain to be repeated in a period $\frac{4l}{c}$ since the waves travel with velocity c.

179. When the whole of a rod AB is moving with velocity v in the direction AB the end A is suddenly brought to rest. To find the subsequent state of the rod.

Up to the instant when the end A is brought to rest the value of u is given by

$$u = vt \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10.47)$$

Let the origin be taken at the end A, and let t be measured from the instant when A is brought to rest. We must first analyse vt into a pair of waves in the form given by (10.15). The result is clearly

$$u = \frac{v}{2c}\{(ct - x) + (ct + x)\} \quad . \quad . \quad . \quad . \quad . \quad (10.48)$$

These are the waves that exist just before A comes to rest. After that instant these waves continue to travel along the rod, and are reflected from the fixed end A and the free end B according to the laws that we have found in the preceding articles.

If we write

$$u_1 = \frac{v}{2c}(ct - x) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10.49)$$

$$u_2 = \frac{v}{2c}(ct + x) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10.50)$$

the wave u_1 travels towards the free end and the wave u_2 travels towards the fixed end. The effects represented by u_1 and u_2 are each spread over the whole length of the rod when $t = 0$, but as t increases the rear end of each wave travels along the rod leaving no effect behind it except what is caused by the other wave. Thus when, in fig. 90, the rear end of the wave u_1 has reached the point K, the front end of this wave has been reflected from B and has reached K′. Thus the whole effect of the wave which was represented by u_1 at the start is now spread over KB and BK′. Likewise the effect of the other wave is spread over K′A and its reflected part over AK. Then between K and K′ the two original waves are still superposed. Therefore the part KK′ is still moving with velocity v exactly as at the start. We have still to find the state of the rod in the parts AK and K′B. Along AK we have the wave u_2 plus the reflection of u_2. Now using (10.40) we see that

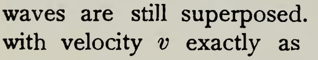


Fig. 90

$$u_2 = -f(ct + x) = \frac{v}{2c}(ct + x) \quad . \quad . \quad . \quad . \quad (10.51)$$

Then the function f is defined by

$$f(z) = -\frac{v}{2c}z$$

Therefore the reflected wave is

$$u'_2 = f(ct - x) = -\frac{v}{2c}(ct - x) \quad . \quad . \quad . \quad . \quad (10.52)$$

Hence along AK

$$\begin{aligned} u &= u_2 + u'_2 \\ &= \frac{v}{2c}(ct + x) - \frac{v}{2c}(ct - x) \\ &= \frac{vx}{c}, \end{aligned} \qquad (10.53)$$

and thus the tension in this portion is

$$P = E\frac{\partial u}{\partial x} = \frac{Ev}{c} = v\sqrt{\frac{Ew}{g}} \qquad (10.54)$$

which is constant. Also the velocity in AK is

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t}\left(\frac{vx}{c}\right) = 0$$

Thus AK is under a constant tension and has no velocity.

To find the state of K′B we must use (10.43). The forward wave itself is

$$u_1 = \varphi(ct - x) = \frac{v}{2c}(ct - x) \qquad (10.55)$$

which defines the function φ.

Therefore the wave reflected from B is

$$u'_1 = \varphi(ct + x) = \frac{v}{2c}(ct + x) \qquad (10.56)$$

Thus the value of u along K′B is

$$\begin{aligned} u &= u_1 + u'_1 \\ &= \frac{v}{2c}(ct - x) + \frac{v}{2c}(ct + x) \\ &= vt \end{aligned} \qquad (10.57)$$

which is precisely the same as at the start. It follows that the length AK is at rest under a uniform tension, and the whole length KB has no tension and has the same uniform velocity v as at the start. The point K, which divides the part in tension from the part with a velocity, travels with a velocity c. When K arrives at B the whole rod is in tension and at rest, and u is given by (10.53). From this instant onwards the waves which were reflected from A and B have now reached the other ends and are again reflected.

To find the next state of the rod we need only repeat the preceding argument, starting from the equation

$$u = \frac{v}{2c}(ct + x) - \frac{v}{2c}(ct - x), \qquad (10.58)$$

instead of from equation (10.48). The wave approaching the fixed end is now

$$u_2 = \frac{v}{2c}(ct + x)$$

and its reflection is

$$u'_2 = -\frac{v}{2c}(ct - x)$$

When the front of this reflected wave has reached K in fig. 90 the displacement in AK is

$$\begin{aligned} u &= u_2 + u'_2 \\ &= \frac{vx}{c} \quad \text{. . . . . (10.59)} \end{aligned}$$

which shows that AK is still in tension and at rest. The displacement in KK′ is given by (10.58) and is therefore in the same state as AK. The wave approaching the free end is

$$u_1 = -\frac{v}{2c}(ct - x)$$

and its reflexion, the front of which has reached K′, is

$$u'_1 = -\frac{v}{2c}(ct + x).$$

Therefore the displacement along K′B is

$$\begin{aligned} u &= u_1 + u'_1 \\ &= -vt \quad \text{. . . . . (10.60)} \end{aligned}$$

Thus the portion K′B has a velocity $-v$ and no tension.

We have found then that in the second state of the rod the portion AK′ has a tensional stress $\frac{Ev}{c}$, just as at the beginning of the first state, while the portion K′B has acquired a velocity v towards A, but has recovered from its tension. When the point K′ reaches A the whole rod will have acquired the velocity v towards A and will be unstrained. This last state of the rod differs from the first state only in having the velocity v in the contrary direction. We have traced the first half of the cycle, and the second half differs from the first only in having a thrust where there previously was a tension and in having the velocities in the direction opposite to those in the first half.

The important conclusion we draw from the problem we have just solved is that the stopping of one end of a rod of any length and cross section, which is moving with a velocity v in the direction of its length, sets up alternately a tension and a thrust each of magnitude

$$P = \frac{v}{c}E = v\sqrt{\frac{wE}{g}} \quad \text{. . . . . . . (10.61)}$$

It is worth while to dwell on this result because the principles of mechanics, applied to an absolutely rigid rod, indicates that the tension set up by stopping one end would be infinite since the whole rod would be stopped instantaneously. This problem gives a very clear picture showing how elasticity prevents the infinite stresses of rigid dynamics.

An elastic body cannot have the velocity of every particle suddenly changed. Only an infinitely small mass has its velocity changed in an infinitely small time.

Since velocities are all relative, it is clear from the result of the problem just solved that, if the velocity of the whole rod were originally $(v + v_1)$, and if the velocity of the end A were suddenly reduced to v_1, the tensions and thrusts set up would be exactly as before, and the velocities relative to the end A would be exactly the same as before. If v_1 were equal to $-v$ we should get the case of a rod originally at rest and the end A suddenly set in motion with velocity v, and still the stresses would be the same as before.

180. A rod with a sudden change of cross-section.

Suppose a single wave travels along a uniform rod with cross-section a_1 towards a point where the section of the rod increases to a_2. It is to be expected that some part of the wave will continue in the same direction along the portion with the larger section and another part will be reflected. Let

Fig. 91

the wave travelling along AK (fig. 91) in the direction AK be represented by

$$u = f(ct - x), \qquad (10.62)$$

and let the wave reflected along KA be

$$u = F(ct + x) \qquad (10.63)$$

Then the total displacement at a point in AK is

$$u_1 = f(ct - x) + F(ct + x) \qquad (10.64)$$

where the function f is given but the function F is so far unknown. Again let the wave transmitted to the part KB be represented by

$$u_2 = \varphi(ct - x). \qquad (10.65)$$

Now at the junction K

$$u_1 = u_2, \qquad (10.66)$$

and the total tensions are equal, that is,

$$a_1 \frac{\partial u_1}{\partial x} = a_2 \frac{\partial u_2}{\partial x}. \qquad (10.67)$$

With the origin at K these give

$$f(ct) + F(ct) = \varphi(ct), \qquad (10.68)$$

and

$$-a_1 f'(ct) + a_1 F'(ct) = -a_2 \varphi'(ct) \qquad (10.69)$$

Integrating the second of these with respect to ct and neglecting a useless constant we get

$$-a_1 f(ct) + a_1 F(ct) = -a_2 \varphi(ct) \qquad (10.70)$$

Solving equations (10.68) and (10.70) for $F(ct)$ and $\varphi(ct)$ we find

$$F(ct) = -\frac{a_2 - a_1}{a_2 + a_1} f(ct), \quad . \quad . \quad . \quad . \quad . \quad . \quad (10.71)$$

$$\varphi(ct) = \frac{2a_1}{a_2 + a_1} f(ct). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10.72)$$

We have now determined the two unknown functions and have consequently determined the displacement in each portion of the rod. Thus

$$u_1 = f(ct - x) - \frac{a_2 - a_1}{a_2 + a_1} f(ct + x) \quad . \quad . \quad . \quad (10.73)$$

and

$$u_2 = \frac{2a_1}{a_2 + a_1} f(ct - x). \quad . \quad . \quad . \quad . \quad . \quad . \quad (10.74)$$

The displacements we have just found are those due to a single wave originally travelling in the direction AK along the portion AK only. If there were also originally a wave

$$u = \psi(ct + x) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10.75)$$

travelling along BK in the direction BK a repetition of the preceding argument would show that the resulting displacements, so far as this wave and its reflexion have travelled, would be

$$u_1 = \frac{2a_2}{a_2 + a_1} \psi(ct + x) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10.76)$$

$$u_2 = \psi(ct + x) + \frac{a_2 - a_1}{a_2 + a_1} \psi(ct - x) \quad . \quad . \quad . \quad (10.77)$$

Then, due to the two waves denoted by f and ψ, the total displacements along the two portions of the rod, up to the points where the reflexions of these waves have travelled, are

$$u_1 = f(ct - x) - \frac{a_2 - a_1}{a_2 + a_1} f(ct + x) + \frac{2a_2}{a_2 + a_1} \psi(ct + x) \quad (10.78)$$

$$u_2 = \psi(ct + x) + \frac{a_2 - a_1}{a_2 + a_1} \psi(ct - x) + \frac{2a_1}{a_2 + a_1} f(ct - x) \quad (10.79)$$

If it happens that

$$\psi(z) = f(z)$$

then the above results become

$$\left.\begin{aligned} u_1 &= f(ct - x) + f(ct + x) \\ u_2 &= u_1 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (10.80)$$

so that in this case the two waves are just the same as those originally on the rod so long as the reflexions from other points, such as fixed or free ends, have not reached the part of the rod we are dealing with.

We can make use of the results we have just proved to solve, for the rod AKB in fig. 91, the problem we solved in Art. 178 for a uniform rod. Let us suppose that the end A is brought to rest when the whole rod has just been moving in the direction AB with velocity v; and we shall assume, to simplify the problem, that the part KB which has the larger section is infinitely longer than AK. With these assumptions we shall find the ultimate tension in the part AK.

Just at the instant before A was brought to rest the displacement at every point of the rod was given by

$$u = \frac{v}{2c}\{(ct + x) + (ct - x)\} \quad . \quad . \quad . \quad . \quad . \quad (10.81)$$

Let u_1 denote the displacement in the portion AK, and u_2 in the portion KB. Then, by the time the rear end of the wave $\frac{v}{2c}(ct - x)$ on AK has travelled to K, the front end of the reflexion of the other wave has arrived at K also. At this instant

$$u_1 = \frac{v}{2c}\{(ct + x) - (ct - x)\} \quad . \quad . \quad . \quad . \quad (10.82)$$

$$u_2 = \frac{v}{2c}\{(ct + x) + (ct - x)\} \quad . \quad . \quad . \quad . \quad (10.83)$$

Now each of the waves in u_1 passes backward and forward between A and K, and every time the wave reaches K a certain fraction of it passes on along the thicker portion, and the fraction r of it is reflected, where

$$r = \frac{a_2 - a_1}{a_2 + a_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10.84)$$

After n reflexions of either of these waves from K its effect is therefore diminished in the ratio r^n, and when n is very great this fraction is negligible. Moreover, the wave $\frac{v}{2c}(ct - x)$ in u_2 does not affect the portion AK because it travels away from K. It follows that the only wave in the wider portion that affects AK after a long time is the wave $\frac{v}{2c}(ct + x)$, which is constantly supplying, across the point of discontinuity K, a wave represented by

$$u_1 = \frac{2a_2}{a_2 + a_1}\frac{v}{2c}(ct + x) \quad . \quad . \quad . \quad . \quad . \quad (10.85)$$

The reflexion of this from A is

$$u_1 = -\frac{2a_2}{a_2 + a_1}\frac{v}{2c}(ct - x), \quad . \quad . \quad . \quad . \quad (10.86)$$

and the part of this that is reflected back again from K along KA is

$$u_1 = -\frac{a_2 - a_1}{a_2 + a_1}\left(-\frac{2a_2}{a_2 + a_1}\right)\frac{v}{2c}(ct + x)$$

$$= r\frac{2a_2}{a_2 + a_1}\frac{v}{2c}(ct + x), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10.87)$$

which is r times the wave that originally passed over through K from KB to AK. To this wave reflected from K must be added the wave that is constantly flowing through K from KB. At this stage then the part contributed to AK from KB is

$$u_1 = \frac{2a_2}{a_2 + a_1}\frac{r}{2c}(ct+x)(1+r) - \frac{2a_2}{a_2+a_1}\frac{v}{2c}(ct-x) \quad . \quad . \quad . \quad . \quad . \quad (10.88)$$

It is not difficult to see that, after the front end of this wave has been reflected n times from K back into the portion AK, the total displacement on AK is

$$u_1 = \frac{2a_2}{a_2+a_1}\frac{v}{2c}(ct+x)\left\{1+r+r^2+\ldots+r^n\right\} - \frac{2a_2}{a_2+a_1}\frac{v}{2c}(ct-x)\left\{1+r+\ldots+r^{n-1}\right\}, \quad . \quad (10.89)$$

and when $n = \infty$ this becomes

$$u_1 = \frac{2a_2}{a_2+a_1}\frac{v}{2c}\left\{(ct+x)-(ct-x)\right\}\frac{1}{1-r} = \frac{a_2}{a_1}\frac{v}{c}x \quad . \quad . \quad . \quad . \quad . \quad (10.90)$$

This means that AK has now been reduced to rest and is subject to a tensional stress given by

$$P_1 = E\frac{\partial u_1}{\partial x} = \frac{a_2}{a_1}\frac{v}{c}E \quad . \quad . \quad . \quad . \quad . \quad (10.91)$$

The total tension across the section is

$$a_1 P_1 = a_2\frac{v}{c}E, \quad . \quad . \quad . \quad . \quad . \quad (10.92)$$

and since the total tension at K is the same in the two portions it follows that the stress in the thicker portion is

$$P_2 = \frac{a_1 P_1}{a_2} = \frac{v}{c}E, \quad . \quad . \quad . \quad . \quad . \quad (10.93)$$

exactly as if K had been initially brought to rest instead of A.

We now see that, although the stress in a uniform rod due to starting or stopping one end does not depend on the cross-section or the length of the rod, yet the stress set up by doing the same to a rod whose section is not constant does depend on the variation of the section. Moreover, it is fairly evident, although we have not worked out a case in detail, that the stress depends on the relative length of the thin and thick portions.

181. Normal modes of oscillation.

Although it is possible to solve a number of useful problems on the longitudinal oscillations of elastic rods by the method we have already used in this chapter, yet there is a much more powerful method of tackling the oscillations of a finite rod. This method consists in analysing the motion into normal modes of oscillation, as we did with the transverse oscillations. We shall now show how to apply this method to longitudinal oscillations. The results will apply equally well, of course, to torsional oscillations.

182. Rod fixed at both ends.

We shall begin by finding the normal modes of oscillation of a rod of length l fixed at both ends. Our object then is to find solutions of equation (10.7) that make the motion of every particle simple harmonic and at the same time fit the end conditions of the rod. The differential equation is

$$\frac{\partial^2 u}{\partial t^2}=c^2\frac{\partial^2 u}{\partial x^2}, \qquad (10.94)$$

and the end conditions are

$$u=0 \text{ where } x=0, \qquad (10.95)$$

and

$$u=0 \text{ where } x=l, \qquad (10.96)$$

We may assume

$$u=\xi\sin(pt+\alpha) \qquad (10.97)$$

where ξ is a function of x only, and p is a constant. Then

$$\frac{\partial^2 u}{\partial t^2}=-p^2\xi\sin(pt+\alpha)$$

and

$$\frac{\partial^2 u}{\partial x^2}=\frac{d^2\xi}{dx^2}\sin(pt+\alpha).$$

Substituting in equation (10.94) we get

$$-p^2\xi\sin(pt+\alpha)=c^2\frac{d^2\xi}{dx^2}\sin(pt+\alpha),$$

whence

$$\frac{d^2\xi}{dx^2}=-\frac{p^2}{c^2}\xi \qquad (10.98)$$

The solution of this last equation is

$$\xi=\text{A}\cos\frac{px}{c}+\text{B}\sin\frac{px}{c} \qquad (10.99)$$

Up to this point the end conditions have not come into the question and the results will be valid for any end conditions.

The condition (10.95) gives

$$0 = A \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10.100)$$

Then condition (10.96) gives

$$B \sin \frac{pl}{c} = 0\,,$$

whence either

$$B = 0\,,$$

or

$$\sin \frac{pl}{c} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10.101)$$

The first of these alternatives makes u zero everywhere and at all times, and therefore gives no motion. The second gives

$$\frac{pl}{c} = s\pi\,, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10.102)$$

where s is any integer.

The period of the s^{th} normal mode is therefore

$$\tau_s = \frac{2\pi}{p}$$

$$= \frac{2l}{sc} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10.103)$$

The period of the first mode is

$$\tau_1 = \frac{2l}{c}$$

which is the time taken for a wave to travel along the rod and back again with its natural velocity c.

Returning to (10.97), the s^{th} normal mode is represented by

$$u_s = B_s \sin \frac{s\pi x}{l} \sin \left(\frac{s\pi ct}{l} + a_s\right) \quad . \quad . \quad . \quad . \quad (10.104)$$

It can be shown that any possible longitudinal motion of the rod can be represented by summing all the values of u expressing the normal modes with suitable values of the constants B and a for each mode. Thus the general value of u is

$$u = \sum_{s=1}^{s=\infty} B_s \sin \frac{s\pi x}{l} \sin \left(\frac{s\pi ct}{l} + a_s\right) \quad . \quad . \quad . \quad (10.105)$$

183. Rod fixed at one end and free at the other.

If the origin be taken at the fixed end (10.100) is still true. The other condition is

$$\frac{\partial u}{\partial x} = 0 \text{ where } x = l;$$

that is,

$$\frac{\partial \xi}{\partial x} = 0 \text{ where } x = l,$$

or

$$B \cos \frac{pl}{c} = 0.$$

Hence the values of p for the normal modes are given by

$$\frac{pl}{c}=\frac{\pi}{2} \text{ or } \frac{3\pi}{2} \text{ or } \frac{5\pi}{2} \text{ etc.} \quad . \quad . \quad . \quad . \quad (10.106)$$

$$=\frac{2s-1}{2}\pi$$

s being an integer as before; and the s^{th} normal mode is expressed by

$$u_s = B_s \sin\frac{(2s-1)\pi x}{2l}\sin\left\{\frac{(2s-1)\pi ct}{2l}+a_s\right\} \quad . \quad (10.107)$$

184. Rod fixed at one end and carrying a finite mass at the other end.

Let the origin be taken at the fixed end and let the mass at the other end have a weight W. Then the condition at the origin still makes (10.100) true. The other condition, assuming no external forces act on W, is obtained from the fact that the acceleration of W is produced by the tension in the rod. That is, at the end $x=l$,

$$\frac{W}{g}\frac{\partial^2 u}{\partial t^2}=-Ea\frac{\partial u}{\partial x} \quad . \quad . \quad . \quad . \quad . \quad . \quad (10.108)$$

where a is the cross-section of the rod. In terms of ξ this becomes

$$-p^2\frac{W}{g}\xi=-Ea\frac{d\xi}{dx} \text{ where } x=l, \quad . \quad . \quad (10.109)$$

or

$$-p^2\frac{W}{g}B\sin\frac{pl}{c}=-Ea\frac{p}{c}B\cos\frac{pl}{c},$$

whence

$$\cot\frac{pl}{c}=\frac{Wc}{Eag}p \quad . \quad . \quad . \quad . \quad . \quad . \quad (10.110)$$

If we write z for $\frac{pl}{c}$ this becomes

$$\cot z=\frac{Wc^2}{Ealg}z$$

$$=\frac{W}{wal}z$$

$$=hz \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10.111)$$

where h is independent of z, and is, in fact, the ratio of the weight W to the weight of the whole rod. Equation (10.111) has an infinite number of positive roots for z, each differing from the preceding one by rather less than π. The first root is less than $\frac{1}{2}\pi$, and is very near $\frac{1}{2}\pi$ when h is very small, and very near zero when h is large. When s is large the s^{th} positive root is approximately $(s-1)\pi$. Each root of (10.111) corresponds to a normal mode of oscillation of the rod, and any motion can be resolved into the sum of a finite or infinite number of normal modes. The following problem will illustrate the method of using these normal modes.

Suppose the rod, whose normal modes we have just found, has been travelling, in an unstrained state, in the direction of its length when the end which we have called fixed is suddenly brought to rest. To find the subsequent state of the rod.

If t is measured from the instant when the end is brought to rest the initial conditions are

$$u = 0 \text{ when } t = 0 \quad \ldots \ldots \quad (10.112)$$

and

$$\frac{\partial u}{\partial t} = v \text{ when } t = 0 \quad \ldots \ldots \quad (10.113)$$

The first of these conditions is satisfied identically by taking $\alpha = 0$ in every mode. Then the s^{th} mode is represented by

$$u_s = B_s \sin \frac{p_s x}{c} \sin p_s t \quad \ldots \ldots \quad (10.114)$$

the values of p_s being obtained from (10.111). Consequently the complete displacement is given by

$$u = \sum_{s=1}^{s=\infty} u_s \quad \ldots \ldots \quad (10.115)$$

To satisfy condition (10.113) we have

$$v = \left(\frac{\partial u}{\partial t}\right)_{t=0} = \sum_{s=1}^{s=\infty} p_s B_s \sin \frac{p_s x}{c}$$

$$= \sum_{s=1}^{s=\infty} C_s \sin \frac{p_s x}{c} \quad \ldots \ldots \quad (10.116)$$

C_s being written for $p_s B_s$.

We find the constants C_s in much the same way as in an ordinary Fourier Series. Thus

$$v \sin \frac{p_s x}{c} = \sin \frac{p_s x}{c} \left\{ C_1 \sin \frac{p_1 x}{c} + C_2 \frac{\sin p_2 x}{c} + \ldots \right\} \quad . \quad (10.117)$$

Now integrating both sides from $x = 0$ to $x = l$ we get

$$\int_0^l v \sin \frac{p_s x}{c}\, dx = \int_0^l \sin \frac{p_s x}{c} \left\{ C_1 \sin \frac{p_1 x}{c} + C_2 \sin \frac{p_2 x}{c} + \ldots \right\} dx \quad (10.118)$$

Now let us write ξ_s for $\sin \dfrac{p_s x}{c}$. Then we know that

$$\frac{d^2 \xi_s}{dx^2} = -\frac{p_s^2}{c^2} \xi_s, \quad \ldots \ldots \quad (10.119)$$

and that, at the end $x = l$,

$$\frac{d\xi_s}{dx} = \frac{W}{Eag} p_s^2 \xi_s \quad \ldots \ldots \quad (10.120)$$

Next writing D for $\frac{d}{dx}$ we find that

$$-\frac{p_1^2}{c^2}\int_0^l \xi_1 \xi_s dx = \int_0^l \xi_s D^2 \xi_1 dx \quad \text{by (10.119)}$$

$$= \Big[\xi_s D \xi_1\Big]_0^l - \int_0^l D\xi_s D\xi_1 \, dx$$

$$= \Big[\xi_s D\xi_1 - \xi_1 D\xi_s\Big]_0^l + \int_0^l \xi_1 D^2 \xi_s \, dx$$

$$= \Big[\xi_s D_1 \xi - \xi_1 D\xi_s\Big]_0^l - \frac{p_s^2}{c^2}\int_0^l \xi_1 \xi_s dx \quad . \quad . \quad (10.121)$$

Therefore

$$\frac{p_s^2 - p_1^2}{c^2}\int_0^l \xi_1 \xi_s dx = \Big[\xi_s D\xi_1 - \xi_1 D\xi_s\Big]_0^l$$

$$= \frac{W}{Eag}(p_1^2 - p_s^2)\,\xi'_1 \xi'_s \quad . \quad . \quad (10.122)$$

the dashes indicating the values of the ξ's at the end $x = l$. Then finally

$$\int_0^l \xi_1 \xi_s dx = -\frac{Wc^2}{Eag}\xi'_1 \xi'_s \quad . \quad . \quad . \quad . \quad (10.123)$$

In the same way we can prove that

$$\int_0^l \xi_m \xi_s dx = -\frac{Wc^2}{Eag}\xi'_m \xi'_s \quad . \quad . \quad . \quad . \quad . \quad (10.124)$$

provided that m is not equal to s. If $m = s$ we get, on integrating by parts,

$$-\frac{p_s^2}{c^2}\int_0^l \xi_s^2 dx = \Big[\xi_s D\xi_s\Big]_0^l - \int_0^l (D\xi_s)^2 dx$$

$$= \frac{Wp_s^2}{Eag}\xi_s'^2 - \frac{p_s^2}{c^2}\int_0^l (1 - \xi_s^2)\,dx$$

$$= \frac{Wp_s^2}{Eag}\xi_s'^2 - \frac{p_s^2}{c^2}l + \frac{p_s^2}{c^2}\int_0^l \xi_s^2 dx,$$

whence

$$\int_0^l \xi_s^2 dx = \tfrac{1}{2}l - \frac{Wc^2}{2\,Eag}\xi_s'^2 \quad . \quad . \quad . \quad . \quad . \quad (10.125)$$

Now equation (10.118) becomes

$$\int_0^l v \sin\frac{p_s x}{c}\,dx = -\frac{Wc^2}{Eag}\xi'_s\{C_1\xi'_1 + C_2\xi'_2 + C_3\xi'_3 + \ldots\}$$

$$+ \tfrac{1}{2}C_s l + \frac{Wc^2}{2\,Eag}C_s\xi_s'^2 \quad . \quad . \quad . \quad . \quad (10.136)$$

But the expression in the brackets on the right hand side of this equation is v. Also the left hand side is

$$v\int_0^l \sin\frac{p_s x}{c}\,dx = -\frac{vc}{p_s}\left[\cos\frac{p_s x}{c}\right]_0^l$$
$$= \frac{vc}{p_s}\left\{1 - \cos\frac{p_s l}{c}\right\}$$

Then finally (10.118) becomes

$$\frac{vc}{p_s}\left(1 - \cos\frac{p_s l}{c}\right) = -\frac{Wc^2 v}{Eag}\sin\frac{p_s l}{c} + \tfrac{1}{2}C_s l + \frac{Wc^2}{2\,Eag}C_s \sin^2\frac{p_s l}{c},$$

which gives

$$C_s = \frac{\dfrac{2\,vc}{p_s}\left(1 - \cos\dfrac{p_s l}{c}\right) + \dfrac{2\,Wc^2 v}{Eag}\sin\dfrac{p_s l}{c}}{l + \dfrac{Wc^2}{Eag}\sin^2\dfrac{p_s l}{c}}$$

$$= \frac{\dfrac{2\,vc}{p_s}}{l + \dfrac{Wc^2}{Eag}\sin^2\dfrac{p_s l}{c}}$$

$$= \frac{\dfrac{2\,vc}{p_s}}{l + \dfrac{c}{p_s}\cos\dfrac{p_s l}{c}\sin\dfrac{p_s l}{c}}$$

$$= \frac{2\,v}{z_s + \cos z_s \sin z_s} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10.127)$$

Therefore

$$B_s = \frac{C_s}{p_s} = \frac{l}{c}\frac{C_s}{z_s}$$
$$= \frac{2\,vl}{cz_s(z_s + \cos z_s \sin z_s)} \quad . \quad . \quad . \quad . \quad (10.128)$$

This last result, together with (10.114) and (10.115), gives the complete analytical expression for u. The values of z_s are the roots of (10.111), and these depend on the ratio of W to the weight of the rod. But, as we have pointed out before, when s is large z_s is nearly $(s-1)\pi$ whatever the value of h may be. It follows that, for large values of s,

$$B_s = \frac{2\,vl}{c(s-1)^2\pi^2} \text{ nearly} \quad . \quad . \quad . \quad . \quad . \quad (10.129)$$

The important terms in the value of u are the earlier terms in the series. It will be seen that the actual calculation of tension in the rod at any time involves a considerable amount of labour, and the calculation of the maximum tension involves still more labour. We shall, however, make use of the energy method to give us an idea of the maximum tension.

When dealing with the case of the rod which carried no weight at the end in Art. 179 we found that a stage was reached at which the whole of the rod was at rest and the tension was constant throughout the rod. That is, the original kinetic energy of the rod was transformed into potential energy uniformly distributed along the rod. It is clear again that, if the rod had no mass itself but carried a mass at the free end, a stage would be reached where the whole kinetic energy would be transformed into potential energy uniformly distributed along the rod. Then let us assume that the same is true for the present case, where a rod which has mass carries a mass at the end. The total weight is $(W+wal)$, and its original kinetic energy is $\frac{1}{2}(W+wal)\frac{v^2}{g}$. The total potential energy of the stretched rod having a uniform tensional stress P is, by (8.3) or (8.5), $\frac{P^2al}{2E}$. Then, equating the potential energy to the kinetic energy, we get

$$\frac{P^2al}{2E}=\frac{1}{2}(W+wal)\frac{v^2}{g},$$

whence

$$\frac{P}{v}=\sqrt{\frac{(W+wal)E}{alg}} \quad . \quad . \quad . \quad . \quad . \quad (10.130)$$

This is obtained on the assumption that every particle comes to rest at the same instant and that the maximum tension occurs at that instant. It is certainly not true that every particle comes to rest at the same instant, and it is quite possible that the maximum tension is slightly above the value given by (10.130) for the reason that the potential energy will not be uniformly distributed along the rod, and therefore the tension will be greater in some places than in others. Nevertheless equation (10.54) shows that the result given by (10.129) is true when $W=0$; and it is clearly correct also when $w=0$. We should therefore expect that it could not be far wrong in other cases.

Since each B contains the factor v it follows that the tension at any instant is proportional to v, and therefore the maximum tension is proportional to v. The result in equation (10.130) agrees with this conclusion.

185. Pulley at the end of a rotating shaft.

The torsional problem that corresponds to the last problem we have worked out is this:— *A rod, which carries at one end a mass, such as a pulley, whose centre of inertia is on the axis of the rod, is* *rotating in an unstrained state with angular velocity ω when the end opposite to the one which carries the mass is suddenly brought to rest. To find the subsequent state of the rod.*

The reader should have no difficulty in modifying the result we have obtained to suit the torsion problem. The angular velocity ω corresponds to v, the twist τ corresponds to the strain $\frac{\partial u}{\partial x}$, and the torque Q corresponds to the total tension Pa. It follows from what we have proved that Q is proportional to ω In particular, the result in (10.54), which applies to the case when there is no mass at the end, is transformed for the rod in torsion into

$$\begin{aligned} \mathrm{Q} &= \mathrm{H}n\frac{\partial\theta}{\partial x} = \mathrm{H}n\frac{\omega}{c_1} \\ &= \mathrm{H}n\omega\sqrt{\frac{w\mathrm{I}}{gn\mathrm{H}}} \\ &= \omega\sqrt{\frac{wn\mathrm{IH}}{g}} \end{aligned}$$

If the rod has a circular section then $\mathrm{H} = \mathrm{I}$ and

$$\mathrm{Q} = \mathrm{I}\omega\sqrt{\frac{wn}{g}}$$

The constant c_1 is a linear velocity, like the constant c in longitudinal motion, and it represents the velocity with which a torsional disturbance travels along the rod.

CHAPTER XI.

THE EQUILIBRIUM OF THIN CURVED RODS.

186. The actions across a section of a curved rod.

A thin rod may be regarded as generated by the motion of a small plane area which moves in such a way that its centre of gravity always travels in the direction of the normal to the area. If the area keeps its direction fixed in space a straight rod is generated. But if the area rotates about an axis in its plane as it moves through space a curved rod is generated. The generating area in any position is the cross-section of the rod, and the path of the centre of gravity we shall call the central line of the rod.

The body could still be called a rod if the area charged its shape or size as it moved provided that these changes were slow. To put it precisely the body would be called a rod if the boundary of a small element between two cross-sections could be formed of lines which were all nearly parallel. For example, a cone whose height is many times greater than the diameter of its base may be regarded as a rod.

To find the stress across any section of a rod we must consider the equilibrium of the whole of the rod on one side of the section as we did with straight beams in Chap. V. In order to analyse the stresses we need three axes of reference passing through the centre of gravity of the section. Let the axis OX be taken normal to the section, and let OY, OZ, be taken in the plane of the section and coincident with the principal axes of inertia of the section.

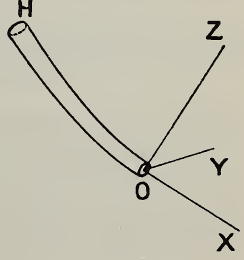


Fig. 92

All the stresses across the section at O on the portion OH (fig. 92) can be resolved into three forces along OX, OY, OZ, respectively, together with three component couples about these axes. The force along OX is a tension; the other two are shearing forces which may conveniently be compounded into a single shearing force inclined to OY and OZ. The couple about OX is a torque which puts a twist in the rod just as it would in a straight rod; the other two couples are bending couples each of which produces a change of curvature of the central line at O, the total change of curvature being the resultant effect of the two changes.

The system of forces and couples across the section through O are *balanced by* the forces on the portion of the rod extending from O through H to the end of the rod. They are also *equivalent to* the forces on the other portion of the rod.

187. Spiral spring with axial pull.

Suppose the central line of the coils of a spiral spring is inclined at the angle $(90^\circ - \varphi)$ with the direction of the axis of the spring. Then φ is the inclination of the coils to the circular sections of the cylinder on which the coils lie. We require to find the stresses across the section at B in fig. 93. To do this we must find the component forces and couples along the principal axes at B which are equivalent to one of the opposing pulls R. Let us suppose that two equal and opposite forces R are introduced at B. The upper R in the figure together with the downward R at B form a couple of moment rR, where r is the radius of the cylinder on which the central line of the coils lies. The other R at B can be resolved into a tension T along the tangent to the coils at B, where

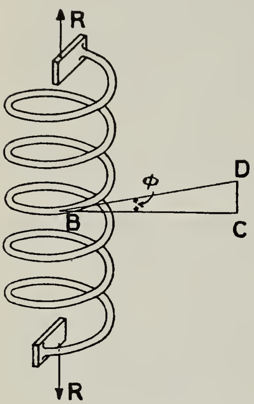


Fig. 93

$$T = R \sin \varphi, \quad . \quad . \quad . \quad (11.1)$$

and a shearing force F in the tangent plane to the coils at B, where

$$F = R \cos \varphi. \quad . \quad . \quad . \quad (11.2)$$

The couple across the section at B acts in the plane containing the axis of the coils and the point B. The axis of this couple is in the direction BC, and the couple can be represented as a vector along BC. Then this vector Rr can be resolved into a torque

$$Q = Rr \cos \varphi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11.3)$$

with axis along BD, together with a bending moment

$$M = Rr \sin \varphi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11.4)$$

with its axis perpendicular to BD and parallel to the plane BCD.

Now springs are usually made either with uniform circular sections, or with sections such that one of the principal axes at B is in the plane BCD. Then the bending couple M acts about one of the principal axes. Now let W denote the work done by the force R in stretching the spring from the state of zero stress to the actual state. Let x be the extension of the spring, that is, the displacement of one end relative to the other. Then the work done by the stretching forces is

$$W = \int_0^x R\, dx \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11.5)$$

This must be equal to the total energy in the spring. This energy is, by (8.15) and (8.24),

$$W = \frac{1}{2}\int \frac{M^2}{EI}\,ds + \frac{1}{2}\int \frac{Q^2}{Hn}\,ds \quad . \quad . \quad . \quad . \quad . \quad (11.6)$$

where ds denotes an element of length of the central line of the coils, and I the moment of inertia of the section about the axis of M. Equating these two values of W we get, since M and Q are constant,

$$\int_0^x R\,dx = \frac{1}{2}\frac{M^2 l}{EI} + \frac{1}{2}\frac{Q^2 l}{Hn},$$

l being the total length of the central line of the coils. Now using (11.3) and (11.4) we get

$$\int_0^x R\,dx = \frac{1}{2}\frac{R^2 r^2 l \sin^2\varphi}{EI} + \frac{1}{2}\frac{R^2 r^2 l \cos^2\varphi}{Hn} \quad . \quad . \quad . \quad (11.7)$$

Differentiating both sides of (11.7) with respect to x we get

$$R = \left\{\frac{R r^2 l \sin^2\varphi}{EI} + \frac{R r^2 l \cos^2\varphi}{Hn}\right\}\frac{dR}{dx}.$$

Therefore
$$\frac{dx}{dR} = r^2 l\left\{\frac{\sin^2\varphi}{EI} + \frac{\cos^2\varphi}{Hn}\right\} \quad . \quad . \quad . \quad . \quad . \quad (11.8)$$

Since the right hand side of (11.8) is constant if we assume that φ is constant, and since $R = 0$ when $x = 0$, we find by integration that

$$x = R r^2 l\left\{\frac{\sin^2\varphi}{EI} + \frac{\cos^2\varphi}{Hn}\right\} \quad . \quad . \quad . \quad . \quad . \quad (11.9)$$

Thus the extension x is proportional to the pull R. If the angle φ is so small that $\sin^2\varphi$ can be neglected, in which case $\cos^2\varphi$ may be regarded as unity, then

$$x = \frac{R r^2 l}{Hn}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11.10)$$

If the wire forming the spring were straightened out and one end held fixed while a torque Rr were applied to the other end, then the twist per unit length in the rod would be τ given by

$$Rr = Hn\tau$$

If a disk of radius r were attached to the free end this disk would turn through the angle $l\tau$ and a point on the rim would move through the distance $lr\tau$, which is the value of x contained in (11.10).

If the wire forming the spring has a circular section of radius a then

$$I = \tfrac{1}{4}\pi a^4$$
$$H = \tfrac{1}{2}\pi a^4$$

Then equation (11.9) becomes

$$x = \frac{R r^2 l}{\pi a^4}\left\{\frac{4\sin^2\varphi}{E} + \frac{2\cos^2\varphi}{n}\right\} \quad . \quad . \quad . \quad . \quad (11.11)$$

The extension given by (11.9) is, of course, that due to the bending and the twisting of the wire. No account has been taken of the extensions due to the tension T or the shear stress F. It is easy to calculate what these amount to, and the results show that they are negligible in comparison with that due to torque if, as is nearly always the case, the dimensions of the section of the coils are small in comparison with the radius r.

188. Spiral spring with the pull parallel to, but not along, the axis.

Let fig. 94 represent a new view of the spring looking along the axis, and let K be the point which represents the line of action of the pulls. Let $OK = c$, and let φ denote the slope of the coils as before. Let B be any point on the central line of the spring, and let KD be the perpendicular from K on the tangent at B to the circular section through B. Now let a pair of opposing forces R be introduced at B parallel to the axis, and another similar pair at D. One of the pulls acting through K (we shall call it the upward pull for definiteness) combined with the downward R through D, forms a couple of magnitude $R \times KD$ acting in that plane which contains KD and is parallel to the axis. The other force through D combined with the downward force through B forms another couple of magnitude $R \times BD$. We have left an upward force R at B. These two couples and the remaining force at B are equivalent to the original single upward force R at K. We have now to find the extension of the spring due to the two couples, the extension due to the force at B being negligible.

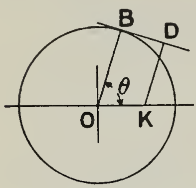


Fig. 94

The couple $R \times BD = R \times OK \sin\theta$ is a purely bending couple. The couple $R \times KD$ has its axis in the direction DB. This can be resolved into a torque Q given by

$$Q = (R \times KD)\cos\varphi \quad . \quad . \quad . \quad . \quad . \quad . \quad (11.12)$$

and a bending couple

$$M_1 = (R \times KD)\sin\varphi \quad . \quad . \quad . \quad . \quad . \quad . \quad (11.13)$$

whose axis is in the plane BCD in fig. 93. The bending couple

$$\begin{aligned} M_2 &= R \times OK \sin\theta \\ &= Rc\sin\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11.14) \end{aligned}$$

has its axis in the direction $\overrightarrow{OB}$. Thus the two bending couples M_1 and M_2 are in perpendicular planes, and their axes coincide with the positions we assumed for the principal axes of the section in the last article. If we assume the same positions for the principal axes again, the equation corresponding to (11.6) is

$$W = \frac{1}{2}\int \frac{M_1{}^2}{EI_1} ds + \frac{1}{2}\int \frac{M_2{}^2}{EI_2} ds + \frac{1}{2}\int \frac{Q^2}{Hn} ds \quad . \quad . \quad . \quad (11.15)$$

Now $$KD = (r - c\cos\theta)$$

whence $$KD^2 = r^2 - 2rc\cos\theta + c^2\cos^2\theta$$

Therefore,

$$\int_0^l KD^2 ds = \int_0^l (r^2 - 2rc\cos\theta + c^2\cos^2\theta) ds$$

But $$\cos^2\theta = \tfrac{1}{2}(1 + \cos 2\theta)$$

Consequently

$$\int_0^l KD^2 ds = (r^2 + \tfrac{1}{2}c^2)\, l + \int_0^l (\tfrac{1}{2}c^2 \cos 2\theta - 2rc\cos\theta)\, ds$$

Now $$ds = (rd\theta)\sec\varphi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11.16)$$

whence it follows that

$$\int_0^l KD^2 ds = (r^2 + \tfrac{1}{2}c^2)\, l + r\sec\varphi\,(\tfrac{1}{4}c^2 \sin 2\theta_1 - 2rc\sin\theta_1) \quad (11.17)$$

where θ_1 is the angle subtended at the axis of the spring by the whole of the coils. If there is an integral number of half coils the terms involving $\sin\theta_1$ and $\sin 2\theta_1$ are zero. In that case

$$\int_0^l KD^2 ds = (r^2 + \tfrac{1}{2}c^2)\, l \quad . \quad . \quad . \quad . \quad . \quad (11.18)$$

If, in the general case, we write

$$\theta_1 = (2n\pi + \alpha),$$

n being the total number of complete coils and α being an angle less than 2π, then we may replace θ_1 by α in equation (11.17).

In a similar way we find that

$$\int_0^l M_2{}^2 ds = \int_0^{\theta_1} R^2 c^2 \sin^2\theta \cdot \sec\varphi\, r d\theta$$

$$= \tfrac{1}{2} R^2 c^2 r \sec\varphi \int_0^{\theta_1} (1 - \cos 2\theta)\, d\theta$$

$$= \tfrac{1}{2} R^2 c^2 r \sec\varphi\,(\theta_1 - \tfrac{1}{2}\sin 2\theta_1)$$

$$= \tfrac{1}{2} R^2 c^2 (l - \tfrac{1}{2} r \sec\varphi \sin 2\alpha) \quad . \quad . \quad . \quad . \quad (11.19)$$

Then finally, making use of (11.17) and (11.19), we get

$$W = \frac{1}{2} R^2 \left\{ \frac{\sin^2\varphi}{EI_1} + \frac{\cos^2\varphi}{Hn} \right\} \left\{ (r^2 + \tfrac{1}{2}c^2)\, l + r\sec\varphi\,(\tfrac{1}{4}c^2 \sin^2 2\alpha - rc\sin\alpha) \right\}$$

$$+ \frac{1}{4}\frac{R^2 c^2}{EI_2}(l - \tfrac{1}{2} r \sec\varphi \sin 2\alpha) \quad . \quad . \quad . \quad . \quad . \quad (11.20)$$

Also

$$W = \int_0^x R dx,$$

and by equating these two values of W and then differentiating with respect to x, as in the last article, we arrive finally at the equation

18

$$\frac{x}{R}=\left\{\frac{\sin^2\varphi}{EI_1}+\frac{\cos^2\varphi}{Hn}\right\}\left\{(r^2+\tfrac{1}{2}c^2)\,l+r\sec\varphi\,(\tfrac{1}{4}c^2\sin 2\alpha-rc\sin\alpha)\right\}$$
$$+\frac{1}{2}\frac{c^2}{EI_2}(l-\tfrac{1}{2}r\sec\varphi\sin 2\alpha) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11.21)$$

The terms containing α are usually much smaller than the other terms because r and c are much less than l. Moreover, these terms vanish completely when the spring has an integral number of half-coils. When these terms are dropped the relation between x and R becomes

$$\frac{x}{R}=(r^2+\tfrac{1}{2}c^2)\,l\left\{\frac{\sin^2\varphi}{EI_1}+\frac{\cos^2\varphi}{Hn}\right\}+\frac{lc^2}{2\,EI_2} \quad . \quad . \quad (11.22)$$

If $c=r$, in which case the pulls R are applied along a generating line of the cylinder on which the central line of the coils lie, then

$$\frac{x}{R}=\tfrac{3}{2}r^2 l\left\{\frac{\sin^2\varphi}{EI_1}+\frac{\cos^2\varphi}{Hn}\right\}+\frac{r^2 l}{2\,EI_2}$$

If φ is so small that $\sin^2\varphi$ can be neglected this last result reduces to

$$\frac{x}{R}=\frac{3}{2}\frac{r^2 l}{Hn}+\frac{r^2 l}{2\,EI_2}=\frac{3}{2}\frac{r^2 l}{Hn}\left\{1+\frac{Hn}{3\,EI_2}\right\}$$

189. Spiral spring under a couple about the axis.

Let us suppose that the spring in fig. 93 is under the action of a pair of opposing couples at the ends, these couples acting in planes perpendicular to the axis of the spring. These actions tighten up or slacken the coils of the spring so that one end rotates about the axis of the spring relative to the other end. As a result of this rotation there may be also a relative axial displacement of the ends, but this we shall not at present take into account.

The couple K acting about the axis of the spring on one end is transmitted as a couple with a parallel axis across the section at B (fig. 93). The couple can be resolved into a torque.

$$Q=K\sin\varphi$$

with axis BD, together with a bending moment

$$M_1=K\cos\varphi$$

with axis perpendicular to BD in the plane BCD.

Let one end of the spring turn through an angle θ relative to the other end. Then the work done by the couple K in producing this angular deformation is

$$W=\int_0^\theta K\,d\theta$$

Equating this to the energy stored in the spring we get

$$\int_0^\theta \mathrm{K}d\theta = \frac{1}{2}\int_0^l \frac{\mathrm{M}_1^2}{\mathrm{EI}_1}ds + \frac{1}{2}\int_0^l \frac{\mathrm{Q}^2}{\mathrm{H}n}ds$$
$$= \frac{1}{2}\frac{l\mathrm{K}^2\cos^2\varphi}{\mathrm{EI}_1} + \frac{1}{2}\frac{l\mathrm{K}^2\sin^2\varphi}{\mathrm{H}n},$$

l being the total length of the coils.

Differentiating with respect to the upper limit θ we get

$$\mathrm{K} = \left\{\frac{l\mathrm{K}\cos^2\varphi}{\mathrm{EI}_1} + \frac{l\mathrm{K}\sin^2\varphi}{\mathrm{H}n}\right\}\frac{d\mathrm{K}}{d\theta},$$

whence

$$\frac{d\theta}{d\mathrm{K}} = l\left\{\frac{\cos^2\varphi}{\mathrm{EI}_1} + \frac{\sin^2\varphi}{\mathrm{H}n}\right\} \quad . \quad . \quad . \quad . \quad . \quad (11.23)$$

Regarding φ as constant this gives, since K is zero when θ is zero,

$$\theta = \mathrm{K}l\left\{\frac{\cos^2\varphi}{\mathrm{EI}_1} + \frac{\sin^2\varphi}{\mathrm{H}n}\right\} \quad . \quad . \quad . \quad . \quad . \quad (11.24)$$

190. Spiral spring under an axial pull and a twisting couple. We have now found in Arts 187 and 189 the extension x and the twist θ of a spiral spring due to an axial pull R and couple K acting separately. We shall now find the effect on the spring produced by the simultaneous action of R and K. When R alone acts there is a twist θ which we have not found. Likewise the couple K, acting alone, produces an extension which we have also not found.

From the preceding results we should expect that the extension produced by R and K acting simultaneously would be a linear function of R and K. That is, we may assume

$$x = a\mathrm{R} + b\mathrm{K} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11.25)$$

Likewise

$$\theta = p\mathrm{K} + q\mathrm{R} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11.26)$$

Now we know a and p, for these are the values of $\frac{x}{\mathrm{R}}$ and $\frac{\theta}{\mathrm{K}}$ in equations (11.9) and (11.24). We have yet to find b and q. We shall first prove that b and q are equal.

The work done by R and K in producing the strains in the spring is

$$\begin{aligned} \mathrm{W} &= \int \mathrm{R}dx + \int \mathrm{K}d\theta \\ &= \int \mathrm{R}\{a d\mathrm{R} + b d\mathrm{K}\} + \int \mathrm{K}\{p d\mathrm{K} + q d\mathrm{R}\} \\ &= \int (a\mathrm{R} + q\mathrm{K})\, d\mathrm{R} + \int (p\mathrm{K} + b\mathrm{R})\, d\mathrm{K} \quad . \quad . \quad . \quad (11.27) \end{aligned}$$

Now let us suppose that R increases from 0 to its final value R_1 while K remains at zero; then K increases to its final value K_1 while R remains at R_1. Then in integrating with respect to R we must put $\mathrm{K} = 0$; and in integrating with respect to K we must put $\mathrm{R} = \mathrm{R}_1$.

Thus
$$W=\int_0^{R_1} aR\,dR+\int_0^{K_1}(pK+bR_1)\,dK$$
$$=\tfrac{1}{2}aR_1^2+\tfrac{1}{2}pK_1^2+bR_1K_1 \quad . \quad . \quad . \quad . \quad . \quad (11.28)$$

If the foregoing operations had been reversed; that is, if K had increased to K_1 while R was zero, and then R increased to R_1 while K remained at K_1, we should have found

$$W=\tfrac{1}{2}aR_1^2+\tfrac{1}{2}pK_1^2+qR_1K_1 \quad . \quad . \quad . \quad . \quad (11.29)$$

Now clearly the energy in the spring in the final state is the same in both cases. By equating the two values of W in (11.28) and (11.29) we get

$$q=b \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11.30)$$

We have now to equate either of the values of W we have just found to the expression for the energy obtained from the stresses and strains in the spring. For this purpose we must find expressions for the bending moment and torque at any point B of the spring.

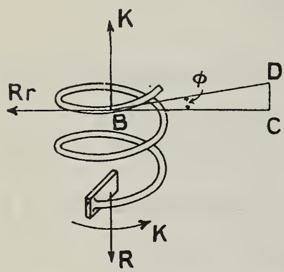


Fig. 95

Let BD be the tangent to the central line of the coils at B, and DC a line parallel to the axis of the spring. Then the force R acting at the lower end in fig. 95 gives rise to a couple acting across the section at B, and its axis is in the direction CB as shown in the figure. Likewise the couple K acting at the lower end, in the direction tending to unwind the spring, has its axis parallel to the axis of the spring. The two lines through B marked Rr and K are vectors representing these couples. Resolving these vectors perpendicular to and along BD into bending moment M_1 and torque Q we get

$$M_1=Rr\sin\varphi+K\cos\varphi \quad . \quad . \quad . \quad . \quad . \quad (11.31)$$
$$Q=K\sin\varphi-Rr\cos\varphi \quad . \quad . \quad . \quad . \quad . \quad (11.32)$$

Then the energy in the spring corresponding to the final values R_1 and K_1 is

$$W=\int_0^l \frac{M_1^2}{2EI_1}\,ds+\int_0^l \frac{Q^2}{2Hn}\,ds$$
$$=\frac{l}{2EI_1}\{R_1r\sin\varphi+K_1\cos\varphi\}^2+\frac{l}{2Hn}\{K_1\sin\varphi-R_1r\cos\varphi\}^2 \quad (11.33)$$

We may now equate the value of W given by (11.28) to the value given by (11.33), and the resulting equation must be identically true for all values of R_1 and K_1. Then regarding φ as constant, we may equate coefficients of R_1^2, K_1^2, and R_1K_1, in the two expressions for W. This gives

$$a = lr^2\left\{\frac{\sin^2\varphi}{EI_1} + \frac{\cos^2\varphi}{Hn}\right\} \quad . \quad . \quad . \quad . \quad . \quad (11.34)$$

$$p = l\left\{\frac{\cos^2\varphi}{EI_1} + \frac{\sin^2\varphi}{Hn}\right\} \quad . \quad . \quad . \quad . \quad . \quad (11.35)$$

$$b = lr\sin\varphi\cos\varphi\left\{\frac{1}{EI_1} - \frac{1}{Hn}\right\} \quad . \quad . \quad . \quad . \quad (11.36)$$

We may rewrite (11.25) and (11.26) thus

$$x = aR + bK \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11.37)$$

$$\theta = pK + bR \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11.38)$$

wherein we now know the values of all the coefficients.

From the last pair of equations we find that, when R alone acts,

$$\left.\begin{aligned} x &= aR \\ \theta &= bR \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11.39)$$

The first of these results we already knew, but the second gives a new result, namely, the amount of twist produced by an axial pull R when no couple K acts.

Likewise, when K alone acts,

$$\left.\begin{aligned} \theta &= pK \\ x &= bK \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11.40)$$

Again we may require to know the extension produced by a given axial pull R when the constraints on the spring are such that one end is not free to twist relatively to the other. In this case K is not zero; it is expressed in terms of R by equating θ to zero.

Thus $$0 = pK + bR$$

Therefore $$K = -\frac{b}{p}R,$$

whence
$$\begin{aligned} x &= aR - \frac{b^2}{p}R \\ &= \left(a - \frac{b^2}{p}\right)R \\ &= \frac{lr^2 R}{Hn\cos^2\varphi + EI_1\sin^2\varphi} \end{aligned} \quad . \quad . \quad . \quad . \quad (11.41)$$

191. The variation of φ.

In the preceding part of this chapter it has been assumed that φ was constant; but it is clear that φ cannot be constant when the spring stretches. The errors resulting from treating φ as constant are only small in any case, as we shall now show by taking account of the variation of φ.

Let the unstretched length of the spring be h; that is, h is the length of the projection of the coils on the axis, and does not include the lenghts of any attached end-pieces. Also let θ denote the angle subtended at the axis by the whole of the coils. Now suppose the surface of the cylinder on which the central line of the coils lies is developed into a plane, and with it the central line itself. This central line develops into a straight line which is the hypotenuse of a right-angled triangle whose sides are h and $r\theta$ as in fig. 96. From this figure we get

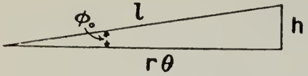


Fig. 96

$$h = l \sin \varphi_0$$

When the spring is stretched so that h becomes $(h+x)$ and φ_0 becomes φ the equation is

$$h + x = l \sin \varphi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11.42)$$

If, now, we want to take account of the variation of φ in such an equation as (11.7) we must make use of (11.42) both in the differentiation to remove the integral, and in the final integral to get the relation between R and x which corresponds to equation (11.9).

Let us write $\quad v = \dfrac{\sin^2 \varphi}{\mathrm{EI}} + \dfrac{\cos^2 \varphi}{\mathrm{H}n}$

$$= \frac{1}{\mathrm{H}n} - \sin^2 \varphi \left(\frac{1}{\mathrm{H}n} - \frac{1}{\mathrm{EI}} \right) \quad . \quad . \quad . \quad . \quad (11.43)$$

Also let $\quad y = h + x = l \sin \varphi \, , \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11.44)$

and $\quad c^2 = \dfrac{1}{l^2} \left(1 - \dfrac{\mathrm{H}n}{\mathrm{EI}} \right).$

Then $\quad v = \dfrac{1}{\mathrm{H}n} \{ 1 - c^2 y^2 \}, \ v_0 = \dfrac{1}{\mathrm{H}n} (1 - c^2 h^2),$

and $\quad dx = dy.$

Consequently (11.7) becomes

$$\int_0^x \mathrm{R} dx = \tfrac{1}{2} \mathrm{R}^2 r^2 l v.$$

Therefore, differentiating with respect to the upper limit,

$$\mathrm{R} = r^2 l \left\{ v \mathrm{R} \frac{d\mathrm{R}}{dx} + \tfrac{1}{2} \mathrm{R}^2 \frac{dv}{dx} \right\}$$

whence

$$v^{-\frac{1}{2}} = r^2 l \left\{ v^{+\frac{1}{2}} \frac{d\mathrm{R}}{dx} + \tfrac{1}{2} v^{-\frac{1}{2}} \mathrm{R} \frac{dv}{dx} \right\}$$

$$= r^2 l \frac{d}{dx} \left(\mathrm{R} v^{\frac{1}{2}} \right)$$

Consequently, by integration,

$$r^2 l \mathrm{R} v^{\frac{1}{2}} = \int_0^x v^{-\frac{1}{2}} dx$$

$$= \int_h^{h+x} \frac{\sqrt{\mathrm{H}n}\,dy}{\sqrt{1 - c^2 y^2}}$$

$$= \frac{\sqrt{\mathrm{H}n}}{c} \Big[\sin^{-1}(cy) \Big]_h^{h+x}$$

$$= \frac{l}{\sqrt{\frac{1}{\mathrm{H}n} - \frac{1}{\mathrm{EI}}}} \{\sin^{-1} c(h+x) - \sin^{-1} ch)\} \quad . \quad (11.45)$$

This last equation expresses R in terms of x. If we expand the right hand side of (11.45) in powers of x and retain only the first power of x we shall arrive at an equation which is identical with (11.9). If, however, we carry the approximation as far as the second power of x the result becomes

$$r^2 l \mathrm{R} v_0^{\frac{1}{2}} v^{\frac{1}{2}} = x\left\{1 + \frac{c^2 hx}{2(1 - c^2 h^2)}\right\}$$

$$= x\left\{1 + \frac{hx}{2 l^2 v_0}\left(\frac{1}{\mathrm{H}n} - \frac{1}{\mathrm{EI}}\right)\right\} \quad . \quad . \quad . \quad (11.46)$$

where v_0 is the value of v when $x = 0$.

Since v_0 and $\left(\frac{1}{\mathrm{H}n} - \frac{1}{\mathrm{EI}}\right)$ are quantities of the same order it is clear that the proportional error due to neglecting the variation of φ is a fraction of the order $\frac{hx}{l^2}$, which is certainly small in every practical case.

192. Conical springs formed with wire of uniform section.

Suppose the central line of a spring lies on a cone. Let the central line be inclined at the angle φ to a circular section of the cone at the point B at which it meets the section. Then an axial force R gives rise to a couple Rr at B, the axis of this couple being BC, the tangent to the circular section. Let BL (fig. 97) be the tangent to the central line and let CL be perpendicular to BL. The couple Rr can be represented by a vector along BC, and this can be resolved into a pair of vectors parallel to BL and LC. These are respectively the torque and the bending moment, and their magnitudes are

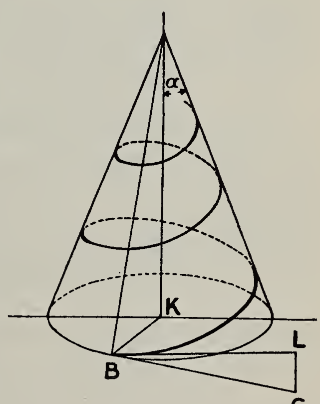


Fig. 97

$$Q = Rr\cos\varphi,$$
$$M = Rr\sin\varphi,$$

just as for a helical spring.

It should be particularly noticed that the axis of the bending moment is in the tangent plane to the cone at B. which is, of course, not perpendicular to the radius through B. If this axis of M happens to coincide with a principal axis of the section of the spring at B then the energy stored in the spring is

$$W = \frac{1}{2}\int\frac{M^2}{EI}ds + \frac{1}{2}\int\frac{Q^2}{Hn}ds \quad . \quad . \quad . \quad . \quad (11.47)$$

Some springs are made, however, with one principal axis of the section along the radius BK of the cone. Then the other principal axis is perpendicular to this radius and at the same time perpendicular to BL. In that case the bending moment must be resolved into components with axes along these principal axes. Let I_1, I_2, denote the moments of inertia of the section about BK and the other principal axis respectively. Then the corresponding bending moments are

$$\left.\begin{aligned} M_1 &= M\cos a = Rr\sin\varphi\cos a \\ M_2 &= M\sin a = Rr\sin\varphi\sin a \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (11.48)$$

a being the semi angle of the cone.

Then the energy is

$$W = \frac{1}{2}\int\frac{M_1^2}{EI_1}ds + \frac{1}{2}\int\frac{M_2^2}{EI_2}ds + \frac{1}{2}\int\frac{Q^2}{Hn}ds \quad . \quad . \quad (11.49)$$

The integrals involved in W cannot be worked out until the variation of r and φ with s is known. Let us assume, as a particular case, that φ is constant.

Fig. 98

Let O be the vertex of the cone and let B′ be another point near B on the central line of the spring, so that $BB' = ds$. Let $OB = l$, $OB' = l + dl$. Let OB′ produced meet the circle throgh B at D (fig. 98), and let BD subtend an angle θ at the point K in fig. 97. Then

$$ds = BB' = BD\sec\varphi = rd\theta\sec\varphi.$$

Also $$dl = -DB' = -rd\theta\tan\varphi.$$

But we know that

$$r = l\sin a.$$

Therefore
$$\begin{aligned} dr &= dl\sin a \\ &= -rd\theta\sin a\tan\varphi \\ &= -ds\sin a\sin\varphi, \end{aligned}$$

whence
$$ds = -\frac{dr}{\sin a\sin\varphi}$$

If the values of r at the wider and narrower ends are r_0 and r_1 equation (11.47) becomes

$$W = -\frac{1}{2}\int_{r_0}^{r_1} {}^2R\left\{\frac{\sin^2\varphi}{EI} + \frac{\cos^2\varphi}{Hn}\right\}\frac{r^2 dr}{\sin\alpha\sin\varphi}$$

$$= \frac{1}{6}\frac{R^2(r_0{}^3 - r_1{}^3)}{\sin\alpha\sin\varphi}\left\{\frac{\sin^2\varphi}{EI} + \frac{\cos^2\varphi}{Hn}\right\}$$

Likewise equation (11.49) gives

$$W = \frac{1}{6}\frac{R^2(r_0{}^3 - r_1{}^3)}{\sin\alpha\sin\varphi}\left\{\frac{\sin^2\varphi\cos^2\alpha}{EI_1} + \frac{\sin^2\varphi\sin^2\alpha}{EI_2} + \frac{\cos^2\varphi}{Hn}\right\} \quad (11.50)$$

By the method used in deriving equation (11.9) from (11.7) we can prove here also that the relation between the force R and the extension x is

$$x = \frac{2W}{R} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11.51)$$

If the section of the spring is a long thin rectangle whose short side lies along the radius of the cone the moment of inertia I_2 is very large compared with I_1. Also it is proved in Chapter VII that H is nearly $4\,I_1$. Consequently, when the term containing I_2 is neglected, equation (11.50) becomes

$$W = \frac{1}{6}\frac{R^2(r_0{}^3 - r_1{}^3)}{I_1\sin\alpha\sin\varphi}\left\{\frac{\sin^2\varphi\cos^2\alpha}{E} + \frac{\cos^2\varphi}{4n}\right\}$$

$$= \frac{2R^2(r_0{}^3 - r_1{}^3)}{hb^3\sin\alpha\sin\varphi}\left\{\frac{\sin^2\varphi\cos^2\alpha}{E} + \frac{\cos^2\varphi}{4n}\right\}, \quad . \quad . \quad (11.52)$$

where h and b denote the long and short sides of the cross-section.

The same methods can be used for any shape of spring. In general, however, both φ and r will vary along the spring, and therefore both these must be treated as variables in the integrals for W.

193. Spring of any form with nearly horizontal coils.

In most spiral springs the coils are so nearly perpendicular to the axis that very little error arises from making this assumption. It is therefore worth while to work out for such a spring an equation connecting R and x which can be used when the radius of the coils and the section of the wire are both variable for the case of an axial force.

Since the angle φ is here assumed to be negligible the bending moment is consequently negligible. Therefore the energy stored in a length ds of the wire when it is subjected to a torque Q is

$$dW = \frac{1}{2}\frac{Q^2}{Hn}ds$$

The total energy stored is thus

$$W = \frac{1}{2}\int_0^l \frac{Q^2}{Hn}\, ds$$
$$= \frac{1}{2}\int_0^l \frac{r^2 R^2}{Hn}\, ds$$
$$= \frac{R^2}{2n}\int_0^l \frac{r^2}{H}\, ds\,.$$

The integral in the last line can be worked out when the radius of the coils r and the shape of the section are known as functions of s.

The other expression for the work stored in the spring is

$$W = \int_0^x R\,dx\,.$$

Equating the two values of W we get

$$\int_0^x R\,dx = \frac{R^2}{2n} C \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11.53)$$

where $$C = \int_0^l \frac{r^2}{H}\, ds\,.$$

Differentiating both sides of equation (11.53) with respect to the upper limit x we get

$$R = C \frac{R}{n}\frac{dR}{dx}\,;$$

whence $$\frac{dR}{dx} = \frac{n}{C}\,,$$

and therefore $$R = \frac{n}{C} x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11.54)$$

Thus the relation between R and x is known when the value of C is worked out, and the value of this integral is a constant for the spring, depending on its shape and size.

This last result can be applied to a thing like a watch spring, where all the coils lie originally in one plane, when a pull is applied perpendicular to this plane at one point, say the innermost end of the spring, and a balancing force and couple at the outermost end.

194. The bending of nearly circular rings in one plane.

The rings that we are about to investigate are thin rings whose middle lines are nearly circular both in the natural and in the strained states. The rings may be closed or open; the general equations will also apply to rods whose central lines form arcs of circles or whole circles.

Let a circle which nearly coincides with the central line of the ring be taken as a curve of reference. Let r be the radius of this circle. In the equations of equilibrium involving the stresses and external forces no appreciable error will arise if we treat the curve as if it were exactly a circle. It is only when we come to express stresses in terms of strains that the deviation from the circle need be considered.

Let (r, θ) be polar co-ordinates of a point on the circle, the pole O being at the centre of the circle. These will be taken as the co-ordinates of a point K on the central line of the ring in the equations of equilibrium. Let M, F, T, denote the bending moment, the shearing force, and the tension at K; and $M + dM$, $F + dF$, $T + dT$, the corresponding quantities at K′, whose coordinates are assumed to be $(r, \theta + d\theta)$.

Let the external forces per unit length of the ring be p acting radially towards the centre and q acting tangentially in the direction in which $d\theta$ is measured. The resultants of these acting on the piece of length $rd\theta$ are $prd\theta$ and $qrd\theta$.

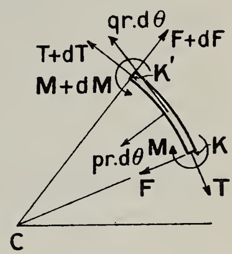


Fig. 99

Resolving all the forces acting on the element KK′ (fig. 99) along the radius through the middle point of KK′ we get, neglecting quantities of higher order than the first,

$$dF - prd\theta - Td\theta = 0 \quad . \quad . \quad (11.55)$$

Again, resolving perpendicular to this middle radius,

$$dT + qrd\theta + Fd\theta = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (11.56)$$

Taking moments about the centre of curvature C of the element KK′, which point coincides nearly with O,

$$dM + rdT + qr^2d\theta = 0 \quad . \quad . \quad . \quad . \quad (11.57)$$

When both sides of the above equations are divided by $d\theta$ the results are

$$\frac{dF}{d\theta} - T = pr \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11.58)$$

$$\frac{dT}{d\theta} + F = -qr \quad . \quad . \quad . \quad . \quad . \quad . \quad (11.59)$$

$$\frac{dM}{d\theta} + r\frac{dT}{d\theta} = -qr^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11.60)$$

It is interesting to notice that the last two of these equations give

$$\frac{1}{r}\frac{dM}{d\theta} = F \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11.61)$$

Since $rd\theta$ is the length of the element of the rod this is the same relation between M and F as in a straight beam.

We want an equation connecting M with p and q. Then we must eliminate F and T from our equations. Eliminating T from (11.58) and (11.59) we get

$$\frac{d^2F}{d\theta^2} + F = r\left(\frac{dp}{d\theta} - q\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (11.62)$$

From (11.61) and (11.62) we get

$$\frac{d^3M}{d\theta^3} + \frac{dM}{d\theta} = r^2\left(\frac{dp}{d\theta} - q\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (11.63)$$

which is the required equation.

We have assumed so far that the ring is in equilibrium. Let us now suppose that it is in motion as, for example, when it is oscillating. Let us assume that the circle of reference, whose radius is r and whose centre is at O, is either fixed or is moving with constant velocity without rotation. We shall now need to take account of the displacement of K from the circle. Let us suppose that the coordinates of K in the unstrained state are ϱ and θ, and let these become $(\varrho + u)$ and $(\theta + \eta)$ during the motion. As we shall not apply our equations to any cases except those in which u and η are always small we shall assume that these quantities are small at once. It follows that the element KK′ has a pair of component accelerations

$$\frac{\partial^2 u}{\partial t^2} \text{ radially outwards}$$

and $$\varrho\frac{\partial^2 \eta}{\partial t^2} \text{ in the direction in which } q \text{ is positive.}$$

Since ϱ differs very little from r we may replace ϱ by r in the latter acceleration; then the tangential acceleration can be written

$$r\frac{\partial^2 \eta}{\partial t^2}.$$

Let w denote the weight of unit volume of the material of the ring, and a the area of the cross-section. Then the forces necessary to give the above component acceleration to the element of length KK′ are

$$\frac{wa}{g} r d\theta \frac{\partial^2 u}{\partial t^2}$$

and $$\frac{wa}{g} r d\theta \cdot r\frac{\partial^2 \eta}{\partial t^2}$$

These two quantities should be inserted on the right hand sides of equations (11.55) and (11.56) respectively. An extra term is required in (11.57) also. These alterations amount to replacing

$$p \text{ by } \left(p + \frac{wa}{g}\frac{\partial^2 u}{\partial t^2}\right)$$

and $$q \text{ by } \left(q - \frac{wa}{g} r \frac{\partial^2 \eta}{\partial t^2}\right)$$

Also, since all the variables are now functions of both θ and t we must write partial instead of ordinary differential coefficients. Then equation (11.63) is altered to

$$\frac{\partial^3 M}{\partial\theta^3}+\frac{\partial M}{\partial\theta}=r^2\left(\frac{\partial p}{\partial\theta}-q\right)+\frac{wa}{g}r^2\frac{\partial^2}{\partial t^2}\left(\frac{\partial u}{\partial\theta}+r\eta\right) \quad . \ (11.64)$$

Equation (11.61) remains unaltered when the rotary inertia is neglected.

195. Relation between bending moment and displacements. The angle between the radius vector of length $(\varrho+u)$ and the normal to the central line of the ring is approximately

$$\begin{aligned}\alpha &= \frac{\partial(\varrho+u)}{\varrho\partial\theta}\\ &=\frac{1}{\varrho}\left\{\frac{\partial\varrho}{\partial\theta}+\frac{\partial u}{\partial\theta}\right\}\\ &=\frac{1}{r}\left\{\frac{\partial\varrho}{\partial\theta}+\frac{\partial u}{\partial\theta}\right\} \text{ approximately} \quad . \ (11.65)\end{aligned}$$

The angle between the two radii which, in the unstrained state, are inclined at $d\theta$, is $d(\theta+\eta)$. Then it follows that the angle between the two tangents at the ends of the element which subtends $d(\theta+\eta)$ at the pole is

$$d(\theta+\eta)-d\alpha$$

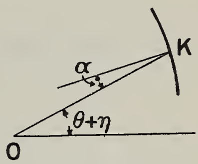


Fig. 100

The length of this element is approximately $rd\theta$, and consequently the curvature is

$$\begin{aligned}\frac{1}{R}&=\frac{d\theta+d\eta-d\alpha}{rd\theta}\\ &=\frac{1}{r}+\frac{1}{r}\frac{\partial\eta}{\partial\theta}-\frac{1}{r}\frac{\partial\alpha}{\partial\theta}\\ &=\frac{1}{r}+\frac{1}{r}\frac{\partial\eta}{\partial\theta}-\frac{1}{r^2}\left\{\frac{\partial^2\varrho}{\partial\theta^2}+\frac{\partial^2 u}{\partial\theta^2}\right\} \quad . \ . \ . \ (11.66)\end{aligned}$$

By putting $u=0$ and $\eta=0$ in this expression we get the curvature of the unstrained rod. The other two terms, namely,

$$\frac{1}{r}\frac{\partial\eta}{\partial\theta}-\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2},$$

represent the increase of curvature due to the displacements. Therefore the bending moment is given by

$$M=EI\left\{\frac{1}{r}\frac{\partial\eta}{\partial\theta}-\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}\right\} \quad . \ . \ . \ . \ . \ (11.67)$$

The bending moment M and the tension T both produce extensional strains in the fibres of the ring, but the strains produced by T are negligible compared with those produced by M since we are assuming thas the thickness b of the ring is negligible compared with the radius. If f_1 is the maximum stress due to M, and f_2 the stress due to T, we have

$$f_1 = \frac{\mathrm{M}b}{\mathrm{I}} \propto \frac{\mathrm{M}}{ab},$$

$$f_2 = \frac{\mathrm{T}}{a}.$$

Now M is, at nearly all points of the ring, of the same order of magnitude as Tr, whence it follows that

$$\frac{f_2}{f_1} \propto \frac{b}{r}.$$

We are now justified in neglecting the extensional strain of the central line, that is, the strain due to T. This strain is equal to the increase of length of the element KK′ divided by the natural length. Equating this strain to zero we get

$$\frac{(\varrho + u)d(\theta + \eta) - \varrho d\theta}{\varrho d\theta} = 0,$$

that is, when second order quantities are neglected,

$$u + \varrho\frac{\partial \eta}{\partial \theta} = 0$$

or

$$u + r\frac{\partial \eta}{\partial \theta} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11.68)$$

This last equation enables us to express the bending moment in the form

$$\mathrm{M} = -\frac{\mathrm{EI}}{r^2}\left\{u + \frac{\partial^2 u}{\partial \theta^2}\right\} \quad . \quad . \quad . \quad . \quad . \quad (11.69)$$

When the ring is not in motion u is a function of θ only and in that case the substitution of the expression for M from equation (11.69) in equation (11.63) gives

$$\frac{d^5 u}{d\theta^5} + 2\frac{d^3 u}{d\theta^3} + \frac{du}{d\theta} = \frac{r^4}{\mathrm{EI}}\left\{q - \frac{dp}{d\theta}\right\}. \quad . \quad . \quad . \quad (11.70)$$

With the same substitution equation (11.64) becomes

$$\frac{\partial^5 u}{\partial \theta^5} + 2\frac{\partial^3 u}{\partial \theta^3} + \frac{\partial u}{\partial \theta} = \frac{r^4}{\mathrm{EI}}\left\{q - \frac{\partial p}{\partial \theta}\right\} - \frac{war^4}{g\mathrm{EI}}\frac{\partial^2}{\partial t^2}\left(\frac{\partial u}{\partial \theta} + r\eta\right),$$

or, in terms of η,

$$\frac{\partial^6 \eta}{\partial \theta^6} + 2\frac{\partial^4 \eta}{\partial \theta^4} + \frac{\partial^2 \eta}{\partial \theta^2} = \frac{r^3}{\mathrm{EI}}\left(\frac{\partial p}{\partial \theta} - q\right) - \frac{war^4}{g\mathrm{EI}}\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2 \eta}{\partial \theta^2} - \eta\right) \quad (11.71)$$

We have now obtained all the differential equations necessary for dealing with the equilibrium and small oscillations of thin rings or thin rods whose equilibrium shapes are nearly circular arcs. For any special problem we have first to solve these equations, and then to determine the constants by means of the conditions at the open ends for incomplete rings, and by means of equivalent conditions for complete rings. We shall now state what these conditions are.

196. Conditions which determine the constants of integration for a rod in equilibrium,

For a rod in equilibrium for which p and q are known it would be possible to find u from equation (11.70), and then, if we knew five independent conditions like the end-conditions of a beam, we could determine the five constants which would arise in integrating this equation of the fifth order. It is, however, easier to solve (11.63) for M, and then, if we require it, to find u from (11.69). This, at least, is the easiest process when the ends are open and not both clamped. The end-conditions for an open ring are much the same as for a beam. The following conditions are true:

at a free end

$$\mathrm{M}=0, \text{ and } \mathrm{F}=\frac{1}{r}\frac{d\mathrm{M}}{d\theta}=0; \quad . \quad . \quad . \quad . \quad (11.72)$$

at a pinned end

$$\mathrm{M}=0, \text{ and } u \text{ is known}; \quad . \quad . \quad . \quad . \quad . \quad (11.73)$$

at a clamped end

$$u \text{ and } \frac{du}{d\theta} \text{ are both known}. \quad . \quad . \quad . \quad . \quad (11.74)$$

Since the preceding rules give only two conditions at each end we get at most only four independent conditions whereas we have five constants to determine. This extra condition is contained in equation (11.58) which we had to differentiate to deduce (11.62). Thus (11.58) and (11.61) give

$$\begin{aligned} r\mathrm{T}+pr^2 &= r\frac{d\mathrm{F}}{d\theta} \\ &= \frac{d^2\mathrm{M}}{d\theta^2} = -\frac{\mathrm{EI}}{r^2}\left\{\frac{d^4u}{d\theta^4}+\frac{d^2u}{d\theta^2}\right\} \quad . \quad . \quad . \quad (11.75) \end{aligned}$$

If T is known at one end of the beam this gives an extra end-condition. Suppose $\mathrm{T}=\mathrm{T}_0$ at this end. Then at that end

$$\frac{d^2\mathrm{M}}{d\theta^2}=r\mathrm{T}_0+pr^2,$$

which is our new condition.

If a rod has a free end on which no finite forces act M can be found completely from equation (11.63) and from the three conditions,

$$\mathrm{M}=0, \quad \frac{d\mathrm{M}}{d\theta}=0, \quad \frac{d^2\mathrm{M}}{d\theta^2}=pr^2, \quad . \quad . \quad . \quad . \quad (11.76)$$

which are all true at the free end.

Again if a rod has an end at which a known finite force is applied, but which is not clamped, this finite force gives rise to a known shearing force F_0 and a known tension T_0. Then the end conditions are

$$M = 0, \quad \frac{dM}{d\theta} = rF_0, \quad \frac{d^2M}{d\theta^2} = rT_0 + pr^2 \quad . \quad . \quad . \quad (11.77)$$

It is thus clear that the stresses in an unclosed ring which has one free end are completely determined without any recourse to the relations between stress and strain. This is equally true, as we already knew, for a straight rod. If, after we have determined M, we wish to find u, we must solve the equation

$$\frac{d^2u}{d\theta^2} + u = -\frac{r^2}{EI} M. \quad . \quad . \quad . \quad . \quad . \quad . \quad (11.78)$$

The general solution of this equation is

$$u = u_1 + A\cos\theta + B\sin\theta, \quad . \quad . \quad . \quad . \quad . \quad (11.79)$$

where u_1 is a particular integral due to the term containing M. The two terms $A \cos \theta + B \sin \theta$ represent merely a bodily displacement of the whole ring, which is, of course, accompanied by no strains. This displacement has no importance since it is purely relative to the point which we have taken as origin, and by taking as origin a point which moves in a different way while the ring is being strained the displacement could be reduced to zero.

If one end is clamped and one pinned, or if both ends are clamped, or both ends pinned, then not only is u known at both ends but also η is known at both ends. If $\theta = \alpha$ and $\theta = \beta$ at the ends, and if the values of η at the two ends are η_0 and η_1, we get

$$\int_\alpha^\beta u d\theta = r\int_\alpha^\beta \frac{d\eta}{d\theta} d\theta$$

$$= r(\eta_1 - \eta_0) \quad . \quad . \quad . \quad . \quad . \quad . \quad (11.80)$$

This is an extra condition which is necessary to replace (11.75) since T_0 is not known in these cases.

In every case of a rod with open ends we know five conditions to determine the five constants in the expression for u. For instance, when both ends are clamped, u and $\frac{du}{d\theta}$ are known at both ends and $(\eta_1 - \eta_0)$ is known.

A complete ring may be regarded as an open ring with both ends joined together so that u and $\frac{du}{d\theta}$ are the same for both ends at the junction. Any point whatever on the ring may be regarded as the junction. If the point $\theta = \alpha$ be regarded as the junction then our conditions can be used in the form

$$\left.\begin{aligned} u &= 0 \\ \frac{du}{d\theta} &= 0 \end{aligned}\right\} \begin{aligned} &\text{where } \theta = a \\ &\text{and where } \theta = 2\pi - a \end{aligned}$$

and

$$\int_0^{2\pi} u d\theta = 0.$$

The preceding five conditions are sufficient to determine the constants in u in any case. It is sometimes convenient to use the conditions in other forms. One condition that we can deduce from the preceding is

$$-\int_0^{2\pi} \mathrm{M} d\theta = \frac{\mathrm{EI}}{r^2}\int_0^{2\pi}\left(\frac{d^2u}{d\theta^2} + u\right)d\theta = \frac{\mathrm{EI}}{r^2}\left[\frac{du}{d\theta} - \eta\right]_0^{2\pi} = 0.$$

197. The strains due to any given forces.

When p and q are given the equations can always be solved to the extent of expressing the bending moment M and the displacement u in terms of integrals. Thus, integrating (11.63) once we get

$$-\left\{\frac{d^2\mathrm{M}}{d\theta^2} + \mathrm{M}\right\} = r^2p - r^2\int q d\theta + \mathrm{A} \quad . \quad . \quad . \quad (11.81)$$

Let the right hand side of this equation be denoted by $\mathrm{A} + f(\theta)$. Then it is shown in the appendix that the solution is

$$-\mathrm{M} = \int_0^{\theta} f(v)\sin(\theta - v)\,dv + \mathrm{A} + \mathrm{B}\cos\theta + \mathrm{C}\sin\theta \quad . \quad (11.82)$$

The three constants A, B, C, have to be determined by the end-conditions.

Let us now suppose that the integral in the expression for M has been determined, so that M may be written

$$-\mathrm{M} = \mathrm{F}(\theta) + \mathrm{A} + \mathrm{B}\cos\theta + \mathrm{C}\sin\theta \quad . \quad . \quad . \quad . \quad (11.83)$$

or

$$\frac{\mathrm{EI}}{r^2}\left\{\frac{d^2u}{d\theta^2} + u\right\} = \mathrm{F}(\theta) + \mathrm{A} + \mathrm{B}\cos\theta + \mathrm{C}\sin\theta$$

If the right hand side of this equation were zero the value of u would be given by the equation

$$\frac{\mathrm{EI}}{r^2}u = \mathrm{H}\cos\theta + \mathrm{K}\sin\theta.$$

The particular integrals corresponding to the terms containing A, B, C, can be found by the usual rules for finding the particular integrals of linear equations. These rules give for the particular integral corresponding to these terms

$$\frac{\mathrm{EI}}{r^2}u = \mathrm{A} + \tfrac{1}{2}\mathrm{B}\theta\sin\theta - \tfrac{1}{2}\mathrm{C}\theta\cos\theta$$

The particular integral corresponding to $\mathrm{F}(\theta)$ is similar to that due to $f(\theta)$ in equation (11.82). Then the total value of u is given by

19

$$\frac{EI}{r^2}u = \int^{\theta} F(v)\sin(\theta - v)\,dv$$
$$+ A + \tfrac{1}{2}\theta(B\sin\theta - C\cos\theta) + H\cos\theta + K\sin\theta \quad (11.84)$$

This is the general expression for u involving the five constants which are to be determined by the end-conditions. It may be more convenient, in easy cases, not to get the result by this method because the integrals may turn out to be clumsy. The method has, however, the advantage of being direct since the whole process is reduced to mere integration.

198. The pressure required to produce a given deformation. If the displacement u is given, equation (11.70) gives at once the value of $q - \frac{dp}{d\theta}$, but does not give either p or q separately. If the value of either p or q is chosen arbitrarily this equation gives the value of the other quantity. Moreover, since we have to perform an integration to get p when u and q are given, there is an indeterminate constant in the expression for p. There is an obvious physical explanation of this fact, for it is clear that a uniform pressure applied to a whole ring, or to a portion of a ring with fixed ends, would cause no deformation of the ring. The deformation is due to the variation of p. It follows then that the constant which would appear in finding p for a portion of a ring with fixed ends is an indeterminable constant.

Suppose, however, that q is zero and that the ring has a free end where both T and M are zero. Then equations (11.81) and (11.75) give

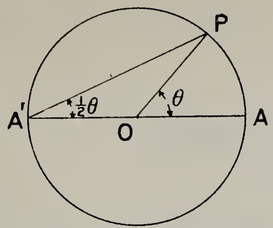
$$rT + r^2p + M = r^2p + A$$

To make M and T vanish together the constant A must be zero. In that case the equation for p at any point is

$$r^2p = \frac{d^2M}{d\theta^2} + M = -\frac{EI}{r^2}\left\{\frac{d^4u}{d\theta^4} + 2\frac{d^2u}{d\theta^2} + u\right\} \quad . \quad . \quad (11.85)$$

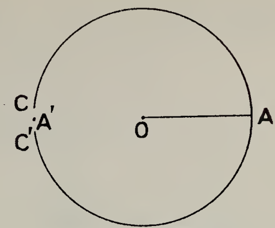
199. The piston ring problem. The problem before us is to determine the form of an open ring of uniform cross section so that, when it is pressed into a cylinder, it will just close up its ends and exert a uniform pressure on the containing cylinder.

Let the radius of the circle into which the central line bends when the ring is in the cylinder be denoted by r; let A (fig. 101) be the point of the ring opposite the centre of the gap. Let the initial line from which θ is measured pass through A, so that this line produced backwards passes through the centre of the gap. Let the pole O be situated half way between the centre of the gap and the point A, the curve of the ring being supposed to be produced through the gap when the ring is free. Let $(r + u)$ be the radius vector in the free ring of the particle which is at (r, θ) in the closed ring.



The Compressed Ring

Fig. 101 a



The Free Ring

Fig. 101 b

In this case it is easy to find M without solving equation (11.81). The uniform pressure along the arc A′P in the closed ring is statically equivalent to an equal uniform pressure distributed along the chord A′P The resultant is a force of magnitude $p \times A'P$ acting through the mid point of A′P and the centre of the ring. The moment of this force about P is

$$\begin{aligned} M &= \tfrac{1}{2} A'P \times p \times A'P \\ &= \tfrac{1}{2} p \times (2r \cos \tfrac{1}{2}\theta)^2 \\ &= pr^2(1 + \cos\theta) \quad \ldots \ldots \quad (11.86) \end{aligned}$$

This is the bending moment at θ in the strained ring, and since u represents an inward displacement in this problem we must replace u by $-u$ in equation (11.69). Then

$$\frac{EI}{r^2}\left(u + \frac{d^2u}{d\theta^2}\right) = M = pr^2(1 + \cos\theta) \quad \ldots \quad (11.87)$$

This equation can be solved by the rules given in the theory of linear differential equations, and its solution is

$$u = \frac{pr^4}{EI}(1 + \tfrac{1}{2}\theta \sin\theta) + A\cos\theta + B\sin\theta \quad \ldots \quad (11.88)$$

The constants A and B depend only on where the pole O is taken relative to the strained ring. We have already chosen this position so that the initial line is an axis of symmetry, and so that u has the same values at $\theta = 0$ and $\theta = \pi$. The first of these conditions gives

$$\frac{du}{d\theta} = 0 \text{ where } \theta = 0,$$

whence $$0 = B.$$

The second gives

$$\frac{pr^4}{EI} + A = \frac{pr^4}{EI} - A$$

which makes $A = 0$. Then

$$u = \frac{pr^4}{EI}(1 + \tfrac{1}{2}\theta \sin\theta) \quad \ldots \ldots \quad (11.89)$$

The angular displacement of each end of the ring when it is closed is

$$\eta_1 = \int_0^{\pi} \frac{u}{r} d\theta$$

$$= \frac{pr^3}{EI}\left[\theta - \tfrac{1}{2}\theta\cos\theta + \tfrac{1}{2}\sin\theta\right]_0^{\pi}$$

$$= \frac{3\pi pr^3}{2EI} \qquad (11.90)$$

Then the gap in the open ring is

$$c = 2r\eta_1 = \frac{3\pi pr^4}{EI} \qquad (11.91)$$

We can now express p in terms of the gap. Thus

$$p = \frac{EIc}{3\pi r^4} \qquad (11.92)$$

If the depth of the ring is denoted by h the pressure per unit *area* between the ring and the cylinder is

$$\frac{p}{h} = \frac{EIc}{3\pi r^4 h} \qquad (11.93)$$

The expression for u in terms of the gap is

$$u = \frac{c}{3\pi}(1 + \tfrac{1}{2}\theta\sin\vartheta) \qquad (11.94)$$

The values of u where $\theta = 0$ and where $\theta = \frac{\pi}{2}$ are respectively

$$\left.\begin{aligned} u_0 &= \frac{c}{3\pi} \\ \text{and} \qquad u_1 &= \frac{c}{3\pi}\left(1 + \frac{\pi}{4}\right)\end{aligned}\right\} \qquad (11.95)$$

The diameters through the gap and perpendicular to the gap in the free ring are respectively

$$\left.\begin{aligned} 2r + 2u_0 &= 2r + \frac{2c}{3\pi} \\ \text{and} \qquad 2r + 2u_1 &= 2r + \frac{2c}{3\pi} + \tfrac{1}{6}c\end{aligned}\right\} \qquad (11.96)$$

The difference between these diameters is ^{1}c.

It is usual for piston rings to have rectangular cross-sections. Let b denote the radial thickness of the ring. Then

$$I = \tfrac{1}{12} b^3 h$$

and the maximum stress, which occurs at the point A, where $\theta = 0$, is

$$f = \tfrac{1}{2} b \cdot \frac{M}{I}$$
$$= \tfrac{1}{2} b \frac{2pr^2}{\tfrac{1}{12} b^3 h}$$
$$= \frac{12pr^2}{b^2 h} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11.97)$$

The ratio of this maximum stress to the pressure per unit area is

$$\frac{fh}{p} = 12 \frac{r^2}{b^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11.98)$$

When we use the value of p in terms of the gap the stress can be put in the form

$$f = \frac{bc}{3\pi r^2} E \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11.99)$$

The piston ring whose central line has the form of the curve given by equation (11.89) could be closed into its circular shape by a pair of opposing forces acting on the open ends, as shown in fig. 102. For the bending moment at K due to the force R at A′ is

$$M = R \times A'N$$
$$= Rr(1 + \cos\theta) \quad (11.100)$$

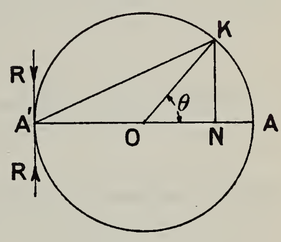


Fig. 102

This is exactly the same bending moment as that due to a uniform pressure p, provided that

$$R = pr. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11.101)$$

Then it follows that the deformation of the ring caused by the force R is exactly the same as that due to the uniform pressure, whence it follows that the force R bends the ring into a perfect circle.

Whether the force R is or is not equal to pr it produces an *inward* radial diplacement abtained by replacing pr by R in equation (11.89), which displacement is

$$u = \frac{Rr^3}{EI} \cdot (1 + \tfrac{1}{2}\theta \sin\theta). \quad . \quad . \quad . \quad . \quad . \quad (11.102)$$

If the force R is reversed an equal outward radial displacement is produced.

The usual method of fitting a piston ring on a piston is to open the ring by a pair of forces similar to the forces R reversed. These forces only increase all the magnitudes of the displacements of the natural ring from the circular shape in the same ratio. That is, the total displacement of the ring from the circular form, when it is deformed by forces R in opening the ring, is given by

$$u = \frac{(R + pr)r^3}{EI} (1 + \tfrac{1}{2}\theta \sin\theta) \quad . \quad . \quad . \quad . \quad (11.103)$$

The ring will go over the piston only when the smallest diameter exceeds the diameter of the closed ring by twice the radial thickness of the ring. This smallest diameter is the one through the gap, and it exceeds the diameter of the closed ring by twice the value of u at the point where $\theta = 0$; that is, by

$$2u_0 = \frac{2(R + pr)r^3}{EI}.$$

Equating u_0 to the thickness we get

$$b = \frac{(R + pr)r^3}{EI} \qquad (11.104)$$

The stress due to the forces R is, by equation (11.97),

$$f_1 = \frac{12Rr}{b^2 h} \qquad (11.105)$$

Also the increase of the gap is

$$c_1 = \frac{3\pi R r^3}{EI} \qquad (11.106)$$

Adding corresponding sides of equations (11.97) and (11.105) we get

$$f + f_1 = \frac{12r(pr + R)}{b^2 h}, \qquad (11.107)$$

and this becomes, by equation (11.104),

$$f + f_1 = \frac{b^2}{r^2}E \qquad (11.108)$$

Thus the sum of the maximum stresses in the ring when it is being fitted on the piston and when it is in the cylinder is constant. To keep the stress as low as possible under all conditions f_1 will have to be made equal to f. Then

$$f = f_1 = \frac{1}{2}\frac{b^2}{r^2}E, \qquad (11.109)$$

and the free gap corresponding to this value of f is

$$c = \frac{3\pi r^2}{Eb.}f$$

$$= \frac{3\pi}{2}b = 4{\cdot}71\,b \qquad (11.110)$$

Since the stress f_1 exists for only a very short time and the stress f for a very long time it is probably much better to let f_1 be greater than f. Probably a good rule would be to make f_1 equal to $2f$. This would give $c = \pi b$.

200. Circular piston ring.

Suppose a piston ring were cut in the form of a perfect circle with a uniform rectangular cross-section. We shall find its shape when it is put into a cylinder, and the forces it exerts on the cylinder.

Let us suppose that the length of the ring is such that, when it is fitted into the cylinder, the open ends just come together without exerting any pressure on each other.

Let the external radius of the ring in the unstrained state be $(r+c)$, and let the internal radius of the cylinder be r. Then, wherever the ring fits the cylinder, the change of curvature of the ring is approximately

$$\frac{1}{r}-\frac{1}{r+c}=\frac{c}{r^2},$$

and therefore, the bending moment is

$$M=\frac{EIc}{r^2}. \quad . \quad . \quad (11.111)$$

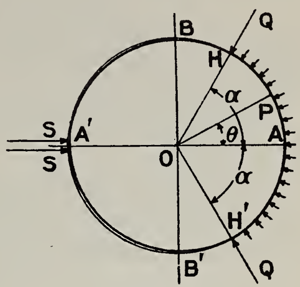


Fig. 103

Since the ends are free, equation (11.85) gives

$$p=\frac{1}{r^2}\left(\frac{d^2M}{d\theta^2}+M\right)$$

$$=\frac{EIc}{r^4} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11.112)$$

Thus the pressure and the bending moment are both constant at all points where the ring fits the cylinder. But the ring cannot fit all along its length for that would require that M should have the same constant value at the free ends, and we know that M is zero there. The problem is to find where the ring does fit the cylinder.

It is easy to see that the free ends must be in contact with the cylinder. and whatever finite forces the cylinder exerts on those ends the bending moment vanishes there. Thus the change of curvature at the ends is zero. It follows that the curvature of the ring near the ends is less than the curvature of the cylinder, and therefore that the ring cannot fit the cylinder near its free ends. Then there must be a region between each end and some point on the ring where there is no contact between the cylinder and the ring. The two regions where there is no contact are from A′ to H and from A′ to H′ in fig. 103. In order that M should be constant between H and H′ it can be shown that there must be finite forces Q at H and H′. We shall now show that all the conditions of the problem are satisfied by the following:—

(1) finite forces S acting on each free end;
(2) a finite force Q acting at each of the points H and H′;
(3) a uniform pressure p per unit length between H and H′;
(4) no contact from A′ to H and from A′ to H′.

Let the angles AOH and AOH′ be each α. Then the bending moment at a point P in the region where the uniform pressure acts, and at angular distance θ from the point opposite the gap, is

$$M = Sr\sin\theta + Qr\sin(\alpha - \theta) + pr^2\{1 - \cos(\alpha - \theta)\} \quad (11.113)$$

This result is obtained by taking moments about P of all the forces acting on the part A'BHP. The part due to the uniform pressure on HP is exactly similar to the bending moment in (11.86).

Writing φ for $(\alpha - \theta)$ in the last equation, and also writing $(\alpha - \varphi)$ for θ, we get

$$\begin{aligned} M &= Sr\sin(\alpha - \varphi) + Qr\sin\varphi + pr^2(1 - \cos\varphi) \\ &= Sr\{\sin\alpha\cos\varphi - \cos\alpha\sin\varphi\} + Qr\sin\varphi + pr^2(1 - \cos\varphi) \quad (11.114) \end{aligned}$$

But we have already found that, in the region between H' and H,

$$M = \frac{EIc}{r^2};$$

therefore

$$\frac{EIc}{r^2} = pr^2 + (Sr\sin\alpha - pr^2)\cos\varphi + (Qr - Sr\cos\alpha)\sin\varphi \quad (11.115)$$

for all values of φ between 0 and 2α. The two sides of this equation will be identically equal provided the following three equations hold:

$$\left.\begin{aligned} pr^2 &= \frac{EIc}{r^2} \\ Sr\sin\alpha - pr^2 &= 0 \\ Qr - Sr\cos\alpha &= 0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (11.116)$$

These three equations are just sufficient to determine the values of p, S, and Q. Thus

$$\left.\begin{aligned} p &= \frac{EIc}{r^4} \\ S &= pr\operatorname{cosec}\alpha = \frac{EIc}{r^3}\operatorname{cosec}\alpha \\ Q &= S\cos\alpha = \frac{EIc}{r^3}\cot\alpha \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (11.117)$$

201. Shape of the ring between A′ and H.

At a point between A′ and H at angular distance θ from A the bending moment is

$$M = Sr\sin\theta = \frac{EIc}{r^2}\frac{\sin\theta}{\sin\alpha}.$$

Therefore the equation for the displacement is

$$\frac{EI}{r^2}\left(\frac{d^2u}{d\theta^2} + u\right) = \frac{EIc}{r^2}\frac{\sin\theta}{\sin\alpha},$$

or

$$\frac{d^2u}{d\theta^2} + u = c\frac{\sin\theta}{\sin\alpha} \quad . \quad . \quad . \quad . \quad . \quad . \quad (11.118)$$

The solution of this equation is

$$u = A\cos\theta + B\sin\theta - \tfrac{1}{2}c\frac{\theta\cos\theta}{\sin\alpha} \quad . \quad . \quad . \quad (11.119)$$

The conditions that this value of u must satisfy are

$$u = c \text{ where } \theta = \pi$$

and

$$\left.\begin{aligned} u &= c \\ \frac{du}{d\theta} &= 0 \end{aligned}\right\} \text{ where } \theta = \alpha$$

These give

$$c = -A + \tfrac{1}{2}c\frac{\pi}{\sin\alpha} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11.120)$$

$$c = A\cos\alpha + B\sin\alpha - \tfrac{1}{2}c\frac{\alpha\cos\alpha}{\sin\alpha} \quad . \quad . \quad . \quad . \quad . \quad (11.121)$$

$$0 = -A\sin\alpha + B\cos\alpha - \frac{c}{2\sin\alpha}(\cos\alpha - \alpha\sin\alpha) \quad . \quad (11.122)$$

Eliminating B from the last two equations we get

$$c\cos\alpha = \left(A - \tfrac{1}{2}c\frac{\alpha}{\sin\alpha}\right)(\cos^2\alpha + \sin^2\alpha) + \tfrac{1}{2}c\cos\alpha$$

$$= A - \tfrac{1}{2}c\frac{\alpha}{\sin\alpha} + \tfrac{1}{2}c\cos\alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11.123)$$

Now adding corresponding sides of (11.120) and (11.123) we get

$$c + c\cos\alpha = \tfrac{1}{2}c\frac{\pi - \alpha}{\sin\alpha} + \tfrac{1}{2}c\cos\alpha\,,$$

whence

$$\begin{aligned} \pi - \alpha &= 2\sin\alpha + \sin\alpha\cos\alpha \\ &= 2\sin\alpha + \tfrac{1}{2}\sin 2\alpha \quad . \quad . \quad . \quad . \quad (11.124) \end{aligned}$$

This equation determines the angle α and therefore the positions of the points H and H′ where the ring comes into contact with the cylinder. An approximate solution can be obtained graphically and then the solution can be improved by analytical methods. This process gives

$$\begin{aligned} \alpha &= 1{\cdot}0025 \text{ radians} \\ &= 57^0\,26' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11.125) \end{aligned}$$

The correctness of this answer can be verified by the tables at once. Thus, with the value of α in (11·125)

$$\pi - \alpha = 2{\cdot}1391$$

and

$$2\sin\alpha + \tfrac{1}{2}\sin 2\alpha = 2{\cdot}1392\,,$$

which are near enough for practical purposes.

By equation (11.120)

$$\begin{aligned} A &= c\left(\frac{\pi}{2\sin\alpha} - 1\right) \\ &= 0{\cdot}8638c \end{aligned}$$

Also either of the equations (11.121) or (11.122) gives

$$B = \tfrac{1}{2} c \left(\sin\alpha + \frac{1}{\sin\alpha} \right)$$
$$= 1{\cdot}0146 c$$

Therefore

$$\frac{u}{c} = 0{\cdot}8638 \cos\theta + 1{\cdot}0146 \sin\theta - 0{\cdot}5933\, \theta \cos\theta \quad . \quad (11.126)$$

The clearance between the ring and the cylinder is $(u - c)$, and this is a maximum where u is a maximum, that is, where

$$\frac{du}{d\theta} = 0,$$

or where

$$-0{\cdot}8638 \sin\theta + 0{\cdot}4213 \cos\theta + 0{\cdot}5933\, \theta \sin\theta = 0 .$$

This is satisfied when $\theta = 1$ or $\theta = 2{\cdot}6$ approximately. The first of these roots is clearly α. The second is the one that makes u a maximum, and since 2·6 radians is approximately 149° this maximum value of u is

$$u = \{-0{\cdot}8638 \cos 31^0 + 1{\cdot}0146 \sin 31^0 + 0{\cdot}5933 \times 2{\cdot}6 \sin 31^0\}\, c$$
$$= 1{\cdot}105 c \quad , \quad . \quad . \quad . \quad (11.127)$$

Thus the maximum clearance is $0{\cdot}105c$ and occurs at 31° from the centre of the gap. Since no other value of θ between α and π makes $\frac{du}{d\theta}$ zero it follows that the value of u in (11.127) is the maximum value, and also that u gradually approaches c as θ varies from 2·6 to α or from 2·6 to π; that is, the ring lies inside the cylinder between A′ and H, and therefore also between A′ and H′. Thus the solution we have obtained satisfies all the conditions of the problem.

202. Piston ring of variable thickness.

Suppose the outside of a piston ring has the form of a perfect circle. We shall find what the value of I must be to make the ring exert uniform pressure on the cylinder.

Let r denote the internal radius of the cylinder and $(r + c)$ the external radius of the free ring. Then the inward radial displacement of every point of the ring is c. Therefore

$$M = \frac{EI}{r^2}\left(u + \frac{d^2u}{d\theta^2}\right)$$
$$= \frac{EIc}{r^2} \quad . \quad . \quad . \quad . \quad (11.128)$$

But by (11.86)

$$M = pr^2 (1 + \cos\theta).$$

Therefore, equating these two values of M, we get

$$\frac{\mathrm{EI}c}{r^2} = pr^2(1 + \cos\theta),$$

whence

$$\mathrm{I} = \frac{pr^4}{\mathrm{E}c}(1 + \cos\theta) \quad . \quad . \quad . \quad . \quad . \quad (11.129)$$

If the ring has a rectangular section with a uniform axial depth h and a variable radial width b this last equation gives the width. Thus

$$b^3 = \frac{12pr^4}{\mathrm{E}ch}(1 + \cos\theta) \quad . \quad . \quad . \quad . \quad . \quad (11.130)$$

Writing p_1 for the pressure per unit area on the ring we have, since p is the pressure per unit length,

$$p_1 h = p$$

whence

$$b^3 = \frac{12p_1 r^4}{\mathrm{E}ch^2}(1 + \cos\theta)$$

$$b = \left(\frac{24p_1 r^4}{\mathrm{E}ch^2}\cos^2\frac{\theta}{2}\right)^{\frac{1}{3}} \quad . \quad . \quad . \quad . \quad . \quad (11.131)$$

It will be noticed that the width b vanishes when $\theta = \pm 180^\circ$, that is, at the free ends. This is due to the fact that the bending moment vanishes and the change of curvature is finite at these ends. There is only one practical way of avoiding wedge-shaped ends to a piston ring and that is by making the radius of curvature of the outside of the ring at its free ends the same as the radius of the inside of the cylinder. Since M is proportional to the change of curvature equation (11.87) shows that the ring whose form is defined by (11.89) has a radius of curvature r where $\theta = \pi$.

203. The eccentric ring.

To save trouble in manufacture piston rings have been made with the inside and the outside both circles whose centres were not quite coincident. The gap is then cut out at the thinnest part. Let us see what pressure the ring exerts assuming that it fits the cylinder everywhere.

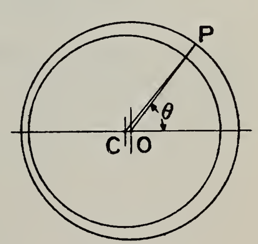


Fig. 104

Let O be the centre of the exterior of the ring, C the centre of the interior, and let $\mathrm{OC} = c$. Then, if P is a point on the exterior and if the radius of the exterior circle is ϱ,

$$\begin{aligned}\mathrm{CP} &= \mathrm{OP} + \mathrm{CO}\cos\theta \\ &= \varrho + c\cos\theta\end{aligned}$$

If the radius of the interior circle is r then the radial thickness of the ring is

$$\begin{aligned} b &= \mathrm{CP} - r \\ &= \varrho + c\cos\theta - r \\ &= k + c\cos\theta, \end{aligned} \qquad \qquad (11.132)$$

k being written for the small difference $(\varrho - r)$.

Let $\varrho - \delta$ denote the internal radius of the cylinder. Then the radial displacement u is everywhere δ. Consequently

$$\mathrm{M} = \frac{\mathrm{EI}\delta}{r^2} \qquad \qquad (11.133)$$

Now p is obtained from equation (11.85). Thus

$$p = \frac{1}{r^2}\left\{\frac{d^2\mathrm{M}}{d\theta^2} + \mathrm{M}\right\} \qquad \qquad (11.134)$$

In the present case the variable factor in M is I, which varies because b varies. Assuming that the axial depth of the ring is constant and equal to h we get

$$\begin{aligned} \mathrm{M} &= \frac{1}{12}\frac{\mathrm{E}\delta b^3 h}{r^2} \\ &= \mathrm{H}b^3, \end{aligned} \qquad \qquad (11.135)$$

H being a constant. Then

$$p = \frac{\mathrm{H}}{r^2}\left\{\frac{d^2(b^3)}{d\theta^2} + b^3\right\}$$

But $$\frac{db^3}{d\theta} = -3b^2 c\sin\theta \text{ by } (11.132);$$

$$\frac{d^2(b^3)}{d\theta^2} = 6bc^2\sin^2\theta - 3b^2c\cos\theta.$$

Therefore

$$\begin{aligned} p &= \frac{\mathrm{H}b}{r^2}\{6c^2\sin^2\theta - 3bc\cos\theta + b^2\} \\ &= \frac{\mathrm{H}b}{r^2}\{k^2 - kc\cos\theta + c^2(6\sin^2\theta - 2\cos^2\theta)\} \end{aligned} \qquad (11.136)$$

If $k = c$ then the thickness vanishes at the gap and therefore M vanishes at the gap. If the calculated M did not vanish at the gap the ring could not fit the cylinder for we know that the actual bending moment must vanish at a free end. Assuming that $k = c$ we get

$$\begin{aligned} p &= \frac{\mathrm{H}bc^2}{r^2}(7 - 8\cos\theta)(1 + \cos\theta) \\ &= \frac{\mathrm{H}c^3}{r^2}(7 - 8\cos\theta)(1 + \cos\theta)^2 \end{aligned} \qquad (11.137)$$

This vanishes at the gap where $\theta = \pi$ and also at the points where $8\cos\theta = 7$, that is, where $\theta = \pm 29^0$. Between the points where $\theta = -29^0$ and $\theta = +29^0$ the value of p given by (11.137) is negative.

But the cylinder cannot supply a negative pressure. It follows then that the ring cannot fit the cylinder even when the free ends taper to sharp edges. Then a ring whose inner and outer boundaries are circles and whose axial depth is constant can in no case fit a cylinder. When such a ring is used as a piston ring the actual distribution of pressure is different from that given by (11.137) for this is obtained on the assumption that the ring is forced to fit the cylinder.

204. Stresses in closed rings.

Suppose a uniform closed ring is subjected to n equal radial forces P distributed at equal angular intervals 2a round the ring.

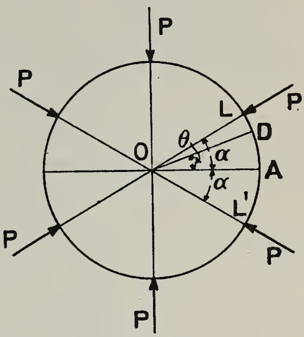


Fig. 105

Six forces are shown in fig. 105 but the method could be used for any number of forces.

We need only consider the portion L′L between two forces. Over the whole of this portion p and q are both zero, and consequently equation (11.70) becomes

$$\frac{d^5u}{d\theta^5}+2\frac{d^3u}{d\theta^3}+\frac{du}{d\theta}=0 \quad . \quad . \quad . \quad . \quad . \quad (11.138)$$

The complete integral of this is given by (11.84). We may write the result in the form

$$u=A+(B+C\theta)\sin\theta+(H+K\theta)\cos\theta \quad . \quad . \quad (11.139)$$

Now it is clear that the ring is symmetrical about the diameter through A. That is, the value of u at θ must be equal to the value of u at $-\theta$ for all values of θ between 0 and α. Thus

$$\begin{aligned} A+(B+C\theta)\sin\theta+(H+K\theta)\cos\theta &\\ &=A+(B-C\theta)\sin(-\theta)+(H-K\theta)\cos(-\theta)\\ &=A-(B-C\theta)\sin\theta+(H-K\theta)\cos\theta. \end{aligned}$$

Since this is an identity, the coefficients of sin θ and of θ cos θ on the two sides must be equal; that is,

$$B=-B,$$
$$K=-K,$$

whence and

$$\left.\begin{aligned} B&=0\\ K&=0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11.140)$$

Therefore equation (11.139) reduces to

$$u=A+C\theta\sin\theta+H\cos\theta, \quad . \quad . \quad . \quad . \quad (11.141)$$

the right hand side now containing only even functions of θ.

We need three conditions to determine the three remaining constants. The following three will suffice:—

$$\left.\begin{aligned}\frac{du}{d\theta}&=0\\ F&=-\tfrac{1}{2}P\end{aligned}\right\}\text{ where } \theta=\alpha; \quad . \quad . \quad . \quad (11.142)$$

and

$$\eta_L-\eta_A=0.$$

The last condition, expressed in terms of u, can be written thus

$$\int_0^\alpha u d\theta=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11.143)$$

Also the second condition is, if u represents the outward displacement,

$$\frac{EI}{r^3}\left(\frac{d^3u}{d\theta^3}+\frac{du}{d\theta}\right)=\tfrac{1}{2}P \text{ where } \theta=\alpha.$$

This combined with the first condition gives

$$\frac{d^3u}{d\theta^3}=\frac{Pr^3}{2\,EI} \text{ where } \theta=\alpha \quad . \quad . \quad . \quad . \quad . \quad (11.144)$$

Now

$$\frac{du}{d\theta}=C(\sin\theta+\theta\cos\theta)-H\sin\theta,$$

$$\frac{d^3u}{d\theta^3}=-C(3\sin\theta+\theta\cos\theta)+H\sin\theta,$$

$$\int_0^\alpha u d\theta=A\alpha+C(\sin\alpha-\alpha\cos\alpha)+H\sin\alpha.$$

Then the three conditions give

$$0=C(\sin\alpha+\alpha\cos\alpha)-H\sin\alpha,$$

$$-\frac{Pr^3}{2\,EI}=C(3\sin\alpha+\alpha\cos\alpha)-H\sin\alpha,$$

$$0=A\alpha+C(\sin\alpha-\alpha\cos\alpha)+H\sin\alpha.$$

The values of the constants satisfying these equations are

$$C=-\frac{1}{4}\frac{Pr^3}{EI}\operatorname{cosec}\alpha,$$

$$H=-\frac{1}{4}\frac{Pr^3}{EI}(1+\alpha\cot\alpha)\operatorname{cosec}\alpha,$$

$$A=\frac{1}{2}\frac{Pr^3}{EI}\cdot\frac{1}{\alpha}.$$

Therefore

$$u=\frac{1}{4}\frac{Pr^3}{EI}\left\{\frac{2}{\alpha}-\theta\sin\theta\operatorname{cosec}\alpha-\cos\theta(1+\alpha\cot\alpha)\operatorname{cosec}\alpha\right\} \quad (11.145)$$

The bending moment is

$$\begin{aligned}M&=-\frac{EI}{r^2}\left(\frac{d^2u}{d\theta^2}+u\right)\\ &=-\frac{EI}{r^2}(A+2C\cos\theta)\\ &=-\tfrac{1}{2}Pr\left(\frac{1}{\alpha}-\frac{\cos\theta}{\sin\alpha}\right) \quad . \quad . \quad . \quad . \quad . \quad (11.146)\end{aligned}$$

The bending moments at A and L are respectively

$$\left.\begin{aligned} M_0 &= \tfrac{1}{2}Pr\left(\frac{1}{\sin\alpha}-\frac{1}{\alpha}\right) \\ \text{and} \quad M_1 &= -\tfrac{1}{2}Pr\left(\frac{1}{\alpha}-\cot\alpha\right) \end{aligned}\right\} \qquad (11.147)$$

The second of these is clearly negative. The ratio of the magnitudes of the bending moments, and therefore also of the maximum stresses, at L and A, are

$$\frac{-M_1}{M_0} = \frac{\dfrac{1}{\alpha}-\cot\alpha}{\dfrac{1}{\sin\alpha}-\dfrac{1}{\alpha}}$$

$$= \frac{\sin\alpha-\alpha\cos\alpha}{\alpha-\sin\alpha} \qquad (11.148)$$

If α is small this ratio is approximately

$$\frac{-M_1}{M_0} = \frac{(\alpha-\frac{1}{6}\alpha^3)-\alpha(1-\frac{1}{2}\alpha^2)}{\alpha-(\alpha-\frac{1}{6}\alpha^3)}$$

$$= 2 \qquad (11.149)$$

Thus if α is small, anything less than 20^0, the stress near one of the forces is approximately twice as great as half way between two forces.

If only two forces P are applied at opposite ends of a diameter then $2\alpha=\pi$ and therefore

$$M = -\tfrac{1}{2}Pr\left(\frac{2}{\pi}-\cos\theta\right) \qquad (11.150)$$

Also

$$\frac{-M_1}{M_0} = \frac{\sin\dfrac{\pi}{2}}{\dfrac{\pi}{2}-\sin\dfrac{\pi}{2}} = \frac{2}{\pi-2} = \frac{7}{4} \qquad (11.151)$$

approximately. Thus even in this extreme case the ratio does not differ much from 2.

205. *A closed uniform ring is subjected to a uniform pressure p per unit length over an angle 2α and a balancing force P on the opposite side of the ring, as shown in fig. 106.*

Let suffixes 1 and 2 be used to indicate quantities in the regions AL and LA′ respectively. The bending moments in these two regions are

$$M_1 = A_1 + B_1\cos\theta + C_1\sin\theta \qquad (11.152)$$

$$M_2 = A_2 + B_2\cos\theta + C_2\sin\theta \qquad (11.153)$$

The shearing forces are F_1, F_2 given by

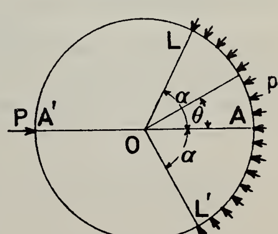


Fig. 106

$$rF_1 = -B_1 \sin\theta + C_1 \cos\theta \quad . \quad . \quad . \quad . \quad (11.154)$$

$$rF_2 = -B_2 \sin\theta + C_2 \cos\theta \quad . \quad . \quad . \quad . \quad (11.155)$$

Now, in consequence of the symmetry about the radius OA, the shearing force is zero at A. Therefore

$$0 = C_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11.156)$$

Moreover the shearing force and bending moment are both continuous in passing the point L. That is,

$$\left.\begin{aligned} F_1 &= F_2 \\ M_1 &= M_2 \end{aligned}\right\} \text{where } \theta = \alpha \quad . \quad . \quad . \quad . \quad . \quad (11.157)$$

Therefore

$$-B_1 \sin\alpha + C_1 \cos\alpha = -B_2 \sin\alpha + C_2 \cos\alpha$$

$$A_1 + B_1 \cos\alpha + C_1 \sin\alpha = A_2 + B_2 \cos\alpha + C_2 \sin\alpha$$

These give, since $C_1 = 0$,

$$B_2 - B_1 = C_2 \cot\alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11.158)$$

$$\begin{aligned} A_2 - A_1 &= -(B_2 - B_1)\cos\alpha - C_2 \sin\alpha \\ &= -C_2 \operatorname{cosec}\alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad (11.159) \end{aligned}$$

Again equation (11.58) gives

$$\frac{dF}{d\theta} = T + pr, \quad . \quad . \quad . \quad . \quad . \quad . \quad (11.160)$$

and since T has the same value on opposite sides of L we find that, when $\theta = \alpha$,

$$\frac{dF_1}{d\theta} - \frac{dF_2}{d\theta} = pr \quad . \quad . \quad . \quad . \quad . \quad . \quad (11.161)$$

Thus $\qquad -B_1 \cos\alpha - C_1 \sin\alpha + B_2 \cos\alpha + C_2 \sin\alpha = pr^2,$

whence $\qquad C_2 \sin\alpha + C_2 \cot\alpha \cos\alpha = pr^2,$

or $\qquad C_2 = pr^2 \sin\alpha \quad . \quad . \quad . \quad . \quad (11.162)$

Of the six constants in equations (11.152) and (11.153) we have determined two, namely C_1 and C_2, and we have two equations for the other four. We need two new equations to determine them completely. The direct method would be to find u_1, u_2, η_1, η_2, and use the conditions that u and $\frac{du}{d\theta}$ are continuous at L; that $\frac{du}{d\theta}$ is zero at A and A′; that u has whatever small value we choose at A; and that η is zero at A and A′. All this is very laborious, and to avoid it the following two equations will be used, the proof being given below

$$\int_0^{\pi} M d\theta = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11.163)$$

$$\int_0^{\pi} M \cos\theta d\theta = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (11.164)$$

To prove the first of these we integrate both sides of the equation

$$M = -\frac{EI}{r^2}\left(\frac{d^2u}{d\theta^2} + u\right)$$
$$= -\frac{EI}{r^2}\left(\frac{d^2u}{d\theta^2} - r\frac{d\eta}{d\theta}\right)$$

Thus
$$-\int_0^{\pi} M d\theta = \frac{EI}{r^2}\left[\frac{du}{d\theta} - r\eta\right]_0^{\pi}$$

Since $\frac{du}{d\theta}$ and η are each zero at both limits, equation (11.163) is proved.

Next to prove equation (11.164).

By integration by parts we get

$$\int_0^{\pi}\frac{d^2u}{d\theta^2}\cos\theta d\theta = \left[\cos\theta\frac{du}{d\theta}\right]_0^{\pi} + \int_0^{\pi}\frac{du}{d\theta}\sin\theta d\theta$$
$$= 0 + \left[u\sin\theta\right]_0^{\pi} - \int_0^{\pi}u\cos\theta d\theta$$
$$= -\int_0^{\pi}u\cos\theta d\theta\,.$$

Therefore

$$\int_0^{\pi}\left(\frac{d^2u}{d\theta^2} + u\right)\cos\theta d\theta = 0\,,$$

from which equation (11.164) follows.

Equation (11.163) is equivalent to

$$\int_0^{\alpha} M_1 d\theta + \int_{\alpha}^{\pi} M_2 d\theta = 0\,,$$

that is, since $C_1 = 0$,

$$A_1\alpha + B_1\sin\alpha + A_2(\pi - \alpha) - B_2\sin\alpha + C_2(1 + \cos\alpha) = 0\,,$$

whence

$$\pi A_2 = \alpha(A_2 - A_1) + (B_2 - B_1)\sin\alpha - C_2(1 + \cos\alpha)$$
$$= -\alpha C_2\operatorname{cosec}\alpha + C_2\cos\alpha - C_2(1 + \cos\alpha)$$

by equations (11.158) and (11.159). Thus

$$A_2 = -\frac{1}{\pi}pr^2(\sin\alpha + \alpha)\,. \quad . \quad . \quad . \quad . \quad (11.165)$$

Again (11.164) is equivalent to

$$\int_0^{\alpha} M_1\cos\theta d\theta + \int_{\alpha}^{\pi} M_2\cos\theta d\theta = 0\,,$$

which gives

$$A_1\sin\alpha + \tfrac{1}{2}B_1(\alpha + \sin\alpha\cos\alpha) - A_2\sin\alpha$$
$$+ \tfrac{1}{2}B_2(\pi - \alpha - \sin\alpha\cos\alpha) - \tfrac{1}{2}C_2\sin^2\alpha = 0\,,$$

whence

$$\pi B_2 = 2(A_2 - A_1)\sin\alpha + (B_2 - B_1)(\alpha + \sin\alpha\cos\alpha) + C_2\sin^2\alpha$$
$$= -2C_2 + C_2(\alpha\cot\alpha + \cos^2\alpha) + C_2\sin^2\alpha$$
$$= pr^2(\alpha\cos\alpha - \sin\alpha) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11.166)$$

From (11.159) and (11.165) we get

$$A_1 = \frac{1}{\pi} pr^2 (\pi - a - \sin a) \quad . \quad . \quad . \quad . \quad (11.167)$$

Also from (11.158) and (11.166)

$$B_1 = -\frac{1}{\pi} pr^2 \{(\pi - a)\cos a + \sin a\} \quad . \quad . \quad . \quad (11.168)$$

Then finally

$$M_1 = \frac{1}{\pi} pr^2 [\pi - a - \sin a - \{(\pi - a)\cos a + \sin a\}\cos\theta]$$

$$M_2 = -\frac{1}{\pi} pr^2 [(\sin a + a) - (a\cos a - \sin a)\cos\theta - \pi\sin a\sin\theta]$$

By considering the equilibrium of the whole ring we find that

$$P = p \times 2r\sin a$$

Then the bending moments in terms of P are

$$\left.\begin{aligned} M_1 &= \frac{Pr}{2\pi\sin a}[\pi - a - \sin a - \{(\pi - a)\cos a + \sin a\}\cos\theta] \\ M_2 &= -\frac{Pr}{2\pi\sin a}[(\sin a + a) - (a\cos a - \sin a)\cos\theta - \pi\sin a\sin\theta] \end{aligned}\right\} (11.169)$$

When a is very small, so that $\frac{\sin a}{a}$ may be taken as unity, then M_2 reduces to

$$M_2 = -\frac{Pr}{2\pi}(2 - \pi\sin\theta), \quad . \quad . \quad . \quad . \quad (11.170)$$

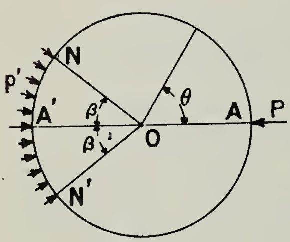


Fig. 107

which agrees with (11.150) when allowance is made for the different position of the line from which θ is measured.

Let us next suppose that a uniform pressure p' acts over an angle 2β and a balancing force P at A, as shown in fig. 107. We can write down the bending moments in this case by merely writing $(\pi - \theta)$ for θ in the previous results. Thus if M_1' and M_2' denote the bending moments in the regions A′N and NA respectively,

$$\left.\begin{aligned} M_1' &= \frac{Pr}{2\pi\sin\beta}[\pi - \beta - \sin\beta + \{(\pi - \beta)\cos\beta + \sin\beta\}\cos\theta] \\ M_2' &= -\frac{Pr}{2\pi\sin\beta}[(\sin\beta + \beta) + (\beta\cos\beta - \sin\beta)\cos\theta - \pi\sin\beta\sin\theta] \end{aligned}\right\} (11.171)$$

The bending moments due to both sets of forces in figs. 106 and 107 are obtained by adding together the bending moments due to each set separately. Moreover, if the force P in fig. 107 is equal to the force

P in fig. 106 then the distributed pressure p would balance the distributed pressure p' if the two forces P were removed. The bending moment at any point due to these distributed pressures without the two forces P is obtained by adding the bending moments given by (11.169) and (11.171), and subtracting the bending moment given by (11.170). This will give three different expressions for the bending moment, one in the region where p acts, another in the region where p' acts, and the third in the region between $\theta = \alpha$ and $\theta = \pi - \beta$ where no pressure acts. For example, the bending moment in the region where p acts is

$$M = \frac{Pr}{2\pi}\left[(\pi - \alpha)\operatorname{cosec}\alpha - \beta\operatorname{cosec}\beta - \left\{(\pi - \alpha)\cot\alpha + \beta\cot\beta\right\}\cos\theta\right] \quad (11.172)$$

206. Oscillations of a ring in its own plane.

A ring can execute oscillations under the action of no forces. Such oscillations are given by equation (11.71) if we put p and q each zero. Let us then put p and q zero, and let us also assume that

$$\eta = \xi \sin ct \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11.173)$$

in that equation, ξ being a function of θ only. Then the equation reduces to

$$\left(\frac{d^6\xi}{d\theta^6} + 2\frac{d^4\xi}{d\theta^4} + \frac{d^2\xi}{d\theta^2}\right)\sin ct = \frac{war^4}{gEI}c^2\left(\frac{d^2\xi}{d\theta^2} - \xi\right)\sin ct$$

The factor $\sin ct$ divides out leaving an equation for ξ in terms of θ.

Next put

$$\xi = A\sin n\theta + B\cos n\theta; \quad . \quad . \quad . \quad . \quad . \quad (11.174)$$

then the last equation gives

$$-n^2(n^2 - 1)^2 = \frac{war^4c^2}{gEI}(-n^2 - 1),$$

or

$$n^2(n^2 \quad 1)^2 = \frac{wr^4c^2}{gk^2E}(n^2 + 1). \quad . \quad . \quad . \quad . \quad (11.175)$$

ak^2 having been written for I.

For a complete ring n must be an integer, for this gives n complete waves in the circle, and η must certainly be a function whose values recur as θ increases by 2π. Moreover n cannot be equal to unity because this corresponds to an oscillation of the whole ring without any alteration of shape. Then n may have any of the values 2, 3, 4, ... The value of c^2 corresponding to any value of n is

$$c^2 = \frac{gk^2E}{wr^4}\frac{n^2(n^2 - 1)^2}{(n^2 + 1)}, \quad . \quad . \quad . \quad . \quad . \quad (11.176)$$

and the period is $\frac{2\pi}{c}$. Thus the normal modes of oscillation for a complete ring are given by

$$\left.\begin{aligned}\eta &= \mathrm{H}\sin ct\sin(n\theta+\mathrm{K})\\ \text{where}\qquad n &= 2,\ 3,\ 4,\ \ldots\ldots\end{aligned}\right\}\qquad (11.177)$$

and c is given by (11.176)

Incomplete ring.

The normal modes for an incomplete ring, or one with open ends, are also given by (11.173) and (11.174), but now n is not necessarily an integer. In fact n has to be found from (11.175). This is a cubic in n^2 giving usually three values of n^2. Let these values be n_1^2, n_2^2, n_3^2. Then

$$\xi = \mathrm{A}_1\sin n_1\theta + \mathrm{B}_1\cos n_1\theta + \mathrm{A}_2\sin n_2\theta + \mathrm{B}_2\cos n_2\theta + \mathrm{A}_3\sin n_3\theta + \mathrm{B}_3\cos n_3\theta \qquad (11.178)$$

The conditions at the ends will give five linear relations between the six constants and one equation to determine the possible values of c, and therefore of the corresponding values of n_1, n_2, n_3. The problem is a very awkward one owing to the fact that the equation for n^2 is a cubic. We shall apply our equation to solve the following easier problem.

Suppose ξ is a given function of θ what pressure will produce the displacement given by the equation

$$\eta = \xi\sin ct\,? \qquad (11.179)$$

It is clear that the pressure p must have the same period as η. Then let

$$p = z\sin ct, \qquad (11.180)$$

z being a function of θ. Now making the substitutions in (11.71) and also putting $q = 0$, we get, after dividing by $\sin ct$,

$$\frac{r^3}{\mathrm{EI}}\frac{dz}{d\theta} = \frac{war^4c^2}{g\mathrm{EI}}\left(\frac{d^2\xi}{d\theta^2} - \xi\right) + \left(\frac{d^6\xi}{d\theta^6} + 2\frac{d^4\xi}{d\theta^4} + \frac{d^2\xi}{d\theta^2}\right)$$

Therefore by integration

$$z + \mathrm{B} = \frac{warc^2}{g}\left\{\frac{d\xi}{d\theta} - \int\xi d\theta\right\} + \frac{\mathrm{EI}}{r^3}\left\{\frac{d^5\xi}{d\theta^5} + 2\frac{d^3\xi}{d\theta^3} + \frac{d\xi}{d\theta}\right\} \quad (11.181)$$

The function ξ cannot be chosen arbitrarily, for it must be such as will satisfy the end-conditions of the open ring. Suppose, for example, that we apply our result to a ring which has small gap at one point. From our investigation on the piston ring we know that the bending moment and shearing force are zero at the free ends provided

$$\frac{d\xi}{d\theta} = \mathrm{A}\,(2 + \theta\sin\theta),$$

θ being measured from the point diametrically opposite the centre of gap. The value of ξ from this last equation is

$$\xi = \mathrm{A}\,(2\theta - \theta\cos\theta + \sin\theta) \qquad (11.182)$$

No constant is needed since ξ must be zero when θ is zero. Substituting this in (11.181) we find

$$z + \mathrm{B} = \frac{warc^2}{g} \mathrm{A}\{2 + 2\theta \sin\theta + 2\cos\theta - \theta^2\} \quad (11.183)$$

Now the bending moment and shearing force are not affected by the acceleration but depend only on the actual form of the curve at any instant. The tension does, however, vary with the acceleration. There are, in fact, *inertia* forces in the tension but none in F or M. Thus equation (11.58) corrected for the inertia, becomes

$$\begin{aligned}\frac{\partial \mathrm{F}}{\partial \theta} - \mathrm{T} &= r\left(p + \frac{wa}{g}\frac{\partial^2 u}{\partial t^2}\right)\\ &= r\left\{p - \frac{war}{g}\frac{\partial}{\partial \theta}\left(\frac{\partial^2 \eta}{\partial t^2}\right)\right\}\\ &= r\left\{p + \frac{warc^2}{g}\frac{d\xi}{d\theta}\sin ct\right\}\\ &= r\sin ct\left\{z + \frac{warc^2}{g}\frac{d\xi}{d\theta}\right\} \quad . \quad . \quad . \quad . \quad . \quad (11.184)\end{aligned}$$

But

$$\begin{aligned}\frac{\partial \mathrm{F}}{\partial \theta} &= \frac{1}{r}\frac{\partial^2 \mathrm{M}}{\partial \theta^2}\\ &= \frac{\mathrm{EI}}{r^2}\left\{\frac{\partial^5 \eta}{\partial \theta^5} + \frac{\partial^3 \eta}{\partial \theta^3}\right\}\\ &= \frac{\mathrm{EI}}{r^2}\sin ct\left\{\frac{d^5 \xi}{d\theta^5} + \frac{d^3 \xi}{d\theta^3}\right\}\\ &= \frac{\mathrm{EI}}{r^2}\sin ct \times (-2\mathrm{A}\cos\theta) \quad . \quad . \quad . \quad . \quad (11.185)\end{aligned}$$

Now one end-condition is that $\mathrm{T} = 0$ at the free end where $\theta = \pi$. Putting $\theta = \pi$ in (11.184), and making use of (11.185), we get, after dividing by sin ct,

$$\frac{2\mathrm{EI}}{r^2}\mathrm{A} = r\left\{z + 2\mathrm{A}\frac{warc^2}{g}\right\} \quad . \quad . \quad . \quad . \quad . \quad (11.186)$$

Also putting $\theta = \pi$ in (11.183) we find

$$z + \mathrm{B} = \frac{warc^2}{g}\mathrm{A}(-\pi^2) \quad . \quad . \quad . \quad . \quad . \quad (11.187)$$

The last two equations give, when z is eliminated,

$$\mathrm{B} = -\frac{warc^2}{g}\mathrm{A}(\pi^2 - 2) - \frac{2\mathrm{EI}}{r^3}\mathrm{A} \quad . \quad . \quad . \quad (11.188)$$

Therefore the general value of z given by (11.183) is

$$z = \frac{warc^2}{g}\mathrm{A}\{2\theta\sin\theta + 2\cos\theta + \pi^2 - \theta^2\} + \frac{2\mathrm{EI}}{r^3}\mathrm{A}. \quad (11.189)$$

The value of z steadily decreases from $\theta=0$ to $\theta=\pi$ and its values at these points are

$$\left.\begin{aligned} z_0 &= (2+\pi^2)\frac{warc^2}{g}\mathrm{A}+\frac{2\mathrm{EI}}{r^3}\mathrm{A} \\ \text{and} \quad z_1 &= -2\frac{warc^2}{g}\mathrm{A}+\frac{2\mathrm{EI}}{r^3}\mathrm{A} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (11.190)$$

If the oscillations are rapid, in which case c is large, it is possible for z_1 to be negative. If that were so there would be two regions near the free ends of the ring where the pressure would be acting outwards while the pressure on the rest acted inwards and vice versa.

207. Application to a piston ring in its cylinder.

Let us modify the last problem by assuming, instead of (11.179), the following equation for η: –

$$\eta = \mathrm{B}_1\xi + \xi \sin ct, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11.191)$$

where ξ still has the value given by (11.182). Now each term in the value of η gives rise to its own pressure, and the total pressure is the sum of the pressures due to the two terms $\mathrm{B}_1\xi$ and $\xi \sin ct$. But we know from the piston ring problem that the displacement $\eta = \mathrm{B}_1\xi$ corresponds to a uniform pressure p_0. The substitution of $-r\mathrm{B}_1\dfrac{d\xi}{d\theta}$ for u in (11.85) gives

$$p_0 = \frac{2\mathrm{EI}}{r^3}\mathrm{AB}_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11.192)$$

Since the pressure corresponding to $\eta = \xi \sin ct$ is $p = z \sin ct$ it follows that the total pressure corresponding to the value of η in (11.191) is

$$\begin{aligned} p &= p_0 + z \sin ct \\ &= \frac{2\mathrm{EI}}{r^3}\mathrm{A}(\mathrm{B}_1+\sin ct)+\frac{warc^2}{g}\mathrm{A}\left\{2\theta\sin\theta+2\cos\theta+\pi^2-\theta^2\right\}\sin ct \quad (11.193) \end{aligned}$$

Let us suppose that the cylinder is slightly out of shape, and let us assume that the form of the cylinder is such that the ring whose shape varies in the way we have assumed in equation (11.191) always fits the cylinder. This will be possible only if p is positive at every point of the ring at all times. When $\sin ct = -1$ then

$$p = p_0 - z,$$

and this will be negative at the point where $\theta=0$ if it is negative anywhere. Then the condition that the ring should actually have the oscillations we have assumed is that p_0 should be greater than the value of z at the point where $\theta=0$, that is,

$$\frac{2\mathrm{EI}}{r^3}\mathrm{AB}_1 > \frac{2\mathrm{EI}}{r^3}\mathrm{A}+(2+\pi^2)\frac{warc^2}{g}\mathrm{A} \quad . \quad . \quad . \quad (11.194)$$

Now in following any reasonable deformation of the cylinder the change of form of the ring must be very much less than its change when it is pressed into the cylinder; that is, the coefficient B_1 in equation (11.191) must be very much greater that the coefficient 1 of $\xi \sin ct$. But (11.194) gives

$$\frac{2\,\mathrm{EI}}{r^3}(\mathrm{B}_1 - 1) > (2 + \pi^2)\frac{warc^2}{g},$$

or, neglecting unity compared with B_1,

$$\frac{2\,\mathrm{EI}}{r^3}\mathrm{B}_1 > (2 + \pi^2)\frac{warc^2}{g}.$$

When ak^2 written for I this becomes

$$\mathrm{B}_1 > \left(1 + \frac{\pi^2}{2}\right)\frac{wr^4c^2}{g\mathrm{E}k^2} \quad . \quad . \quad . \quad . \quad . \quad (11.195)$$

Suppose the ring has a rectangular section of radial thickness b; and let n be written for the frequency of the oscillations, that is, for the number of oscillations per second. Thus

$$n = \frac{c}{2\pi},$$

and

$$k^2 = \tfrac{1}{12}b^2;$$

therefore

$$\mathrm{B}_1 > 24\pi^2(2 + \pi^2)\frac{wr^4n^2}{g\mathrm{E}b^2}.$$

This gives the relation between the frequency n and the quantity B_1, which is the ratio of the deformation produced by the constant pressure p_0 to the maximum additional deformation due to the oscillations.

Suppose the engine is running at N revolutions per second, and suppose also that there are s complete undulations on the cylinder in the length of the stroke. Then the ring has to make $2\,\mathrm{N}s$ complete oscillations per second. This is the *mean* frequency; but since the motion of the piston is approximately simple harmonic its maximum speed is approximately $\frac{1}{2}\pi$ times its mean speed. Therefore the maximum frequency of the ring is $\pi s\mathrm{N}$. Substituting this for n in the last equation we get

$$\mathrm{B}_1 > 24\pi^4(2 + \pi^2)\frac{wr^4s^2\mathrm{N}^2}{g\mathrm{E}b^2} \quad . \quad . \quad . \quad . \quad (11.196)$$

Let us apply this to a cast iron ring and assume that

$$\mathrm{E} = 16 \times 10^6 \text{ lbs/sq. inch},$$
$$w = 0{\cdot}26 \text{ lb/cub inch},$$
$$r = 3 \text{ inches},$$
$$b = 0{\cdot}15 \text{ inch},$$
$$g = 32 \times 12 \text{ in inch units:}$$

Also let us suppose that

$$\mathrm{N} = 30.$$

Then the condition that the ring should always follow the undulations of the cylinder is

$$B_1 > 24\pi^4(2+\pi^2)\frac{0{\cdot}26 \times 3^4 \times 30^2}{(32\times 12)\times 16\times 10^6 \times 0{\cdot}15^2}s^2$$

or $$> 4{\cdot}0\, s^2 \text{ approximately.}$$

If there are six complete undulations in the stroke then the maximum amplitude of the oscillations must be less than $\frac{1}{144}$ of the deformation that the ring undergoes when it is squeezed into the cylinder. If the gap in the free ring is 24 millimetres it is possible for the ring to oscillate so that the gap varies between 0 and 0·35 millimetre If there were more than six undulations of this magnitude the resilience of the ring would not be sufficient to keep it in contact with the cylinder. If there were twelve undulations per stroke contact would be maintained only if the gap kept within the range from zero to one twelfth of a millimetre.

208. The extension of a ring.

We have thus far assumed that the rings we have been dealing with had no extension. Wherever there is a tension, there must, however, be an extension, but this will affect the deformation of a closed ring only in adding a constant to u everywhere. The addition of a constant to u everywhere is accompanied by a constant addition to the bending moment also. This does not, however, disturb the equilibrium of any element of the ring, for it merely adds a pair of balancing couples to the ends of the element. The shearing force remains unchanged by the addition to u. It follows then that the increase of length of a ring due to the tension, even when this tension is variable, must be distributed so as to add the same amount to every radius vector, for in this way the equilibrium is undisturbed.

Suppose T is the tension at any point of a ring, and let s denote the extension of the middle line of the arc which extends from 0 to θ. Then the extensional strain at θ is $\frac{ds}{rd\theta}$, and therefore, if a denotes the area of the cross-section,

$$T = Ea\frac{ds}{rd\theta},$$

whence $$s = \frac{r}{Ea}\int T d\theta.$$

The whole extension of a closed ring is

$$s_1 = \frac{r}{Ea}\int_0^{2\pi} T d\theta,$$

and the consequent increase in the radius is

$$u_1 = \frac{r}{2\pi Ea}\int_0^{2\pi} T d\theta. \qquad (11.198)$$

209. Stresses in a rotating wheel.

Suppose a rotating wheel has a number of straight uniform spokes separated by equal angles 2α. We shall regard the rim as a thin ring, and we shall make the assumption (which cannot possibly be quite true) that the spokes extend to the centre. The problem is to determine the deformation and stresses in the rim and spokes due to a given angular velocity ω. Let a and a_1 denote the cross-sections of the rim and of a spoke respectively. Each spoke applies a pull P to the rim, and the extension of each spoke is due to the reaction P of the ring on the spoke and to the tension set up by the centrifugal force in the spoke itself. Equation (11.58) can be used to get the tension in the rim provided we use $-\frac{wa}{g}r\omega^2$ for p. The actual p is zero and the expression we propose to substitute for p is really the correction for the acceleration in the radial direction. Thus we get

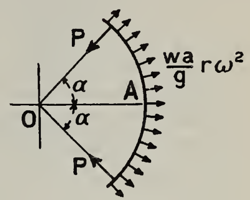


Fig. 108

$$T=\frac{dF}{d\theta}+\frac{wa}{g}r^2\omega^2.$$

Now let u_1 denote the increase of the radius vector due to the extension of the rim, and u_2 the increase due to bending. Then, since u_2 is due to the forces P only when u_1 is supposed to be zero, its value, taken from equation (11.145), is

$$u_2=\frac{1}{4}\frac{Pr^3}{EI}\left\{\frac{2}{\alpha}-\frac{\theta\sin\theta}{\sin\alpha}-\frac{\sin\alpha+\alpha\cos\alpha}{\sin^2\alpha}\cos\theta\right\} \quad . \quad . \quad (11.199)$$

Therefore

$$\frac{dF}{d\theta}=-\frac{EI}{r^3}\left\{\frac{d^2u_2}{d\theta^2}+\frac{d^4u_2}{d\theta^4}\right\}$$

$$=-\tfrac{1}{2}P\frac{\cos\theta}{\sin\alpha} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11.200)$$

Then the equation for T is

$$T=-\tfrac{1}{2}P\frac{\cos\theta}{\sin\alpha}+\frac{war^2\omega^2}{g} \quad . \quad . \quad . \quad . \quad (11.201)$$

To get u_1 we use an equation similar to (11.198) but instead of integrating from 0 to 2π we may integrate from 0 to α. Then

$$u_1=\frac{r}{Ea\alpha}\int_0^\alpha\left\{\frac{war^2\omega^2}{g}-\tfrac{1}{2}P\frac{\cos\theta}{\sin\alpha}\right\}d\theta$$

$$=\frac{wr^3\omega^2}{Eg}-\frac{Pr}{2Ea\alpha} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11.202)$$

When $\theta = a$ the value of u_2 is

$$u_2 = \frac{1}{4}\frac{Pr^3}{EI}\left\{\frac{2}{a} - \frac{a}{\sin^2 a} - \cot a\right\}$$

Therefore at the same point

$$\begin{aligned} u &= u_1 + u_2 \\ &= \frac{wr^3\omega^2}{Eg} - \frac{Pr}{2Eaa} + \frac{1}{4}\frac{Pr^3}{EI}\left\{\frac{2}{a} - \frac{a}{\sin^2 a} - \cot a\right\} \end{aligned} \quad . \ (11.203)$$

Again let v denote the extension of a length x of a spoke, the length x being measured from the centre. The tension in a spoke at x is the tension P applied by the rim plus the tension due to the acceleration of the portion of the spoke between the rim and x. Let T_1 be the tension at x, $(T_1 + dT_1)$ the tension at $(x + dx)$. Then $-dT_1$ is the force which gives the acceleration $x\omega^2$ to the element of length dx and mass $\frac{wa_1}{g}dx$. Therefore

$$-dT_1 = \left(\frac{wa_1}{g}dx\right)x\omega^2$$

$$T_1 = -\frac{1}{2}\frac{wa_1\omega^2}{g}x^2 + K$$

The constant K is determined by the fact that $T_1 = P$ where $x = r$.

Then
$$T_1 = P + \frac{wa_1\omega^2}{2g}(r^2 - x^2) \quad . \ . \ . \ . \ (11.204)$$

Now
$$Ea_1\frac{dv}{dx} = T_1$$

$$= P + \frac{wa_1\omega^2}{2g}(r^2 - x^2),$$

whence
$$Ea_1 v = Px + \frac{wa_1\omega^2}{2g}\left(r^2 x - \tfrac{1}{3}x^3\right).$$

No constant need be added because $v = 0$ where $x = 0$.

The value of v where $x = r$ must be equal to the displacement u of the rim. Equating the values of Ev and Eu at the end of a spoke we get

$$\frac{Pr}{a_1} + \frac{wr^3\omega^2}{3g} = \frac{wr^3\omega^2}{g} - \frac{Pr}{2aa} + \frac{1}{4}\frac{Pr^3}{I}\left\{\frac{2}{a} - \frac{a}{\sin^2 a} - \cot a\right\},$$

whence, writing $k^2 a$ for I,

$$P\left\{\frac{4a}{a_1} + \frac{2}{a} + \frac{r^2}{k^2}\left(\cot a + \frac{a}{\sin^2 a} - \frac{2}{a}\right)\right\} = \frac{8}{3}\frac{war^2\omega^2}{g} \quad (11.205)$$

This equation determines P. Then the bending moment is

$$\begin{aligned} M &= -\frac{EI}{r^2}\left(\frac{d^2u}{d\theta^2}+u\right) \\ &= -\frac{EI}{r^2}\left(\frac{d^2u_2}{d\theta^2}+u_2+u_1\right) \\ &= -\tfrac{1}{2}Pr\left(\frac{1}{a}-\frac{\cos\theta}{\sin a}\right)+\frac{Pk^2}{2ra}-\frac{wrak^2\omega^2}{g} \end{aligned}$$

The value of M at one of the spokes is probably negative and its value is M′ given by

$$-M' = \tfrac{1}{2}Pr\left(\frac{1}{a}-\cot a\right)-\frac{Pk^2}{2ra}+\frac{wrak^2\omega^2}{g} \quad . \quad . \quad (11.206)$$

The value of the tension at the same point is

$$T' = \frac{war^2\omega^2}{g}-\tfrac{1}{2}P\cot a \quad . \quad . \quad . \quad . \quad . \quad (11.207)$$

The maximum stress across a section of the rim near the end of a spoke is, assuming M′ to be negative, and assuming that the section is rectangular and has a radial width b,

$$\begin{aligned} f' &= \frac{T'}{a}-\frac{bM'}{2ak^2} \\ &= \frac{wr^2\omega^2}{g}\left(1+\frac{b}{2r}\right)+\frac{P}{4a}\left\{\frac{rb}{k^2}\left(\frac{1}{a}-\cot a\right)-\frac{b}{ra}-2\cot a\right\} \quad (11.208) \end{aligned}$$

Again the maximum stress across a section half way between two spokes, where $\theta=0$, provided that the bending moment is positive, is

$$\begin{aligned} f'' &= \frac{T}{a}+\frac{bM}{2ak^2} \\ &= \frac{wr^2\omega^2}{g}\left(1-\frac{b}{2r}\right)+\frac{P}{4a}\left\{\frac{rb}{k^2}\left(\frac{1}{\sin a}-\frac{1}{a}\right)+\frac{b}{ra}-\frac{2}{\sin a}\right\} \quad (11.209) \end{aligned}$$

The maximum stress in a spoke occurs at the hub where $x=0$ and its value is

$$f_1 = \frac{P}{a_1}+\frac{wr^2\omega^2}{2g} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11.210)$$

If a is not greater than $\frac{\pi}{6}$ our equations can be simplified by expanding the functions of a in powers of a. Thus

$$\begin{aligned} \cot a+\frac{a}{\sin^2 a}-\frac{2}{a} &= \frac{2a^3}{45}\left(1+\frac{a^2}{84}+\dots\right) \\ &\doteq \frac{2a^3}{45} \text{ approximately} \end{aligned}$$

Also

$$\cot a = \frac{1 - \frac{a^2}{2} + \frac{a^4}{24}}{a\left(1 - \frac{a^2}{6} + \frac{a^4}{120} \ldots\right)}$$

$$= \frac{1}{a}\left(1 - \tfrac{1}{3}a^2 - \tfrac{1}{45}a^4 \ldots\right)$$

Therefore

$$\frac{1}{a} - \cot a = \tfrac{1}{3}a \text{ approximately}$$

Again

$$\frac{1}{\sin a} - \frac{1}{a} = \frac{1}{a\left(1 - \frac{a^2}{6} + \frac{a^4}{120}\right)} - \frac{1}{a}$$

$$= \tfrac{1}{6}a \text{ approximately}$$

Consequently equations (11.205), (11.208), and (11.209) may be written

$$P\left\{\frac{2a}{a_1} + \frac{1}{a} + \frac{r^2a^3}{45k^2}\right\} = \frac{4}{3}\frac{war^2\omega^2}{g}$$

$$f' = \frac{wr^2\omega^2}{g}\left(1 + \frac{b}{2r}\right) + \frac{P}{4a}\left\{\frac{rba}{3k^2} - \frac{b}{ra} - 2\cot a\right\}$$

$$= \frac{wr^2\omega^2}{g}\left\{\left(1 + \frac{b}{2r}\right) + \frac{1}{3}\frac{\frac{rba}{3k^2} - \frac{b}{ra} - 2\cot a}{\frac{2a}{a_1} + \frac{1}{a} + \frac{r^2a^3}{45k^2}}\right\} \quad . \quad (11.211)$$

$$f'' = \frac{wr^2\omega^2}{g}\left\{\left(1 - \frac{b}{2r}\right) + \frac{1}{3}\frac{\frac{rba}{6k^2} + \frac{b}{ra} - \frac{2}{\sin a}}{\frac{2a}{a_1} + \frac{1}{a} + \frac{r^2a^3}{45k^2}}\right\} \quad . \quad . \quad (11.212)$$

Suppose $a = 3a_1$, $r = 7b$, $a = \frac{1}{6}\pi$, corresponding to six spokes, then the last two equations give, for a rectangular cross-section,

$$f' = 1{\cdot}44\frac{wr^2\omega^2}{g}$$

$$f'' = 1{\cdot}05\frac{wr^2\omega^2}{g}$$

If the accurate values of the functions of a are used instead of the approximate values obtained from the expansions the values of f' and f'' are each increased by less than one per cent.

If a is very great compared with a_1, so that we may regard $\frac{a}{a_1}$ as infinity and regard the spokes as having no effect on the rim, then the stresses are independent of θ, and the maximum stress in the rim is the stress at the inner edge. Its value is the value of f' when $\frac{a}{a_1}=\infty$, which value is

$$f'=\frac{wr^2\omega^2}{g}\left(1+\frac{b}{2r}\right)$$

$$=1\cdot07\frac{wr^2\omega^2}{g} \quad \ldots\ldots\ldots (11.213)$$

It is rather remarkable that the effect of the spokes on the rim is to increase, instead of decreasing, the maximum stress in the rim.

Let us take the other extreme case and suppose that the rim is very thin in comparison with the spokes. We assume that $\frac{a}{a_1}, \frac{b}{r}, \frac{k}{r}$, are all zero. Then

$$f'=1\cdot07\frac{wr^2\omega^2}{g} \quad \ldots\ldots (11.214)$$

$$f''=0\cdot93\frac{wr^2\omega^2}{g} \quad \ldots\ldots (11.215)$$

The greatest stress that can possibly occur in a spoke will occur when $\frac{\mathrm{P}}{a_1}$ has its greatest value, and this happens when $\frac{a}{a_1}=\infty$. The equations (11.205) and (11.210) give

$$f_1=\left(\frac{2}{3}+\frac{1}{2}\right)\frac{wr^2\omega^2}{g}=\frac{7}{6}\frac{wr^2\omega^2}{g} \quad \ldots\ldots (11.216)$$

210. Thin curved rod bent in one plane.

The assumptions we make are that the central line of the rod lies in one plane before and during the strain, and that all the forces acting on the rod are also in that plane.

Let a small piece AB (fig. 109) of the central line of the rod be bent into A′B′. Let the length of AB be ds and its radius of curvature ϱ_0, and let the corresponding quantities for A′B′ be ds' and ϱ. Let φ_0 denote the inclination of AB to a line passing through any two particles on the unstrained central line and φ the inclination of A′B′ to the line passing through the same two particles in the strained state.

The bending moment and the shearing force at A′ are denoted by M and F, and the resultant tension across the section at A′ is denoted by T; the corresponding quantities at B′ are M + dM, F + dF, T + dT,

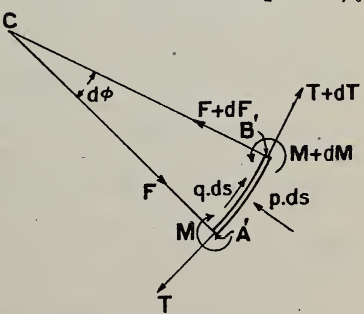


Fig. 109

as shown in fig. 109. In this figure CA′, CB′, are normals to the central line, and consequently the limiting value of CA′ as A′ approaches B′ is the radius of curvature ϱ. Moreover, the angle A′CB′ is $d\varphi$.

Let external forces qds, pds, act parallel and perpendicular to A′B′, the pressure p being reckoned positive when it acts towards the centre of curvature.

Now if the forces on A′B′ are in equilibrium we get, by resolving parallel to the tangent at B′,

$$(T + dT) - T\cos d\varphi - F\sin d\varphi + qds = 0,$$

which becomes, when higher powers of $d\varphi$ than the first are neglected,

$$(T + dT) - T - Fd\varphi + qds = 0,$$

whence

$$\frac{dT}{ds} - \frac{F}{\varrho} + q = 0 \quad . \quad . \quad . \quad . \quad (11.217)$$

Similarly, by resolving along B′C we find

$$(F + dF) - F\cos d\varphi + T\sin d\varphi + pds = 0,$$

whence

$$\frac{dF}{ds} + \frac{T}{\varrho} + p = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (11.218)$$

Again, by taking moments about B′ and neglecting small quantities of the second order, we get

$$dM + Fds = 0,$$

whence

$$\frac{dM}{ds} + F = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11.219)$$

The shearing force F can be eliminated from the equations (11.217), (11.218), (11.219). The two equations resulting from this elimination are

$$\frac{dT}{ds} + \frac{1}{\varrho}\frac{dM}{ds} + q = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (11.220)$$

$$\frac{d^2M}{ds^2} - \frac{T}{\varrho} - p = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (11.221)$$

211. The strain energy in a curved rod.

It has been pointed out by Lord Rayleigh * that the strain energy of a thin curved plate cannot be accurately expressed in terms of the strains and changes of curvature of the middle surface. It is just as true that the energy in a curved rod cannot be expressed in terms of the strain and change of curvature of the central line. The real difficulty is that, when the energy is expressed in powers of the thickness, the term containing the third power is different according to the way in which the straining forces are applied. Since that part of the energy which is due to the bending moment is proportional to the third power of the thickness it might seem that energy principles could

* *Proceedings of the London Mathematical Soc.* XX, 1889. Also, *Scientific Papers*, Vol III, page 162.

not be used for a thin curved rod or plate. This, however, is not true; although we do not get a unique expression for the strain energy in a thin rod, we do, nevertheless, get unique equations of equilibrium, and there must be a form of energy equation consistent with these equations. This expression for the energy, whether it is correct or not, is just as accurate as the equations of equilibrium. We shall find this expression for the energy in a thin curved rod in a state of strain, and we shall then show that the result we get is consistent with the equations of equilibrium. The difficulties to which Lord Rayleigh called attention resulted from the taking of too close a view of the stresses in a plate or rod. If we take the stress resultants to be the couple M and the forces T and F these difficulties disappear.

Let us find the work done in straining the element A′B′ in fig. 109 by actions that are capable of producing its strains. We may regard dM and dT as zero since the actual state of strain could be produced by equal couples M and equal tensions T at A′ and B′, and a suitable pressure p, which must act in the direction away from C if T is positive.

While the element of rod is being strained we may suppose that the ends A′ and B′ remain on a fixed straight line. This ensures that the shearing forces F and F $+ d$F do no work.

Let the longitudinal strain of A′B′ be α, so that

$$ds' = (1 + \alpha)\, ds.$$

Also let

$$\varphi = \varphi_0 + \eta, \qquad (11.222)$$

whence

$$\frac{d\varphi}{ds} - \frac{d\varphi_0}{ds} = \frac{d\eta}{ds} \qquad (11.223)$$

Let the element A′B′ take additional infinitesimal strains represented by $\delta\alpha$ and $\delta\eta$. Then the increase of length of A′B′ is $\delta\alpha\, ds$, and therefore the work done by T acting at the two ends is

$$\mathrm{T}\delta\alpha ds.$$

Also, one end of A′B′ rotates relatively to the other through the angle $d(\delta\eta)$. Consebuently thework done by the two couples M in this rotation is

$$\mathrm{M}d(\delta\eta) = \mathrm{M}\frac{d(\delta\eta)}{ds}ds$$

Therefore the total work done by T and M on the element is

$$\left\{\mathrm{T}\delta\alpha + \mathrm{M}\frac{d(\delta\eta)}{ds}\right\}ds. \qquad (11.224)$$

Now

$$\frac{ds}{\varrho_0} = d\varphi_0$$

and

$$\frac{(1+\alpha)ds}{\varrho} = d\varphi = d\varphi_0 + d\eta.$$

Therefore
$$\left(\frac{1}{\varrho}-\frac{1}{\varrho_0}+\frac{a}{\varrho}\right)=\frac{d\eta}{ds}$$

Let c be written for the change of curvature. Then the last equation becomes

$$\frac{d\eta}{ds}=c+\frac{a}{\varrho} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11.225)$$

An equation similar to this will also hold between the increments $\delta\eta$, δc, etc. Thus

$$\frac{d\,\delta\eta}{ds}=\delta c+\delta\left(\frac{a}{\varrho}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11.226)$$

Thus the whole work done on the rod in the infinitesimal displacements is

$$\delta V=\int\left\{T\delta a+M\delta c+M\delta\left(\frac{a}{\varrho}\right)\right\}ds, \quad . \quad . \quad (11.227)$$

the integral being taken over the whole length of the rod.

In nearly every actual problem $\delta\left(\frac{a}{\varrho}\right)$ will be much smaller than δc. In those rare cases where it is not so $M\delta\left(\frac{a}{\varrho}\right)$ will be much smaller than $T\delta a$. Consequently there are very few cases in which it is necessary to take account of $M\delta\left(\frac{a}{\varrho}\right)$.

Now our stress-strain relations are

$$T=EAa,$$
$$M=EIc,$$

A and I being respectively the area and moment of inertia of the cross section of the rod. Therefore

$$\delta V=\int T\frac{\delta T}{EA}ds+\int M\frac{\delta M}{EI}ds+\int M\delta\left(\frac{a}{\varrho}\right)ds$$
$$=\int\delta\left\{\frac{1}{2}\frac{T^2}{EA}+\frac{1}{2}\frac{M^2}{EI}\right\}ds+\int M\delta\left(\frac{a}{\varrho}\right)ds$$

Thus the total strain energy in the rod is

$$V=\int\left(\frac{1}{2}\frac{T^2}{EA}+\frac{1}{2}\frac{M^2}{EI}\right)ds+\int\left\{\int Md\left(\frac{a}{\varrho}\right)\right\}ds, \quad . \quad (11.228)$$

the integral
$$\int Md\left(\frac{a}{\varrho}\right)$$

being taken over the range from the beginning to the end of the strain at any point of the rod, and the other integrals from end to end of the rod. If the second integral in (11.228) can be neglected the remaining expression for the energy has exactly the same form as for a rod originally straight.

212. Energy expression deduced from equations of equilibrium.

Let us suppose that, owing to very slight changes in p and q, the whole rod takes a small additional strain from the equilibrium position. Let the component displacements of the infinitesimal element A′B′ be δu in the direction of qds and δv in the direction of pds. Then the work done by the applied forces pds and qds in this displacement is

$$p\,ds\,\delta v + q\,ds\,\delta u,$$

and this, by equations (11.220) and (11.221) is equal to

$$\left(\frac{d^2\mathrm{M}}{ds^2} - \frac{\mathrm{T}}{\varrho}\right) ds\,\delta v - \left(\frac{d\mathrm{T}}{ds} + \frac{1}{\varrho}\frac{d\mathrm{M}}{ds}\right) ds\,\delta u.$$

The total work done on the whole rod in these additional displacements is

$$\delta\mathrm{V} = \int\left\{\left(\frac{d^2\mathrm{M}}{ds^2} - \frac{\mathrm{T}}{\varrho}\right)\delta v - \left(\frac{d\mathrm{T}}{ds} + \frac{1}{\varrho}\frac{d\mathrm{M}}{ds}\right)\delta u\right\} ds \quad . \ (11.229)$$

This must be the additional strain energy put into the rod when the state of strain is slightly altered. We have to show that this new expression for δV is identical with the one in (11.227). In order to show this we need to get the relations connecting δu and δv with δc and δa.

Let coordinate axes AX, AY, be taken along the tangent and principal normal to the curve of the unstrained central line at a point A of its length. Let B be any other particle on the unstrained central line, and let its coordinates be x, y; let the length of the arc AB be s. Let B′ be the strained position of the particle B and let its coordinates relative to AX, AY, be x', y'. Let ψ denote the inclination to OX of the tangent at B before strain, and ψ' the inclination of the tangent at B′ to the same axis. Let ϱ and ϱ' be the radii of curvature of the central line at B before and after strain. Suppose the displacement from B to B′ has components v parallel to the principal normal at B and u parallel to the tangent at B. These displacements are assumed to be infinitesimal since we are going to make them finally identical with δv and δu.

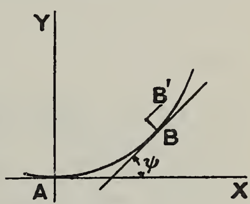


Fig. 110

The equations connecting the coordinates and displacements are

$$x' = x + u\cos\psi - v\sin\psi \quad . \ . \ . \ . \ (11.230)$$

$$y' = y + u\sin\psi + v\cos\psi \quad . \ . \ . \ . \ (11.231)$$

As before, a denotes the longitudinal strain and ds' is the strained length of ds.

Now $$\sin\psi' = \frac{dy'}{ds'} = \frac{1}{1+\alpha}\frac{dy'}{ds}$$

$$= \frac{1}{1+\alpha}\left\{\frac{dy}{ds} + \sin\psi\frac{du}{ds} + \cos\psi\frac{dv}{ds} + (u\cos\psi - v\sin\psi)\frac{d\psi}{ds}\right\}$$

$$= \frac{1}{1+\alpha}\left\{\cos\psi\left(\frac{dv}{ds}+\frac{u}{\varrho}\right) + \sin\psi\left(1+\frac{du}{ds}-\frac{v}{\varrho}\right)\right\} \quad (11.232)$$

Also $$\frac{dx'}{ds} = \frac{dx}{ds} + \cos\psi\frac{du}{ds} - \sin\psi\frac{dv}{ds} - \frac{1}{\varrho}(u\sin\psi + v\cos\psi) \quad . \quad . \quad . \quad (11.233)$$

and $$\frac{ds'}{ds} = \frac{ds'}{dx'}\frac{dx'}{ds}$$

$$= \sec\psi'\frac{dx'}{ds} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11.234)$$

Now let B be made to coincide with A. Then $\psi = 0$, and ψ' is a small angle determined by (11.232). Thus, putting $\psi = 0$ in (11.232),

$$(1+\alpha)\psi' = \frac{dv}{ds} + \frac{u}{\varrho} \quad . \quad . \quad . \quad . \quad . \quad (11.235)$$

Again putting $\psi = 0$ and $\sec\psi' = 1$ in (11.233) and (11.234) we get

$$1+\alpha = \frac{ds'}{ds} = \frac{dx'}{ds}$$

$$= 1 + \frac{du}{ds} - \frac{v}{\varrho}, \quad . \quad . \quad . \quad . \quad . \quad (11.236)$$

whence $$\alpha = \frac{du}{ds} - \frac{v}{\varrho}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11.237)$$

α being the longitudinal strain at the point where u and v are the displacements.

Now returning to (11.232) and assuming that B is near A, but not coincident with A, we get, by using (11.236),

$$(1+\alpha)(\sin\psi' - \sin\psi) = \cos\psi\left(\frac{dv}{ds} + \frac{u}{\varrho}\right)$$

or, approximately,

$$(1+\alpha)(\psi' - \psi) = \frac{dv}{ds} + \frac{u}{\varrho}$$

Now $(\psi' - \psi)$ is the rotation of the tangent at B, and is the angle we have previously called η, or differs from it by a constant. Therefore

$$(1+\alpha)\eta = \frac{dv}{ds} + \frac{u}{\varrho} \quad . \quad . \quad . \quad . \quad . \quad (11.238)$$

Equations (11.237) and (11.238) will clearly remain true if we replace the whole strains and displacements by increments of these quantities. That is, if $\delta\alpha$ and $\delta\eta$ are increments of α and η corresponding to displacements δu and δv, then

$$\delta\alpha = \frac{d(\delta u)}{ds} - \frac{\delta v}{\varrho}, \quad . \quad . \quad . \quad . \quad . \quad (11.239)$$

and

$$(1 + \delta\alpha)\,\delta\eta = \frac{d(\delta v)}{ds} + \frac{\delta u}{\varrho}.$$

When the second order quantity $\delta\alpha\delta\eta$ is neglected this last equation becomes

$$\delta\eta = \frac{d(\delta v)}{ds} + \frac{\delta u}{\varrho} \quad . \quad . \quad . \quad . \quad . \quad . \quad (11.240)$$

We are now in a position to transform the right hand side of (11.229). Thus, if s be zero at one end of the rod and l at the other, integration by parts gives

$$-\int_0^l \frac{dT}{ds}\,\delta u\,ds = -\Big[T\delta u\Big]_0^l + \int_0^l T\,\frac{d(\delta u)}{ds}\,ds$$

Therefore

$$-\int_0^l \left(\frac{dT}{ds}\,\delta u + \frac{T}{\varrho}\,\delta v\right) ds = -\Big[T\delta u\Big]_0^l + \int_0^l T\left\{\frac{d(\delta u)}{ds} - \frac{\delta v}{\varrho}\right\} ds$$

$$= -\Big[T\delta u\Big]_0^l + \int_0^l T\,\delta\alpha\,ds$$

Again

$$\int_0^l \frac{d^2M}{ds^2}\,\delta v\,ds = \left[\frac{dM}{ds}\,\delta v\right]_0^l - \int_0^l \frac{dM}{ds}\,\frac{d(\delta v)}{ds}\,ds$$

Therefore

$$\int_0^l \left(\frac{d^2M}{ds^2}\,\delta v - \frac{\delta u}{\varrho}\,\frac{dM}{ds}\right) ds = \left[\frac{dM}{ds}\,\delta v\right]_0^l$$

$$- \int_0^l \frac{dM}{ds}\left\{\frac{d(\delta v)}{ds} + \frac{\delta u}{\varrho}\right\} ds$$

$$= \left[\frac{dM}{ds}\,\delta v\right]_0^l - \int_0^l \frac{dM}{ds}\,\delta\eta\,ds$$

$$= \left[\frac{dM}{ds}\,\delta v - M\,\delta\eta\right]_0^l + \int_0^l M\,\frac{d(\delta\eta)}{ds}\,ds$$

Then finally equation (11.229) becomes

$$\delta V = \left[\frac{dM}{ds}\,\delta v - M\delta\eta - T\delta u\right]_0^l$$

$$+ \int_0^l \left\{T\delta\alpha + M\,\frac{d(\delta\eta)}{ds}\right\} ds \quad . \quad . \quad . \quad (11.241)$$

The terms at the boundary are zero in consequence of the boundary conditions. Take, for example, the term $T\delta u$ at the boundary. If either end is free then T is zero at that end; but if either end is fixed so that u cannot change there then δu is zero. Thus the product $T\delta u$ is zero in any case at both ends. Again each end is clamped or not clamped. At a clamped end M is not zero but $\delta\eta$ must be zero. At an unclamped end M must be zero, and therefore the product $M\delta\eta$ is zero. The other term is $-F\delta v$, which is zero at both ends for similar reasons. Thus finally

$$\delta V=\int_0^l\left\{T\delta\alpha+M\frac{d(\delta\eta)}{ds}\right\}ds \quad . \quad . \quad . \quad . \quad (11.242)$$

which is identical with (11.227).

Some idea of the error introduced by neglecting the term

$$\int M\delta\left(\frac{\alpha}{\varrho}\right)ds \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11.243)$$

in equation (11.227) can be got by observing that errors of the same magnitude exist already owing to taking slightly inaccurate expressions for the curvature.

Consider for example, the curvature of the rod in fig. 110 at the point A. It is usual to take the curvature at A as correctly given by the equation

$$\frac{1}{\varrho}=\frac{d^2y'}{dx^2}$$

whereas actually the curvature is

$$\frac{1}{\varrho}=\frac{d^2y'}{dx'^2}.$$

But $dx'=(1+\alpha)\,dx$, and therefore the correct expression for curvature is

$$\frac{1}{\varrho}=\frac{1}{1+\alpha}\frac{d}{dx}\left(\frac{1}{1+\alpha}\frac{dy'}{dx}\right)$$

$$=\frac{1}{(1+\alpha)^2}\frac{d^2y'}{dx^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11.244)$$

at the point where $x=0$.

If therefore M is proportional to the change of curvature we ought to take the equation

$$\frac{M}{EI}=\frac{1}{(1+\alpha)^2}\frac{d^2y'}{dx^2}-\frac{d^2y}{dx^2} \quad . \quad . \quad . \quad . \quad (11.245)$$

instead of the usual equation

$$\frac{M}{EI}=\frac{d^2y'}{dx^2}-\frac{d^2y}{dx^2}$$

The difference between these two expressions for M introduces errors into the energy expression of just the same order as the term in

(11.243). Unless therefore we make use of a precise expression for the curvature it is useless to retain the term in (11.243) in the expression for the energy. We may therefore use, in all cases, the approximate equation

$$\delta V = \int (T\delta\alpha + M\delta c)\,ds$$

and assume

$$T = EA\alpha,$$
$$M = EIc;$$

whence we find

$$\delta V = \int (EA\alpha\delta\alpha + EIc\delta c)\,ds$$
$$= \int \left\{ \tfrac{1}{2} EA\delta(\alpha^2) + \tfrac{1}{2} EI\delta(c^2) \right\} ds,$$

and therefore

$$V = \int (\tfrac{1}{2} EA\alpha^2 + \tfrac{1}{2} EIc^2)\,ds$$
$$= \int (\tfrac{1}{2} T\alpha + \tfrac{1}{2} Mc)\,ds \quad . \quad . \quad . \quad . \quad . \quad . \quad (11.246)$$

just as for a naturally straight rod.

CHAPTER XII.

SPHERES AND CYLINDERS.

213. Sphere with radial displacement only.

Suppose a uniform sphere or spherical shell is subjected to radial forces only, such as internal or external pressures. It is required to find the strain in terms of the displacement.

Let the forces alter the distance of a particle from the centre of the sphere from r to $(r+u)$. Then clearly the radial strain is

$$\alpha = \frac{du}{dr} \qquad \text{(12.1)}$$

The circumference of any great circle on the surface of the sphere of radius $(r+u)$ has been changed from $2\pi r$ to $2\pi(r+u)$. Therefore the circumferential strain is

$$\beta = \frac{2\pi u}{2\pi r} = \frac{u}{r} \qquad \text{(12.2)}$$

This is the strain in every direction perpendicular to the radius r. If then we take two perpendicular axes in a plane touching the sphere of radius $(r+u)$, and denote the strains in the directions of these axes by β and γ, we have

$$\beta = \gamma = \frac{u}{r} \qquad \text{(12.3)}$$

Thus the three extensional strains in the directions of three perpendicular axes are

$$\frac{du}{dr}, \quad \frac{u}{r}, \quad \frac{u}{r}.$$

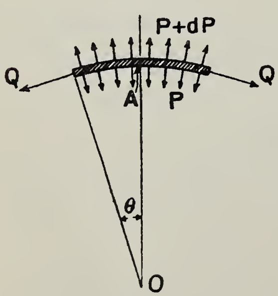


Fig. 111

Relative to the same set of axes the shear strains are obviously zero.

Instead of using the general equations of equilibrium it is more instructive to work out the equations of equilibrium afresh for the case of the sphere. Let P, Q, denote the radial and circumferential tensions in the sphere. Then let us consider the equilibrium of a small circular portion of a shell of radius r and thickness dr. Let the diameter of this circle subtend 2θ at the centre of the sphere. Then the radius of the circle is approximately $r\theta$, and its circumference $2\pi r\theta$. A force Qdr acts on each unit length of this circumference and it is everywhere inclined at $\frac{\pi}{2} - \theta$ to the middle radius OA (Fig. 111). Therefore the resultant of all these forces is

$$Q\,dr \times 2\pi r\theta \cos\left(\frac{\pi}{2} - \theta\right) = 2\pi r\theta^2 Q\,dr \text{ approximately}.$$

Again a force P acts on each unit of the inner surface, giving a total force $\pi(r\theta)^2 P$ on this inner surface. Although all the elements of this total force are not quite parallel the resultant differs from $\pi r^2\theta^2 P$ by a quantity of smaller order than θ^3. Consequently we may take this as the resultant. Likewise the resultant pull on the outer curved surface may be taken as $\pi\theta^2(r + dr)^2(P + dP)$, or, as it may be written, $\pi\theta^2\{r^2P + d(r^2P)\}$. Thus the difference of the pulls on the two faces is $\pi\theta^2 d(r^2P)$ acting along the outward radius, and this must balance the force $2\pi r\theta^2 Q$ which acts along the inward radius, Hence

$$\pi\theta^2 d(r^2P) = 2\pi r\theta^2 Q\,dr,$$

whence

$$\frac{d(r^2P)}{dr} = 2rQ \quad \ldots \ldots \quad (12.4)$$

We may write this if we choose,

$$\frac{d(r^2P)}{d(r^2)} = Q \quad \ldots \ldots \quad (12.5)$$

Since the stresses in the directions of the strains α, β, γ, are P, Q, Q, the relations between stresses and strains are, by equation (2.14),

$$\begin{aligned} E\alpha &= P - \sigma(Q + Q) \\ &= P - 2\sigma Q, \\ E\beta &= Q - \sigma(P + Q); \end{aligned}$$

that is,

$$E\frac{du}{dr} = P - 2\sigma Q, \quad \ldots \ldots \quad (12.6)$$

$$E\frac{u}{r} = (1 - \sigma)Q - \sigma P. \quad \ldots \ldots \quad (12.7)$$

We can solve equations (12.4), (12.6), (12.7), for u, P, Q.

To eliminate u from (12.6) and (12.7), we multiply (12.7) by r and differentiate. Thus we get

$$E\frac{du}{dr} = (1 - \sigma)\frac{d(rQ)}{dr} - \sigma\frac{d(rP)}{dr}. \quad \ldots \ldots \quad (12.8)$$

Now subtracting (12.6) from (12.8) we find

$$0 = (1 - \sigma)\frac{d(rQ)}{dr} - \sigma\frac{d(rP)}{dr} - P + 2\sigma Q \quad \ldots \quad (12.9)$$

Then the elimination of Q from (12.4) and (12.9) gives

$$\tfrac{1}{2}(1-\sigma\frac{d^2(r^2P)}{dr^2} - \sigma\frac{d(rP)}{dr} - P + \frac{\sigma}{r}\frac{d(r^2P)}{dr} = 0 \quad . \quad . \quad (12.10)$$

Let $r^2P = y$. Then

$$\frac{d}{dr}(rP) = \frac{d}{dr}\left(\frac{y}{r}\right) = \frac{1}{r}\frac{dy}{dr} - \frac{1}{r^2}y.$$

Therefore (12.10) becomes

$$\tfrac{1}{2}(1-\sigma)\frac{d^2y}{dr^2} - \frac{\sigma}{r}\frac{dy}{dr} + \frac{\sigma y}{r^2} - \frac{y}{r^2} + \frac{\sigma}{r}\frac{dy}{dr} = 0, \quad . \quad . \quad (12.11)$$

whence

$$\frac{d^2y}{dr^2} - 2\frac{y}{r^2} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (12.12)$$

This is a homogeneous linear equation whose solution is

$$y = Ar^2 + \frac{B}{r}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12.13)$$

from which

$$P = A + \frac{B}{r^3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12.14)$$

Equation (12.4) now gives

$$Q = \frac{1}{2r}\frac{d}{dr}\left\{Ar^2 + \frac{B}{r}\right\} = A - \frac{B}{2r^3} \quad . \quad . \quad . \quad . \quad (12.15)$$

214. Thick sphere.

The preceding solution can be applied to the problem of a homogeneous body whose boundaries are two concentric spherical surfaces, these surfaces being subjected to pressures which are uniform over each surface. Let the inner and outer radii be a and b, and the pressures on the inner and outer surfaces p and q. Then our conditions are

$$\left.\begin{array}{l} P = -p \text{ when } r = a \\ P = -q \text{ when } r = b \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad (12.16)$$

Therefore, on substituting these values of P in (12.14), we get

$$-p = A + \frac{B}{a^3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12.17)$$

$$-q = A + \frac{B}{b^3}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12.18)$$

whence

$$B = \frac{a^3b^3}{b^3-a^3}(q-p) \quad . \quad . \quad . \quad . \quad . \quad (12.19)$$

$$A = -\frac{b^3q - a^3p}{b^3-a^3} \quad . \quad . \quad . \quad . \quad . \quad (12.20)$$

Thus the general expressions for P and Q are

$$P = \frac{1}{b^3-a^3}\left\{-b^3q + a^3p + \frac{a^3b^3}{r^3}(q-p)\right\} \quad . \quad . \quad (12.21)$$

$$Q=\frac{1}{b^3-a^3}\left\{-b^3q+a^3p-\frac{a^3b^3}{2r^3}(q-p)\right\} \quad . \quad . \quad . \quad (12.22)$$

It is worth while to notice that, if $q=p$, then $P=-p$ for all values of r. This we could easily foresee because the material is in that case under a hydrostatic thrust.

If the sphere is subject to an internal pressure p only, q being zero, then

$$P=-\frac{a^3p}{b^3-a^3}\left\{\frac{b^3}{r^3}-1\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (12.23)$$

$$Q=\frac{a^3p}{b^3-a^3}\left\{1+\frac{b^3}{2r^3}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (12.24)$$

If a is very small compared with b we get the approximate equations

$$P=-\frac{a^3p}{b^3}\left(\frac{b^3}{r^3}-1\right)=-a^3p\left(\frac{1}{r^3}-\frac{1}{b^3}\right), \quad . \quad . \quad (12.25)$$

$$Q=\frac{a^3p}{b^3}\left(1+\frac{b^3}{2r^3}\right)=a^3p\left(\frac{1}{b^3}+\frac{1}{2r^3}\right). \quad . \quad . \quad . \quad (12.26)$$

At points in the material where $\left(\frac{r}{b}\right)^3$ is small these reduce to

$$\left.\begin{aligned}P&=-\frac{a^3p}{r^3}\\ Q&=\frac{a^3p}{2r^3}\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (12.27)$$

These may be regarded as the stresses in a body of any shape at points near a spherical hole inside which there is a pressure p, provided that the outer surface of the body is free from pressure, and provided that every point of this outer surface is at a greater distance from the centre of the hole than four or five diameters of the hole.

215. Thick cylinder.

This problem is similar to the problem of the thick sphere. The assumptions we now make are that the strain in the direction of the axis is either zero or constant, and that the displacement perpendicular to the axis is radial and depends only on the radius. The first of these assumptions implies that plane sections perpendicular to the axis remain plane during the strain.

Taking the z-axis along the axis of the cylinder and denoting the unstrained distance of a particle from the axis by r and the strained distance by $r+u$, the radial strain is

$$\alpha=\frac{du}{dr} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12.28)$$

Also the strain in the direction perpendicular to r and to the axis is, as for the sphere,

$$\beta=\frac{u}{r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12.29)$$

The strain parallel to the axis is

$$\gamma = \text{constant} \quad . \quad . \quad . \quad . \quad . \quad . \quad (12.30)$$

From the conditions of the problem there are no shear strains relative to the axes of α, β, γ. Let the tensional strains in the directions of α, β, γ, be P, Q, R. Then, by (2.14),

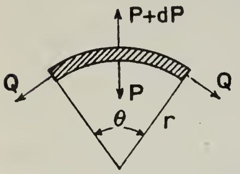


Fig. 112

$$E\frac{du}{dr} = P - \sigma(Q + R) \quad . \quad . \quad . \quad . \quad . \quad . \quad (12.31)$$

$$E\frac{u}{r} = Q - \sigma(R + P) \quad . \quad . \quad . \quad . \quad . \quad . \quad (12.32)$$

$$E\gamma = R - \sigma(P + Q) \quad . \quad . \quad . \quad . \quad . \quad . \quad (12.33)$$

For the equilibrium of an element of dimensions dr, $r\theta$, dz, we get, by resolving parallel to the middle radius and assuming that θ is small,

$$(P + dP) \times (r + dr)\theta dz - Pr\theta dz = (2Q\,dr\,dz)\sin\frac{\theta}{2}$$

$$= Q dr dz \theta; \quad . \quad . \quad . \quad (12.34)$$

that is,

$$\theta dz d(Pr) = Q dr dz \theta,$$

whence

$$\frac{d(Pr)}{dr} = Q, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12.35)$$

From equations (12.31), (12.32), and (12.35) we can eliminate P and Q. Thus, from (12.31) and (12.32)

$$P - \sigma(Q + R) = E\frac{du}{dr}$$

$$= \frac{d}{dr}\{rQ - \sigma r(R + P)\} \quad . \quad . \quad . \quad (12.36)$$

from which we get, using (12.35) and writing y for Pr,

$$\frac{y}{r} - \sigma\left\{\frac{dy}{dr} + R\right\} = \frac{d}{dr}\left\{r\frac{dy}{dr} - \sigma r R - \sigma y\right\}$$

$$= r\frac{d^2y}{dr^2} + \frac{dy}{dr} - \sigma\frac{d(rR)}{dr} - \sigma\frac{dy}{dr};$$

that is,

$$r\frac{d^2y}{dr^2} + \frac{dy}{dr} - \frac{y}{r} = \sigma\left\{\frac{d(rR)}{dr} - R\right\}$$

$$= \sigma r\frac{dR}{dr} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12.37)$$

We have not yet made use of (12.33). This gives, since γ is constant,

$$0 = \frac{dR}{dr} - \sigma\frac{d(P + Q)}{dr}$$

Therefore

$$\frac{dR}{dr}=\sigma\left\{\frac{dP}{dr}+\frac{dQ}{dr}\right\}$$

$$=\sigma\left\{\frac{d}{dr}\left(\frac{y}{r}\right)+\frac{d^2y}{dr^2}\right\}$$

$$=\sigma\left\{\frac{1}{r}\frac{dy}{dr}-\frac{y}{r^2}+\frac{d^2y}{dr^2}\right\} \quad . \quad . \quad . \quad . \quad . \quad (12.38)$$

Eliminating R from (12.37) and (12.38) we find

$$r(1-\sigma^2)\left\{\frac{d^2y}{dr^2}+\frac{1}{r}\frac{dy}{dr}-\frac{y}{r^2}\right\}=0 \quad . \quad . \quad . \quad . \quad (12.39)$$

This is a homogeneous linear equation whose solution is

$$y=Ar+\frac{B}{r}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12.40)$$

whence

$$P=A+\frac{B}{r^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12.41)$$

and

$$Q=\frac{d}{dr}(Pr)=A-\frac{B}{r^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (12.42)$$

If the cylinder is subjected to internal and external pressures p and q at radii a and b we find, in the same way as for the sphere,

$$P=\frac{1}{b^2-a^2}\left\{-b^2q+a^2p+\frac{a^2b^2}{r^2}(q-p)\right\} \quad . \quad . \quad (12.43)$$

Also

$$Q=\frac{d(Pr)}{dr}$$

$$=\frac{1}{b^2-a^2}\left\{-b^2q+a^2p-\frac{a^2b^2}{r^2}(q-p)\right\} \quad . \quad . \quad (12.44)$$

It will be seen from the actual values of P and Q that

$$P+Q=\frac{-2(b^2q-a^2p)}{b^2-a^2} \quad . \quad . \quad . \quad . \quad . \quad (12.45)$$

which is independent of r as well as of z. It now follows from (12.33) that

$$R=\text{constant}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12.46)$$

an assumption which is often illegitimately made in books on strength of materials in order to shorten the investigation.

The result in (12.46) follows also from (12.37) and (12.38), for it happens that y can be eliminated at once from these equations, giving

$$(1-\sigma^2)\frac{dR}{dr}=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12.47)$$

whence

$$R=\text{constant} \quad . \quad . \quad . \quad . \quad . \quad (12.48)$$

The constant value of R depends on the value of γ, which is at present quite arbitrary. But suppose that we are dealing with a long cylinder closed at both ends and subjected to the internal and external pressures p and q; then on each end there is a pressure p over a circle of radius a and a pressure q over a circle of radius b. Thus the total axial pull at each end is

$$\pi a^2 p - \pi b^2 q$$

and this must equal the total tension across a section of the cylinder. Therefore

$$\pi(b^2 - a^2)\,\mathrm{R} = \pi(a^2 p - b^2 q)\,, \quad . \quad . \quad . \quad . \quad (12.49)$$

whence

$$\mathrm{R} = \frac{a^2 p - b^2 q}{b^2 - a^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (12.50)$$

This happens to be the common constant term in the expressions for P and Q.

Also

$$\mathrm{P} + \mathrm{Q} = 2\mathrm{R} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12.51)$$

216. Rotating cylinder.

A homogeneous circular cylinder, either solid or hollow, rotates with constant angular velocity ω about its axis. The problem before us is to find the stresses due to this rotation

The exact solution to the problem of a finite rotating cylinder with free ends has never yet been worked out, but there are three distinct solutions to the rotating cylinder problem each of which, however, leaves some stress at the surface of the cylinder. Of the three solutions given below the first and simplest is the most unreal; the second gives a good approximation to the stresses in a long cylinder at points not too near the ends; and the third, which is Chree's solution, gives a good approximation to the stresses in a rotating disk, that is, a cylinder whose length is much shorter than its diameter.

For every case of the rotating cylinder equations (12.31), (12.32), (12.33), are true. Equation (12.34) has to be modified to allow for the centrifugal force. Thus the centrifugal force on the element shown in fig. 112 is

$$(\varrho r\theta dr dz) r\omega^2\,,$$

ϱ being the mass per unit volume of the material of the disk. Since this acts in the direction of $(\mathrm{P} + d\mathrm{P})$ it has to be added to the left hand side of (12.34). Then equation (12.35) becomes

$$\frac{d(\mathrm{P}r)}{dr} = \mathrm{Q} - \varrho r^2 \omega^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (12.52)$$

217. First solution; purely radial strain.

In this case we assume that $\gamma = 0$ in (12.33). Eliminating u from (12.31) and (12.32) we get

$$P-\sigma(Q+R)=\frac{d}{dr}\cdot r\{Q-\sigma(R+P)\},$$

which becomes, by the aid of (12.33),

$$(1-\sigma^2)P-\sigma(1+\sigma)Q=\frac{d}{dr}r\{(1-\sigma^2)Q-\sigma(1+\sigma)P\}$$

Dividing this by $(1+\sigma)$ we get

$$(1-\sigma)P-\sigma Q=\frac{d}{dr}r\{(1-\sigma)Q-\sigma P\} \quad . \quad . \quad . \quad (12.53)$$

Now (12.52) and (12.53) give, when y is written for Pr,

$$(1-\sigma)\frac{y}{r}-\sigma\frac{dy}{dr}-\sigma\varrho r^2\omega^2=\frac{d}{dr}r\left\{(1-\sigma)\left(\frac{dy}{dr}+\varrho r^2\omega^2\right)-\sigma\frac{y}{r}\right\}$$

$$=\frac{d}{dr}\left\{(1-\sigma)\left(r\frac{dy}{dr}+\varrho r^3\omega^2\right)-\sigma y\right\}$$

$$=(1-\sigma)\left(r\frac{d^2y}{dr^2}+\frac{dy}{dr}\right)+3(1-\sigma)\varrho r^2\omega^2-\sigma\frac{dy}{dr};$$

that is,

$$r^2\frac{d^2y}{dr^2}+r\frac{dy}{dr}-y=-\frac{3-2\sigma}{1-\sigma}\varrho r^3\omega^2 \quad . \quad . \quad . \quad (12.54)$$

The complete solution of this equation is

$$y=Ar+\frac{B}{r}-\frac{1}{8}\frac{3-2\sigma}{1-\sigma}\varrho r^3\omega^2; \quad . \quad . \quad . \quad (12.55)$$

whence

$$P=A+\frac{B}{r^2}-\frac{1}{8}\frac{3-2\sigma}{1-\sigma}\varrho r^2\omega^2, \quad . \quad . \quad . \quad . \quad (12.56)$$

and

$$Q=\frac{d(Pr)}{dr}+\varrho r^2\omega^2$$

$$=A-\frac{B}{r^2}-\frac{1}{8}\frac{1+2\sigma}{1-\sigma}\varrho r^2\omega^2 \quad . \quad . \quad . \quad . \quad (12.57)$$

Also

$$R=\sigma(P+Q)$$

$$=\sigma\left\{2A-\frac{1}{2}\frac{1}{1-\sigma}\varrho r^2\omega^2\right\} \quad . \quad . \quad . \quad . \quad . \quad (12.58)$$

Now suppose the cylinder is hollow, and that its inner and outer radii are a and b. Then, putting

$$P=0 \text{ where } r=a$$
$$\text{and where } r=b$$

we get, on writing C for the constant $\frac{1}{8}\frac{3-2\sigma}{1-\sigma}\varrho\omega^2$,

$$\left.\begin{aligned} 0&=A+\frac{B}{a^2}-Ca^2,\\ 0&=A+\frac{B}{b^2}-Cb^2;\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (12.59)$$

whence and

$$\left.\begin{aligned} A&=(b^2+a^2)C,\\ B&=-a^2b^2C.\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (12.60)$$

Thus
$$P = C\left\{b^2 + a^2 - \frac{a^2b^2}{r^2} - r^2\right\}$$
$$= C\frac{(b^2 + a^2)r^2 - a^2b^2 - r^4}{r^2}$$
$$= C\frac{(r^2 - a^2)(b^2 - r^2)}{r^2}, \quad . \quad . \quad . \quad . \quad . \quad (12.61)$$
$$Q = C\left\{b^2 + a^2 + \frac{a^2b^2}{r^2} - \frac{1 + 2\sigma}{3 - 2\sigma}r^2\right\} \quad . \quad . \quad (12.62)$$
The greatest value of Q occurs at the inner boundary where $r = a$, and this value of Q is
$$Q_0 = C\left\{2b^2 + a^2 - \frac{1 + 2\sigma}{3 - 2\sigma}a^2\right\} \quad . \quad . \quad . \quad . \quad (12.63)$$
If now $\frac{a^2}{b^2}$ is very small, we find that
$$Q_0 = 2Cb^2 \text{ nearly} \quad . \quad . \quad . \quad . \quad . \quad . \quad (12.64)$$
The only boundary condition in a solid cylinder is
$$P = 0 \text{ where } r = b \quad . \quad . \quad . \quad . \quad . \quad . \quad (12.65)$$
There is, however, one other condition which is equivalent to a boundary condition, namely, that P is finite at the centre where $r = 0$. Thus the constant B is zero, and this makes both P and Q finite at $r = 0$. When B is zero equation (12.59) gives
$$A = b^2C.$$
It is worth while to notice that the correct values of the constants A and B for this case could have been got merely by putting zero for a^2 in the values given by (12.60). It follows then that the stresses in a solid cylinder can be got by putting zero for a^2 in (12.61) and (12.62). These stresses are
$$P = C(b^2 - r^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11.66)$$
$$Q = C\left\{b^2 - \frac{1 + 2\sigma}{3 - 2\sigma}r^2\right\} \quad . \quad . \quad . \quad . \quad . \quad (12.67)$$
Here the maximum value of Q, as well as of P, is
$$Q_0 = Cb^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12.68)$$
This is just half the value of Q_0 in equation (12.64), which was obtained on the assumption that the cylinder had a very small cylindrical hole coaxial with the outer boundary.

Since the stress Q at the inner boundary of a hollow cylinder or at the centre of a solid one is the greatest stress in the cylinder we see how much the small central hole weakens the cylinder. It is advisable then to avoid central holes, or even holes that are not very near the centre, in rapidly rotating cylinders and disks.

The axial stress in a hollow cylinder is

$$R = \sigma\left\{2\,(a^2+b^2)\,C - \frac{1}{2}\frac{1}{1-\sigma}\varrho r^2\omega^2\right\}$$

$$= \frac{1}{4}\frac{\sigma}{1-\sigma}\varrho\omega^2\left\{(3-2\sigma)(a^2+b^2) - 2r^2\right\} \quad . \quad . \quad (12.69)$$

Therefore the total axial pull across a section is

$$F = \int_a^b R\,2\pi r\,dr$$

$$= \frac{1}{2}\frac{\pi\sigma}{1-\sigma}\varrho\omega^2\left\{\tfrac{1}{2}(3-2\sigma)(b^4-a^4) - \tfrac{1}{2}(b^4-a^4)\right\}$$

$$= \tfrac{1}{2}\pi\sigma\varrho\omega^2(b^4-a^4) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12.70)$$

The mean axial stress across a normal section is

$$R_1 = \frac{F}{\pi\,(b^2-a^2)}$$

$$= \tfrac{1}{2}\sigma\varrho\omega^2(b^2+a^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad (12.71)$$

It is clear then why the preceding solution is very unreal. It gives a variable axial stress which gives rise to a resultant axial force that is not zero. In the next solution we make this resultant axial pull zero.

218. Second solution; resultant axial pull vanishes.

This solution, as we have previously remarked, applies very well to all except the parts near the ends of a cylinder whose length is much greater than its diameter.

We need only superpose on the last solution a uniform axial compressive stress R_1 given by (12.71). This stress, uniformly distributed over the ends, produces a uniform longitudinal strain γ and a strain $\sigma\gamma$ in all perpendicular directions, but does not affect the stresses P and Q. Thus P and Q are exactly as for the first solution and R is diminished by R_1. In this case then

$$R = \sigma\{P+Q\} - R_1$$

$$= \sigma\left\{2\,(b^2+a^2)\,C - \frac{1}{2}\frac{1}{1-\sigma}\varrho r^2\omega^2\right\} - \tfrac{1}{2}\sigma\varrho\omega^2(b^2+a^2)$$

$$= \frac{1}{4}\frac{\sigma}{1-\sigma}\varrho\omega^2\left\{b^2+a^2-2r^2\right\} \quad . \quad . \quad . \quad . \quad (12.72)$$

219. Third solution, applicable to a thin disk.

For this solution we satisfy the equations of internal equilibrium and make all the stresses zero at the surface of the cylinder except the radial stress P over the curved surface. There is certainly a solution in which this last stress also vanishes, but this solution probably involves very complicated functions, and possibly functions that are still unknown. The solution we are about to deduce involves only simple algebraic expressions, and has the merit of being very nearly true for a thin disk at all points except near the curved surface, and here the error is not important since the stresses have certainly not their maximum values in this neighbourhood.

In deducing equation (12.52) we tacitly assumed, what was certainly true for the first two cases, that the shear stress was zero over a section perpendicular to the z-axis. In the present case, where R is to be zero at all points of the end sections, this shear stress may not be zero at points inside the material. There may, in fact, be a radial shear stress on the sections perpendicular to the z-axis, accompanied by shear stress parallel to the z-axis on cylindrical surfaces coaxial with the boundary cylinders.

Let S denote the shear stress at (r, z) inside the material; and let u, w denote the radial and axial displacements of the particle originally at (r, z). It is understood that z is measured from the middle section of the cylinder.

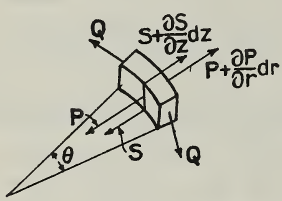


Fig. 113a

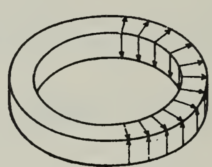


Fig. 113b

Now all our stresses and displacements are functions of z as well as of r. Instead of (12.52) we must now use an equation in which account is taken of the shear force on the two surfaces of the element shown in fig. 112 which are perpendicular to the z-axis. All the stresses acting perpendicular to the z-axis are shown in fig. 113a. Resolving radially for the equilibrium of this element we get

$$\theta dz \frac{\partial}{\partial r}(\mathrm{P}r)\,dr + r\theta dr \frac{\partial \mathrm{S}}{\partial z}\,dz = 2\,\mathrm{Q}\,dr dz \sin\frac{\theta}{2} - (\varrho r\theta dr dz) r\omega^2,$$

that is, since $\sin\frac{\theta}{2} = \frac{\theta}{2}$ nearly,

$$\frac{\partial(\mathrm{P}r)}{\partial r} + r\frac{\partial \mathrm{S}}{\partial z} - \mathrm{Q} = -\varrho r\omega^2 . \quad . \quad . \quad . \quad . \quad (12.73)$$

This equation takes the place of (12.52), and it involves a new unknown. We therefore need a new equation.

We can get our new equation by resolving the forces on the same element in the direction parallel to the axis of the cylinder.

The stresses in the z-direction are shown in fig. 114. The equation expressing equilibrium is

$$\theta dz \frac{\partial(\mathrm{S}r)}{\partial r} dr + \theta r dr \frac{\partial \mathrm{R}}{\partial z} dz,$$

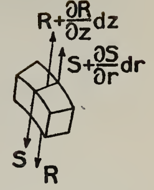


Fig. 114

whence
$$\frac{\partial(\mathrm{S}r)}{\partial r} + r\frac{\partial \mathrm{R}}{\partial z} = 0 \quad . \quad . \quad . \quad (12.74)$$

If we had no new unknowns except S in our equations this one new equation would be enough. But we have to remember that the strain γ in (12.33) is also unknown now, whereas in the first solution it was assumed to be constant. We still need then another equation. The required equation is the one expressing S in terms of displacements. Thus, if w denotes the axial displacement of a particle,

$$\mathrm{S} = n\left(\frac{\partial w}{\partial r} + \frac{\partial u}{\partial z}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12.75)$$

We have also to remember that

$$\gamma = \frac{\partial w}{\partial z} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12.76)$$

Let us now try to get a solution by assuming that R is zero everywhere. Then equation (12.74) gives

$$\frac{\partial(\mathrm{S}r)}{\partial r} = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12.77)$$

from which it follows that Sr is not a function of r. Let us further assume that S is zero, and leave the justification of this step till a later stage. These assumptions are, at any rate, consistent with equation (12.74).

With our new assumptions the four unknowns u, w, P, Q, must satisfy the five equations

$$\mathrm{E}\frac{\partial u}{\partial r} = \mathrm{P} - \sigma\mathrm{Q} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12.78)$$

$$\mathrm{E}\frac{u}{r} = \mathrm{Q} - \sigma\mathrm{P} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12.79)$$

$$\mathrm{E}\frac{\partial w}{\partial z} = -\sigma(\mathrm{P} + \mathrm{Q}) \quad . \quad . \quad . \quad . \quad . \quad . \quad (12.80)$$

$$0 = \frac{\partial w}{\partial r} + \frac{\partial u}{\partial z} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12.81)$$

$$\frac{\partial(r\mathrm{P})}{\partial r} = \mathrm{Q} - \varrho r^2\omega^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (12.82)$$

Since the number of equations exceeds the number of unknowns we shall have to show that these equations are consistent.

Eliminating u from (12.78) and (12.79) we get

$$P-\sigma Q=\frac{\partial}{\partial r}\{r(Q-\sigma P)\} \quad . \quad . \quad . \quad . \quad . \quad (12.83)$$

From (12.82) and (12.83) we now get

$$P-\sigma Q=\frac{\partial(rQ)}{\partial r}-\sigma(Q-\varrho r^2\omega^2),$$

whence

$$P=\frac{\partial(rQ)}{\partial r}+\sigma\varrho\omega^2r^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (12.84)$$

Multiplying through this last equation by r and then differentiating with respect to r we find

$$\frac{\partial(rP)}{\partial r}=\frac{\partial}{\partial r}\left\{r\frac{\partial(rQ)}{\partial r}\right\}+3\sigma\varrho\omega^2r^2 \quad . \quad . \quad . \quad . \quad (12.85)$$

When y is written for rQ the equation obtained by eliminating rP from (12.82) and (12.85) is

$$\frac{y}{r}-\varrho\omega^2r^2=\frac{\partial}{\partial r}\left(r\frac{\partial y}{\partial r}\right)+3\sigma\varrho\omega^2r^2,$$

whence

$$r\frac{\partial^2y}{\partial r^2}+\frac{\partial y}{\partial r}-\frac{y}{r}=-(1+3\sigma)\varrho\omega^2r^2 \quad . \quad . \quad . \quad (12.86)$$

The solution of this is

$$y=Ar+\frac{B}{r}-\tfrac{1}{8}(1+3\sigma)\varrho\omega^2r^3, \quad . \quad . \quad . \quad . \quad (12.87)$$

where A and B are not functions of r, but may be functions of z.

For a disk without a central hole B must be zero or Q would be infinite at the centre. We shall first deal with a disk with no central hole and assume therefore that B is zero. Then we get

$$Q=\frac{y}{r}=A-\tfrac{1}{8}(1+3\sigma)\varrho\omega^2r^2. \quad . \quad . \quad . \quad . \quad (12.88)$$

and equation (12.84) gives

$$P=A-\tfrac{1}{8}(3+\sigma)\varrho\omega^2r^2 \quad . \quad . \quad . \quad . \quad . \quad (12.89)$$

In order to complete the solution we must find the value of A.

Differentiating through (12.81) with respect to z we get

$$\frac{\partial^2w}{\partial z\partial r}+\frac{\partial^2u}{\partial z^2}=0;$$

that is,

$$E\frac{\partial^2u}{\partial z^2}=-\frac{\partial}{\partial r}\left(E\frac{\partial w}{\partial z}\right)$$

By means of equations (12.79) and (12.80) this last equation becomes

$$\frac{\partial^2}{\partial z^2}\{r(Q-\sigma P)\}=\sigma\frac{\partial}{\partial r}(P+Q) \quad . \quad . \quad . \quad . \quad (12.90)$$

This equation gives a relation between quantities that have already been found. It may or may not be true. This equation can, in fact, be regarded as the criterion which tests whether the four unknowns we started with can satisfy the five equations (12.78) to (12.82). If equation (12.90) can be satisfied our equations are consistent.

Now substituting for P and Q in (12.90) we get

$$(1-\sigma) r \frac{d^2 A}{dz^2} = -\sigma(1+\sigma)\varrho\omega^2 r, \quad \ldots \quad (12.91)$$

whence

$$\frac{d^2 A}{dz^2} = -\frac{\sigma(1+\sigma)}{1-\sigma}\varrho\omega^2 \quad \ldots \quad (12.92)$$

Because r has disappeared from the last equation it agrees with what we already knew, namely, that A is a function of z only. If r could not be removed from the equation our original five equations would have been inconsistent. The disappearance of r justifies, in fact, the assumptions that a solution was possible in which R and S were both zero.

Suppose z is measured from the middle section of the disk, so that P and Q are even functions of z. Then A is an even function of z, and therefore (12.92) gives

$$A = \frac{1}{2}\frac{\sigma(1+\sigma)}{1-\sigma}\varrho\omega^2 (K - z^2), \quad \ldots \quad (12.93)$$

K being a constant.

Thus

$$P = \frac{1}{2}\frac{\sigma(1+\sigma)}{1-\sigma}\varrho\omega^2 (K - z^2) - \tfrac{1}{8}(3+\sigma)\varrho\omega^2 r^2, \quad . \quad (12.94)$$

and

$$Q = \frac{1}{2}\frac{\sigma(1+\sigma)}{1-\sigma}\varrho\omega^2 (K - z^2) - \tfrac{1}{8}(1+3\sigma)\varrho\omega^2 r^2 \quad . \quad (12.95)$$

At the curved surface of the disk, where $r = a$, the value of P is P_1 given by

$$P_1 = \frac{1}{2}\frac{\sigma(1+\sigma)}{1-\sigma}\varrho\omega^2 (K - z^2) - \tfrac{1}{8}(3+\sigma)\varrho\omega^2 a^2 \quad . \quad (12.96)$$

Now let the constant K be determined so that the resultant effect of P_1 is zero over a thin rectangular strip of the curved surface with the long sides parallel to the axis and the other sides in the plane ends of the disk; that is, if the length of the disk is $2h$, K is determined by the equation

$$\int_{-h}^{h} P_1 \, dz = 0,$$

the factor representing the width of the strip being omitted.

The last equation gives

$$\frac{\sigma(1+\sigma)}{1-\sigma}\varrho\omega^2 (Kh - \tfrac{1}{3}h^3) - \tfrac{1}{4}(3+\sigma)\varrho\omega^2 a^2 h = 0,$$

whence $$\frac{\sigma(1+\sigma)}{1-\sigma}K = \frac{1}{3}\frac{\sigma(1+\sigma)}{1-\sigma}h^2 + \tfrac{1}{4}(3+\sigma)a^2.$$

Thus we get finally

$$P = \frac{1}{2}\frac{\sigma(1+\sigma)}{1-\sigma}\varrho\omega^2(\tfrac{1}{3}h^2 - z^2) + \tfrac{1}{8}(3+\sigma)\varrho\omega^2(a^2 - r^2), \quad . \quad . \quad . \quad (12.97)$$

$$Q = \frac{1}{2}\frac{\sigma(1+\sigma)}{1-\sigma}\varrho\omega^2(\tfrac{1}{3}h^2 - z^2) + \tfrac{1}{8}\varrho\omega^2\{(3+\sigma)a^2 - (1+3\sigma)r^2\}. \quad (12.98)$$

The solution we have now arrived at satisfies all the conditions of the problem of the rotating cylinder except one; it does not make P zero at the curved surface of the cylinder, although it does make the mean value of P zero over every thin strip parallel to the axis. It is clear therefore that, in the case of a thin disk, the effect of the surface force P_1 must be negligible at points whose distance from the curved edge is more than three or four times the thickness of the disk; that is, the stresses given by (12.97) and (12.98) must be very accurate at points not near the edge. We may therefore regard this as a satisfactory solution to the problem of a thin rotating disk.

It has been worth while to work out Chree's solution in full because we can see precisely to what extent it fails to satisfy the conditions of the problem. It is, as we have pointed out, a very accurate solution for a thin disk. But for such a disk the terms involving h and z are certainly small in comparison with those involving a and r. If then we drop the terms containing h and z we get an approximate solution to the rotating disk problem, although it is a worse approximation than Chree's solution. Dropping the terms in h and z we get

$$P = \tfrac{1}{8}(3+\sigma)\varrho\omega^2(a^2 - r^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12.99)$$

$$Q = \tfrac{1}{8}\varrho\omega^2\{(3+\sigma)a^2 - (1+3\sigma)r^2\} \quad . \quad . \quad . \quad (12.100)$$

Since the maximum stresses occur at the centre of the disk, and since the error at the centre in neglecting h and z is less than one per cent if a is greater than $6h$ and σ is $\frac{1}{3}$, it is clear that we may take these last expressions for the stresses in any disk whose diameter is greater than about five times the thickness. Moreover, since the terms in h and z represent a first approximation to the errors in (12.99) and (12.100), we may be fairly sure that the whole error is something of the same order as this approximation.

220. Thin disk again.

The final results in (12.99) and (12.100) could have been obtained very much more easily than by going through Chree's solution. These final results give P and Q as functions of r only. If we had started by assuming that P and Q were functions of r only we could have got their values from equations (12.78), (12.79), and (12.82). We have, in fact, used these equations to get (12.87), (12.88), (12.89), and the only difference between the method now suggested and the one actually used is that A and B would be constants and not functions of z. These constants could then be found by assuming that P is zero at the curved boundaries.

The method suggested here amounts, in effect, to using P and Q, not for the actual stresses in the disk, but for the z-means of these stresses. If we introduce P′ and Q′ for these z-means, which are defined by the equations

$$P' = \frac{1}{2h}\int_{-h}^{h} P\,dz, \quad \ldots \quad (12.101)$$

$$Q' = \frac{1}{2h}\int_{-h}^{h} Q\,dz, \quad \ldots \quad (12.102)$$

then equations (12.78), (12.79), (12.82), are all that we need to determine their values. Thus let

$$u' = \frac{1}{2h}\int_{-h}^{h} u\,dz \quad \ldots \quad (12.103)$$

Then from (12.78) we get

$$E\frac{1}{2h}\int_{-h}^{h}\frac{\partial u}{\partial r}\,dz = \frac{1}{2h}\int_{-h}^{h}(P - \sigma Q)\,dz$$
$$= P' - \sigma Q' \quad \ldots \quad (12.104)$$

But
$$\frac{1}{2h}\int_{-h}^{h}\frac{\partial u}{\partial r}\,dz = \frac{\partial}{\partial r}\left\{\frac{1}{2h}\int_{-h}^{h} u\,dz\right\}$$
$$= \frac{du'}{dr} \quad \ldots \quad (12.105)$$

since the operators $\frac{\partial}{\partial r}$ and $\int dz$ are independent. Therefore (12.104) becomes

$$E\frac{du'}{dr} = P' - \sigma Q', \quad \ldots \quad (12.106)$$

exactly the same form of equation as (12.78), the only difference being that the functions involved are functions of r only.

In the same way we can deduce from (12.79) and (12.82) two exactly similar equations with dashed letters. The solution of these three equations is then expressed by (12.87). The mean stresses are therefore

$$Q' = \frac{y}{r} = A + \frac{B}{r^2} - \tfrac{1}{8}(1 + 3\sigma)\varrho\omega^2 r^2 \quad \ldots \quad (12.107)$$

$$P' = \frac{dy}{dr} + \sigma\varrho\omega^2 r^2$$
$$= A - \frac{B}{r^2} - \tfrac{1}{8}(3 + \sigma)\varrho\omega^2 r^2 \quad \ldots \quad (12.108)$$

It is clear now that, in dealing with thin disks, we may regard P and Q as functions of r only and treat R and S as zero. Although the values of P and Q that we get in this way are the z-means of the stresses there is no doubt that, at points not near the edge, the actual stresses vary so little across the thickness that these z-means differ by insignificant amounts from the actual stresses at the middle section of the disk.

221. Disk with a central hole.

Let the inner and outer radii of the hole be b and a respectively. The mean stresses are given by (12.107) and (12.108). We have only only to determine A and B so as to satisfy the conditions

$$P' = 0 \begin{cases} \text{where } r = a \\ \text{and } \quad r = b \end{cases}$$

These conditions give

$$\left.\begin{aligned} 0 &= A - \frac{B}{a^2} - \tfrac{1}{8}(3+\sigma)\varrho\omega^2 a^2, \\ 0 &= A - \frac{B}{b^2} - \tfrac{1}{8}(3+\sigma)\varrho\omega^2 b^2; \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (12.109)$$

whence

$$\left.\begin{aligned} A &= \tfrac{1}{8}\varrho\omega^2(3+\sigma)(a^2+b^2) \\ B &= \tfrac{1}{8}\varrho\omega^2(3+\sigma)a^2b^2 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (12.110)$$

Then finally

$$P' = \tfrac{1}{8}\varrho\omega^2(3+\sigma)\left\{a^2+b^2-\frac{a^2b^2}{r^2}-r^2\right\}, \quad . \quad . \quad . \quad . \quad (12.111)$$

$$Q' = \tfrac{1}{8}\varrho\omega^2(3+\sigma)\left\{a^2+b^2+\frac{a^2b^2}{r^2}\right\} - \tfrac{1}{8}\varrho\omega^2(1+3\sigma)r^2 \quad (12.112)$$

The mean circumferential or hoop stress at the edge of the hole, where $r = b$, is

$$\begin{aligned} Q' &= \tfrac{1}{8}\varrho\omega^2(3+\sigma)\left\{a^2+b^2+\frac{a^2b^2}{b^2}\right\} - \tfrac{1}{8}\varrho\omega^2(1+3\sigma)b^2 \\ &= \tfrac{1}{4}\varrho\omega^2\{(3+\sigma)a^2-(1-\sigma)b^2\}, \quad . \quad . \quad . \quad . \quad (12.113) \end{aligned}$$

which becomes, when $\dfrac{b^2}{a^2}$ is negligible,

$$Q' = \tfrac{1}{4}(3+\sigma)\varrho\omega^2 a^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12.114)$$

This is the maximum stress in the disk, and it is twice as great as the maximum stress when there is no central hole.

It is worth while to notice that, although the conditions by which the constants are determined are not the same in the two cases, the stresses in a solid disk, that is, the stresses given by (12.99) and (12.100), can be got from (12.111) and (12.112) by putting $b = 0$ in the last pair of equations. In order to find the stresses at the centre of a solid disk it is necessary to put $b=0$ *before* putting $r=0$.

We see again here, as in Art. 217, that the effect of a small circular hole at the centre is to double the stress at that point. It is shown in Art. 231 in the next chapter that this is a particular case of a general theorem. It is there shown that, if a small circular hole is made at a point in a plate where the principal stresses would be equal if there were no hole, the maximum stress is thereby doubled.

222. Rotating disk of variable thickness.

We may use the method of the last article to deal with a rotating disk the thickness of which is a function of the distance r from the axis.

Let P′ and Q′ denote the mean radial and hoop stresses at r, and let u' denote the mean radial displacement across the thickness of the plate. Let $2h$ be the thickness at r, and $2(h+dh)$ at $(r+dr)$.

Let us consider the motion of a small element of dimensions $dr \times r\theta$, the angle θ being infinitesimal. The inward radial pull on the cylindrical surface at r is $2hr\theta P'$, and the outward radial pull on the cylindrical surface at $(r+dr)$ is $2\theta\{hrP'+d(hrP')\}$. Therefore, since the mass is $2\rho hr\theta dr$ and its acceleration $r\omega^2$, the equation of motion is

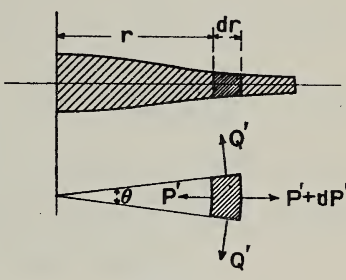


Fig. 115

$$2hdrQ' \times 2\sin\frac{\theta}{2} - 2\theta d(hrP') = 2\rho hr\theta dr \times r\omega^2,$$

whence, on putting $\frac{1}{2}\theta$ for $\sin\frac{1}{2}\theta$ and dividing by $2\theta dr$, we get

$$hQ' - \frac{d(hrP')}{dr} = \rho hr^2\omega^2 \quad . \quad . \quad . \quad . \quad (12.115)$$

We may also assume that there is the same relation between mean stresses and mean strains as between actual stresses and actual strains. Thus

$$E\frac{du'}{dr} = P' - \sigma Q' \quad . \quad . \quad . \quad . \quad . \quad (12.116)$$

$$E\frac{u'}{r} = Q' - \sigma P' \quad . \quad . \quad . \quad . \quad . \quad (12.117)$$

The elimination of u' from the last two equations gives

$$P' - \sigma Q' = \frac{d}{dr}(rQ' - \sigma rP') \quad . \quad . \quad . \quad . \quad (12.118)$$

Equations (12.115) and (12.118) determine P′ and Q′ when h is given as a function of r. Thus from (12.115) we get, on multiplying by $\frac{r}{h}$ and differentiating,

$$\frac{d(rQ')}{dr}-\frac{d}{dr}\left\{\frac{r}{h}\frac{d(hrP')}{dr}\right\}=3\varrho r^2\omega^2 \quad . \quad . \quad (12.119)$$

When the values of Q′ and $\dfrac{d(rQ')}{dr}$ from (12.115) and (12.119) are substituted in (12.118) the last equation becomes

$$P'-\frac{\sigma}{h}\frac{d(hrP')}{dr}-\sigma\varrho r^2\omega^2$$
$$=\frac{d}{dr}\left\{\frac{r}{h}\frac{d(hrP')}{dr}\right\}+3\varrho r^2\omega^2-\sigma\frac{d(rP')}{dr}$$

When y is written for hrP′ the last equation can be put in the form

$$r^2\frac{d^2y}{dr^2}+r\frac{dy}{dr}\left(1-\frac{r}{h}\frac{dh}{dr}\right)+y\left(\sigma\frac{r}{h}\frac{dh}{dr}-1\right)=-(3+\sigma)\varrho hr^3\omega^2, \quad (12.120)$$

from which y can be found when h is given.

If

$$h=cr^{-\beta} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12.121)$$

equation (12.120) reduces to

$$r^2\frac{d^2y}{dr^2}+(1+\beta)r\frac{dy}{dr}-(1+\sigma\beta)y=-(3+\sigma)\varrho hr^3\omega^2 \quad . \quad (12.122)$$

This is a homogenous-linear equation whose solution is

$$y=Ar^{q_1}+Br^{q_2}-\frac{3+\sigma}{8-(3+\sigma)\beta}\varrho cr^{3-\beta}\omega^2, \quad . \quad (12.123)$$

where q_1, q_2, are the roots of the equation

$$q^2+\beta q-1-\sigma\beta=0 \quad . \quad . \quad . \quad . \quad . \quad (12.124)$$

Therefore

$$P'=\frac{y}{hr}=\frac{yr^{\beta-1}}{c}$$
$$=\frac{A}{c}r^{q_1+\beta-1}+\frac{B}{c}r^{q_2+\beta-1}-\frac{3+\sigma}{8-(3+\sigma)\beta}\varrho r^2\omega^2 \quad (12.125)$$

Also, by (12.115),

$$Q'=q_1\frac{A}{c}r^{q_1+\beta-1}+q_2\frac{B}{c}r^{q_2+\beta-1}-\frac{1+3\sigma}{8-(3+\sigma)\beta}\varrho r^2\omega^2 \quad . \quad (12.126)$$

If q_2 is the smaller root of (12.124) and if $(1+\sigma\beta)$ is positive then $(q_2+\beta)$ is negative; for (12.124) can be written in the form:—

$$(q+\beta)^2-\beta(q+\beta)-1-\sigma\beta=0,$$

from which it follows that

$$(q_1+\beta)(q_2+\beta)=-(1+\sigma\beta)=\text{negative}.$$

Therefore $(q_1+\beta)$ and $(q_2+\beta)$ have opposite signs, and thus the smaller of the two must be negative. Consequently, if B were not zero, the term containing B in the stresses P′ and Q′ would be infinite where $r=0$. Since the stresses must be finite at the centre of a complete disk it is necessary that B be zero for such a disk. The other constant A is then determined by the conditions at the outer boundary of the disk. For a disk with a central hole the constants A and B are determined by the conditions at the two circular boundaries.

Suppose $P'=0$ where $r=a$ for a complete disk of radius a. Then since B is zero,

$$0=\frac{A}{c}a^{q_1+\beta-1}-\frac{3+\sigma}{8-(3+\sigma)\beta}\varrho a^2\omega^2,$$

whence $$\frac{A}{c}=\frac{3+\sigma}{8-(3+\sigma)\beta}\varrho\omega^2 a^{3-q_1-\beta}.$$

Therefore

$$P'=\frac{3+\sigma}{8-(3+\sigma)\beta}\varrho\omega^2a^2\left\{\left(\frac{r}{a}\right)^{q_1+\beta-1}-\frac{r^2}{a^2}\right\} \quad . \quad . \quad . \quad (12.127)$$

$$Q'=\frac{3+\sigma}{8-(3+\sigma)\beta}\varrho\omega^2a^2\left\{q_1\left(\frac{r}{a}\right)^{q_1+\beta-1}-\frac{1+3\sigma}{3+\sigma}\frac{r^2}{a^2}\right\} \quad (12.128)$$

223. Disk with uniform stress.

We may put $P'=Q'$ in the equations for the disk with variable thickness provided we regard the thickness $2h$ as an unknown quantity which has to be determined from the equations. If we put $P'=Q'=d$ in equations (12.116), (12.117), these equations give

$$\frac{du'}{dr}=\frac{u'}{r},$$

whence $$\frac{1}{r}\frac{du'}{dr}-\frac{u'}{r^2}=0, \quad \text{or} \quad \frac{d}{dr}\left(\frac{u'}{r}\right)=0,$$

and therefore

$$\frac{u'}{r}=\text{constant}=C \quad . \quad . \quad . \quad . \quad . \quad . \quad (12.129)$$

Now (12.117) gives

$$P'=Q'=\frac{E}{1-\sigma}\frac{u'}{r}=\frac{EC}{1-\sigma}, \quad . \quad . \quad . \quad . \quad (12.130)$$

which shows that the two stresses are not only equal but constant throughout the disk.

Again equation (12.115) gives

$$hp-p\frac{d(hr)}{dr}=\varrho hr^2\omega^2,$$

or $$-pr\frac{dh}{dr}=\varrho hr^2\omega^2,$$

whence $$\frac{1}{h}\frac{dh}{dr}=-\frac{\varrho}{p}r\omega^2.$$

Integrating this we get

$$\log_e h = -\frac{\varrho}{2p} r^2 \omega^2 + \log_e H .$$

Therefore

$$h = He^{-\frac{\varrho r^2 \omega^2}{2p}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12.131)$$

It should be observed that h vanishes only when r is infinite. Consequently the disk with uniform stress is really impossible in practice. Nevertheless if a finite disk were constructed with the thickness given by (12.131) we could be sure that the stresses would be everywhere less than if the disk were infinite.

CHAPTER XIII.

STRETCHING OF THIN PLATES.

224. Equations of equilibrium.

In this book a flat plate will be understood to mean a body bounded by two surfaces symmetrically situated on opposite sides of a plane, the distance between these surfaces, measured perpendicular to the plane, being small in comparison with the dimensions of the body parallel to the plane. The distance from boundary to boundary measured perpendicular to the middle plane is called the thickness, and will be denoted by $2h$. When the plate is strained the surface containing the same particles as were originally in the middle plane will be called the *middle surface.* When h is constant the plate is a *uniform* plate. In all cases except where the contrary is specially stated the plates with which we deal will be assumed to be uniform.

In this chapter we shall deal only with problems in which the middle surface remains plane during strain.

Let the origin be taken at some point of the middle surface, and let the z-axis be the normal to the middle surface at this point; so that the xy plane is the middle surface itself.

Our assumptions are that the mean values of the stresses S_1 and S_2 across the thickness of the plate are zero; that is,

$$\frac{1}{2h}\int_{-h}^{h} S_1\,dz = 0;\quad \frac{1}{2h}\int_{-h}^{h} S_2\,dz = 0 \qquad (13.1)$$

Now let us suppose that equations (2.24) and (2.25) are both multiplied by $\frac{dz}{2h}$ and then integrated from $-h$ to $+h$. In this way S_1 and S_2 disappear from the equations, and the other stresses and forces will then be represented by their mean values across the thickness. Thus if P_1, P_2, S, now denote the mean values of the stresses P_1, P_2, S_3, equations (2.24) and (2.25) give

$$\left.\begin{aligned} \frac{\partial P_1}{\partial x} + \frac{\partial S}{\partial y} &= \varrho\,(f_1 - X) \\ \frac{\partial P_2}{\partial y} + \frac{\partial S}{\partial x} &= \varrho\,(f_2 - Y) \end{aligned}\right\} \qquad (13.2)$$

In these equations the accelerations f_1, f_2, and the body forces X, Y, have also their mean values with respect to z, and are therefore functions of x and y only.

Let a function of φ be now introduced such that

$$S = -E\frac{\partial^2\varphi}{\partial x\partial y} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13.3)$$

Then equations (13.2) become

$$\frac{\partial}{\partial x}\left(P_1 - E\frac{\partial^2\varphi}{\partial y^2}\right) = \varrho(f_1 - X) \quad . \quad . \quad . \quad . \quad . \quad (13.4)$$

$$\frac{\partial}{\partial y}\left(P_2 - E\frac{\partial^2\varphi}{\partial x^2}\right) = \varrho(f_2 - Y) \quad . \quad . \quad . \quad . \quad . \quad (13.5)$$

From these we get

$$\left.\begin{aligned} P_1 &= E\frac{\partial^2\varphi}{\partial y^2} + \int\varrho(f_1 - X)\,dx \\ P_2 &= E\frac{\partial^2\varphi}{\partial x^2} + \int\varrho(f_2 - Y)\,dy \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (13.6$$

Let us write

$$\int\varrho(f_1 - X)\,dx = P'_1\,; \quad \int\varrho(f_2 - Y)\,dy = P'_2 \quad . \quad . \quad . \quad (13.7)$$

Then

$$\left.\begin{aligned} P_1 &= E\frac{\partial^2\varphi}{\partial y^2} + P'_1 \\ P_2 &= E\frac{\partial^2\varphi}{\partial x^2} + P'_2 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13.8)$$

Again another assumption we make is that P_3 is zero at each surface of the plate. This means that no pressures or tensions are applied at the surface in the direction of the normal to the middle surface. Since P_3 is zero at $z = -h$ and at $z = +h$ it must be very small for intermediate values of z. We may then assume that it is negligible in comparison with the stresses P_1, P_2, and S. If, therefore, u and v denote the displacements of a particle in the middle surface, equations (2.14), (2.15), (2.19), give

$$\frac{\partial u}{\partial x} = \frac{1}{E}(P_1 - \sigma P_2)$$

$$= \frac{\partial^2\varphi}{\partial y^2} - \sigma\frac{\partial^2\varphi}{\partial x^2} + \frac{1}{E}(P'_1 - \sigma P'_2) \quad . \quad . \quad . \quad (13.9)$$

$$\frac{\partial v}{\partial y} = \frac{\partial^2\varphi}{\partial x^2} - \sigma\frac{\partial^2\varphi}{\partial y^2} + \frac{1}{E}(P'_2 - \sigma P'_1) \quad . \quad . \quad . \quad (13.10)$$

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{1}{n}S = \frac{2(1+\sigma)}{E}S$$

$$= -2(1+\sigma)\frac{\partial^2\varphi}{\partial x\partial y} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13.11)$$

Operating on these three equations by $\frac{\partial^2}{\partial y^2}$, $\frac{\partial^2}{\partial x^2}$, and $-\frac{\partial^2}{\partial x \partial y}$ respectively, and adding the results, so as to eliminate u and v, we get,

$$0 = \frac{\partial^4 \varphi}{\partial x^4} + \frac{\partial^4 \varphi}{\partial y^4} + 2\frac{\partial^4 \varphi}{\partial x^2 \partial y^2} + \frac{1}{E}\frac{\partial^2}{\partial y^2}(P'_1 - \sigma P'_2) + \frac{1}{E}\frac{\partial^2}{\partial x^2}(P'_2 - \sigma P'_1) \quad . \quad . \quad (13.12)$$

Since we write $\nabla_1{}^2$ for the operator $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, and since

$$\nabla_1{}^2(\nabla_1{}^2 \varphi) = \frac{\partial^4 \varphi}{\partial x^4} + 2\frac{\partial^4 \varphi}{\partial x^2 \partial y^2} + \frac{\partial^4 \varphi}{\partial y^4}, \quad . \quad . \quad . \quad (13.13)$$

we may write equation (13.12) in the form

$$E\nabla_1{}^4 \varphi = -\frac{\partial^2}{\partial y^2}(P'_1 - \sigma P'_2) - \frac{\partial^2}{\partial x^2}(P'_2 - \sigma P'_1) \quad . \quad (13.14)$$

If f_1, f_2, X, Y, are given functions of x and y then the expression on the right hand side of this last equation is a known function of x and y. In that case the function φ is determined by (13.14), and therefore all the stresses are known.

It is usually convenient to divide φ into two terms, a particular integral and a complementary function, just as for an ordinary linear differential equation. Thus let φ_1 be any function of x and y such that

$$E\nabla_1{}^4 \varphi_1 = -\frac{\partial^2}{\partial y^2}(P'_1 - \sigma P'_2) - \frac{\partial^2}{\partial x^2}(P'_2 - \sigma P'_1) \quad . \quad (13.15)$$

and let us put

$$\varphi = \varphi_1 + \varphi_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13.16)$$

in (13.14). With this substitution (13.14) reduces to

$$E\nabla_1{}^4 \varphi_2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13.17)$$

The general value of φ_2 satisfying this equation must be added to the particular integral φ_1 to complete the solution of (13.14).

It might appear that, unless an arbitrary function of y is added on the right of the first of equations (13.6) and an arbitrary function of x on the right of the second of equations (13.6), we shall not get the complete solution of our equations. We shall show that it makes no difference whether these arbitrary functions are added or not.

Let P'_1 and P'_2 still represent the simplest integrals we can find to satisfy equations (13.7). Then let (13.6) be written

$$\left.\begin{aligned} P_1 &= E\frac{\partial^2 \varphi}{\partial y^2} + P'_1 + \frac{d^2 \psi_1}{dy^2}, \\ P_2 &= E\frac{\partial^2 \varphi}{\partial x^2} + P'_2 + \frac{d^2 \psi_2}{dx^2}, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (13.18)$$

ψ_1 and ψ_2 being arbitrary functions of y and x respectively. If now we put

$$\xi = \varphi + \psi_1 + \psi_2 \quad \text{. . . . . . . (13.19)}$$

then the two equations (13.18) become

$$\left.\begin{aligned} P_1 &= E\frac{\partial^2 \xi}{\partial y^2} + P'_1 , \\ P_2 &= E\frac{\partial^2 \xi}{\partial x^2} + P'_2 . \end{aligned}\right\} \quad \text{. . . . . . . (13.20)}$$

Moreover

$$\begin{aligned} S &= -E\frac{\partial^2 \varphi}{\partial x \partial y} \\ &= -E\frac{\partial^2 \xi}{\partial x \partial y} \quad \text{. . . . . (13.21)} \end{aligned}$$

because $$\frac{\partial \psi_1}{\partial x} = 0 , \quad \frac{\partial \psi_2}{\partial y} = 0 \quad \text{. . . . . . . (13.22)}$$

Thus ξ has taken the place of φ in the expressions for the stresses; consequently equation (13.14) becomes

$$E\triangle_1{}^4 \xi = \frac{\partial^2}{\partial y^2}(P'_1 - \sigma P'_2) + \frac{\partial^2}{\partial x^2}(P'_2 - \sigma P'_1) \quad \text{. . (13.23)}$$

This has the same particular integral as before since the expression on the right is exactly the same as in (13.14). Let

$$\xi = \varphi_1 + \varphi_2$$

be the complete solution of (13.23), φ_2 being the complete solution of (13.17). When this value of ξ is substituted in (13.20) and (13.21) the expressions for the stresses are exactly the same as if ψ_1 and ψ_2 were taken as zero. It follows then that any particular integrals will serve in equations (13.7).

Thus we find that the stresses are completely determined by (13.3) and (13.8), φ being the most general function of x and y which satisfies (13.14). The arbitrary functions that occur in this complete value of φ are determined by the boundary conditions. The simplest boundary conditions are that u and v are given at all points of the edge of the plate. A different possible set of boundary conditions is fixed if given forces act on the edge of the plate parallel to the plane of the middle surface. These forces on the edge can be resolved at every point into a tension perpendicular to the edge and a shear stress parallel to the edge. Thus we see that two conditions are enough to settle the state of the edge, and since each condition requires an arbitrary function we should expect two arbitrary functions in the complete expression for φ. There are, as we shall show, two arbitrary functions in the solution of (13.17).

225. Plate with no accelerations and no body forces.

When the plate is at rest and acted on by no body forces then equation (13.14) reduces to

$$\nabla_1{}^4 \varphi = 0 \qquad (13.24)$$

the same form as (13.17).

To solve this equation put

$$\nabla_1{}^2 \varphi = \psi \qquad (13.25)$$

Then equation (13.24) becomes

$$\nabla_1{}^2 \psi = 0, \qquad (13.26)$$

the same equation as (7.19), which we had to solve in the torsion problem. We there found that the solution of the equation is

$$\psi = \text{real part of } f(x+iy),$$

$f(x+iy)$ denoting any function of $(x+iy)$. It is worth while to give another proof of this result. For this purpose let

$$z = x + iy,$$
$$w = x - iy.$$

Then, regarding ψ as a function of z and w,

$$d\psi = \frac{\partial\psi}{\partial z}dz + \frac{\partial\psi}{\partial w}dw$$
$$= \frac{\partial\psi}{\partial z}(dx + i\,dy) + \frac{\partial\psi}{\partial w}(dx - i\,dy)$$

Therefore, putting $dy = 0$ and dividing by dx, we get

$$\frac{\partial\psi}{\partial x} = \frac{\partial\psi}{\partial z} + \frac{\partial\psi}{\partial w}.$$

Likewise, by putting $dx = 0$ we get

$$\frac{\partial\psi}{\partial y} = i\left(\frac{\partial\psi}{\partial z} - \frac{\partial\psi}{\partial w}\right).$$

Again, by a repetition of these operations,

$$\frac{\partial^2\psi}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial\psi}{\partial x}\right)$$
$$= \left(\frac{\partial}{\partial z} + \frac{\partial}{\partial w}\right)\frac{\partial\psi}{\partial x}$$
$$= \frac{\partial}{\partial z}\left(\frac{\partial\psi}{\partial z} + \frac{\partial\psi}{\partial w}\right) + \frac{\partial}{\partial w}\left(\frac{\partial\psi}{\partial z} + \frac{\partial\psi}{\partial w}\right)$$
$$= \frac{\partial^2\psi}{\partial z^2} + 2\frac{\partial^2\psi}{\partial z\,\partial w} + \frac{\partial^2\psi}{\partial w^2} \qquad (13.27)$$

In the same way

$$\frac{\partial^2\psi}{\partial y^2} = i\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial w}\right) i\left(\frac{\partial\psi}{\partial z} - \frac{\partial\psi}{\partial w}\right)$$
$$= -\left(\frac{\partial^2\psi}{\partial z^2} - 2\frac{\partial^2\psi}{\partial z\,\partial w} + \frac{\partial^2\psi}{\partial w^2}\right) \qquad (13.28)$$

Consequently equation (13.26) becomes

$$\left(\frac{\partial^2 \psi}{\partial z^2} + 2\frac{\partial^2 \psi}{\partial z \partial w} + \frac{\partial^2 \psi}{\partial w^2}\right) - \left(\frac{\partial^2 \psi}{\partial z^2} - 2\frac{\partial^2 \psi}{\partial z \partial w} + \frac{\partial^2 \psi}{\partial w^2}\right) = 0,$$

or

$$\frac{\partial^2 \psi}{\partial z \partial w} = 0.$$

Integrating with respect to z, the other variable w being a constant in this operation, we get

$$\frac{\partial \psi}{\partial w} = a \text{ constant when } w \text{ is constant}$$
$$= \mathrm{F}(w),$$

$\mathrm{F}(w)$ being any function of w whatever.

Again, integrating this last equation with respect to w find

$$\psi = \int \mathrm{F}(w)\,dw + f(z)$$
$$= \mathrm{F}_1(w) + f(z), \quad \ldots \ldots \quad (13.29)$$

$\mathrm{F}_1(w)$ being the arbitrary function of w obtained on integrating the arbitrary function $\mathrm{F}(w)$.

Thus the complete solution of (13.26) is

$$\psi = \mathrm{F}_1(x - iy) + f(x + iy).$$

Now since ψ must be a real quantity the imaginary part of the right hand side of this last equation must be zero. By expanding the functions F_1 and f in powers of iy it is easily seen that the imaginary terms in their sum cannot vanish unless the two functions are identical, that is, unless

$$\mathrm{F}_1(x) = f(x);$$

in that case the real parts of $\mathrm{F}_1(x - iy)$ and $f(x + iy)$ are identical, also. Therefore

$$\psi = f(x - iy) + f(x + iy)$$
$$= \text{twice the real part of } f(x + iy).$$

If we had written $\frac{1}{2} f(z)$ instead of $f(z)$ in (13.29) we should have lost none of the generality of our result and should have finally got

$$\psi = \text{the real part of } f(x + iy).$$

Then we may write the solution of (13.26) in the form

$$\psi = f(x + iy) \quad \ldots \ldots \quad (13.30)$$

on the assumption that the imaginary terms have to be rejected in order that ψ should be real. It should be noticed, however, that equation (13.30) as it stands is a correct integral of (13.26). It is only because a complex value of ψ has no meaning that we reject the imaginary part. This imaginary part need not be rejected till the final integral is reached.

Substituting in (13.25) for ψ we get

$$\nabla_1{}^2 \varphi = f(x + iy) \quad \ldots \ldots \quad (13.31)$$

It is easy to verify that a particular integral of this is

$$\varphi = \tfrac{1}{2}xf(x+iy).$$

This means that

$$\nabla_1{}^2\{\tfrac{1}{2}xf(x+iy)\} = f(x+iy) \quad . \quad . \quad . \quad . \quad (13.32)$$

Subtracting corresponding sides of (13.32) and (13.31) we find

$$\nabla_1{}^2\{\varphi - \tfrac{1}{2}xf(x+iy)\} = 0$$

The solution of this is, by the same argument as for ψ,

$$\varphi - \tfrac{1}{2}xf(x+iy) = F(x+iy)$$

$F(x+iy)$ being a new arbitrary function of $(x+iy)$. Therefore finally

$$\varphi = F(x+iy) + \tfrac{1}{2}xf(x+iy), \quad . \quad . \quad . \quad . \quad (13.33)$$

the real part only of the right hand side being admissible since φ has to be real.

The functions f and F in (13.33) are arbitrary and independent of each other. They are the two functions needed to satisfy the two boundary conditions.

It is possible to express the solution in (13.33) in many different forms, all of which are, of course, equivalent. For example

$$\begin{aligned}\varphi &= F(x+iy) + \tfrac{1}{2}(x+iy-iy)f(x+iy) \\ &= F(x+iy) + \tfrac{1}{2}(x+iy)f(x+iy) - \tfrac{1}{2}iyf(x+iy).\end{aligned}$$

This is equivalent to

$$\varphi = F_2(x+iy) + yf_2(x+iy) \quad . \quad . \quad . \quad . \quad (13.34)$$

F_2 and f_2 being arbitrary functions.

Still another form is

$$\begin{aligned}\varphi &= F_2(x+iy) + (x-iy)\,f_2(x+iy) \\ &= F_2(x+iy) + (x-iy)(x+iy)\frac{f_2(x+iy)}{(x+iy)} \\ &= F_2(x+iy) + (x^2+y^2)\,f_3(x+iy) \quad . \quad . \quad . \quad . \quad (13.35)\end{aligned}$$

226. The expression for φ in polar coordinates.

If r and θ be polar coordinates with the origin of the x-y coordinates as pole we get

$$\begin{aligned}x+iy &= r\cos\theta + ir\sin\theta \\ &= re^{i\theta} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13.36)\end{aligned}$$

Therefore equation (13.35) can be written

$$\varphi = F_2(re^{i\theta}) + r^2 f_3(re^{i\theta}) \quad . \quad . \quad . \quad . \quad . \quad (13.37)$$

Whatever forms we give to the functions F_2 and f_3 we shall get the solution to some problem concerning a stretched plate. Suppose we take

$$\begin{aligned}F_2(re^{i\theta}) &= A_n r^n e^{in\theta} + B_n r^{-n} e^{-in\theta} \\ &= A_n r^n(\cos n\theta + i\sin n\theta) + B_n r^{-n}(\cos n\theta - i\sin n\theta) \\ &= (A_n r^n + B_n r^{-n})\cos n\theta + i(A_n r^n - B_n r^{-n})\sin n\theta \quad . \quad (13.38)\end{aligned}$$

In this equation A_n and B_n may be any constants real or complex. Suppose then that

$$A_n = a_n - ib_n,$$
$$B_n = c_n + id_n.$$

Then the real part of $F_2(re^{i\theta})$ is

$$(a_n r^n + c_n r^{-n})\cos n\theta + (b_n r^n + d_n r^{-n})\sin n\theta \quad . \quad . \quad (13.39)$$

The function $f_3(re^{i\theta})$ can contain similar terms. Then the most complete expression for φ contains the factor $\cos n\theta$ is

$$\varphi_n = (a_n r^n + c_n r^{-n} + p_n r^{n+2} + q_n r^{2-n})\cos n\theta \quad . \quad . \quad (13.40)$$

There is a similar expression with a factor $\sin n\theta$.

There is one exceptional case, namely the case where $n = 1$. It should be noticed that, for this case, $r^{2-n} = r^n$, which reduces two terms in the bracket in (13.40) to one term. To compensate for the term thus lost we can find another term. It is best to return to the form given in (13.33). If here we take

$$f(x+iy) = 2q \log re^{i\theta}$$
$$= 2q\{\log r + i\theta\} \quad . \quad . \quad . \quad . \quad (13.41)$$

we get

$$\tfrac{1}{2}xf(x+iy) = qr\cos\theta\{\log r + i\theta\} \quad . \quad . \quad . \quad (13.42)$$

The terms in φ corresponding to this are

$$qr\log r\cos\theta + kr\theta\cos\theta \quad . \quad . \quad . \quad . \quad (13.43)$$

Therefore the complete expression for φ which has the form $f(r)\cos\theta$ is

$$\varphi_1 = (ar + cr^{-1} + pr^3 + qr\log_e r)\cos\theta \quad . \quad . \quad (13.44)$$

Moreover another possible value of φ is

$$\varphi_0 = (A + Br^2)\log r + (C + Dr^2)\theta + Kr^2 + H \quad . \quad . \quad (13.45)$$

A more general value of φ is obtained by adding together all the partial expressions for φ. Thus if n is an integer and if

$$C_n = a_n r^n + b_n r^{-n} + c_n r^{n+2} + d_n r^{2-n}, \quad . \quad . \quad . \quad (13.46)$$

$$S_n = h_n r^n + k_n r^{-n} + p_n r^{n+2} + q_n r^{2-n}, \quad . \quad . \quad . \quad (13.47)$$

with the condition that, when $n = 1$, the factor r^{2-n} is to be replaced by $r\log_e r$, then

$$\varphi = \varphi_0 + r\theta(H\cos\theta + K\sin\theta) + \sum_{n=1}^{n=\infty}(C_n\cos n\theta + S_n\sin n\theta) \quad (13.48)$$

gives an expression for φ which is complete enough to satisfy the conditions of most practical problems. In many problems a few terms only are enough. In exceptional cases fractional values of n would be needed.

227. The stresses in terms of polar coordinates.

In dealing with a plate with a circular boundary it is convenient to use polar coordinates throughout with the pole at the centre of the circle. It is necessary therefore to express the stresses on radial and circular sections in terms of polar coordinates. Let R and T denote the mean tensional stresses across sections perpendicular to r and along r respectively. Also let F denote the mean shear stress on the sections on which R and T act, as shown in fig. 116.

If the y-axis be taken along the radius vector r then T is the value of E $\dfrac{\partial^2\varphi}{\partial y^2}$ in this position. Again if the y-axis be put perpendicular to r the stress R is the value of E $\dfrac{\partial^2\varphi}{\partial y^2}$ in the new position.

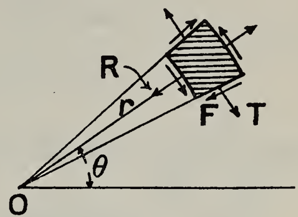


Fig. 116

Now $$d\varphi = \frac{\partial\varphi}{\partial r}dr + \frac{\partial\varphi}{\partial\theta}d\theta$$

But

$$r^2 = x^2 + y^2,$$
$$y = x\tan\theta;$$

consequently

$$r\,dr = x\,dx + y\,dy,$$
$$dy = dx\tan\theta + x\sec^2\theta\, d\theta,$$

the last of which gives, since $x = r\cos\theta$,

$$r d\theta = dy\cos\theta - dx\sin\theta.$$

Therefore

$$d\varphi = \frac{\partial\varphi}{\partial r}\left(\frac{x}{r}dx + \frac{y}{r}dy\right) + \frac{\partial\varphi}{\partial\theta}\left(\frac{\cos\theta}{r}dy - \frac{\sin\theta}{r}dx\right)$$
$$= \frac{\partial\varphi}{\partial r}\left(\cos\theta\, dx + \sin\theta\, dy\right) + \frac{1}{r}\frac{\partial\varphi}{\partial\theta}\left(\cos\theta\, dy - \sin\theta\, dx\right) \qquad (13.49)$$

Putting $dy = 0$ in this, thus keeping y constant, we get, after division by dx,

$$\frac{\partial\varphi}{\partial x} = \cos\theta\frac{\partial\varphi}{\partial r} - \frac{\sin\theta}{r}\frac{\partial\varphi}{\partial\theta} \quad . \; . \; . \; . \; . \; (13.50)$$

Also by putting $dx = 0$ we find

$$\frac{\partial\varphi}{\partial y} = \sin\theta\frac{\partial\varphi}{\partial r} + \frac{\cos\theta}{r}\frac{\partial\varphi}{\partial\theta} \quad . \; . \; . \; . \; . \; (13.51)$$

Thus we find that

$$\left.\begin{aligned}\frac{\partial}{\partial x} &= \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\\ \frac{\partial}{\partial y} &= \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\end{aligned}\right\} \quad . \; . \; . \; . \; (13.52)$$

Consequently

$$\frac{\partial^2\varphi}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial\varphi}{\partial y}\right)=\sin\theta\frac{\partial}{\partial r}\left(\frac{\partial\varphi}{\partial y}\right)+\frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\left(\frac{\partial\varphi}{\partial y}\right)$$

$$=\sin\theta\left\{\sin\theta\frac{\partial^2\varphi}{\partial r^2}+\frac{\cos\theta}{r}\frac{\partial^2\varphi}{\partial r\partial\theta}-\frac{\cos\theta}{r^2}\frac{\partial\varphi}{\partial\theta}\right\}$$

$$+\frac{\cos\theta}{r}\left\{\sin\theta\frac{\partial^2\varphi}{\partial\theta\partial r}+\cos\theta\frac{\partial\varphi}{\partial r}+\frac{\cos\theta}{r}\frac{\partial^2\varphi}{\partial\theta^2}-\frac{\sin\theta}{r}\frac{\partial\varphi}{\partial\theta}\right\}$$

$$=\sin^2\theta\frac{\partial^2\varphi}{\partial r^2}+\cos^2\theta\left(\frac{1}{r}\frac{\partial\varphi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2}\right)$$

$$+2\sin\theta\cos\theta\left(\frac{1}{r}\frac{\partial^2\varphi}{\partial r\partial\theta}-\frac{1}{r^2}\frac{\partial\varphi}{\partial\theta}\right)\quad\text{. . . . (13.53)}$$

Putting $\theta=\frac{\pi}{2}$ in this, thus taking the y-axis along r, we get

$$\mathrm{T}=\mathrm{E}\frac{\partial^2\varphi}{\partial y^2}=\mathrm{E}\frac{\partial^2\varphi}{\partial r^2}\quad\text{. . . . . . . (13.54)}$$

Next putting $\theta=0$ we get

$$\mathrm{R}=\mathrm{E}\frac{\partial^2\varphi}{\partial y^2}=\mathrm{E}\left(\frac{1}{r}\frac{\partial\varphi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2}\right)\quad\text{. . . . (13.55)}$$

The shear stress F is the value of $-\mathrm{E}\frac{\partial^2\varphi}{\partial x\partial y}$ when the x-axis is along r. But

$$\frac{\partial^2\varphi}{\partial x\partial y}=\frac{\partial}{\partial x}\left(\frac{\partial\varphi}{\partial y}\right)$$

$$=\cos\theta\frac{\partial}{\partial r}\left(\frac{\partial\varphi}{\partial y}\right)-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\frac{\partial\varphi}{\partial y}\right)$$

$$=\cos\theta\left\{\sin\theta\frac{\partial^2\varphi}{\partial r^2}+\frac{\cos\theta}{r}\frac{\partial^2\varphi}{\partial r\partial\theta}-\frac{\cos\theta}{r^2}\frac{\partial\varphi}{\partial\theta}\right\}$$

$$-\frac{\sin\theta}{r}\left\{\sin\theta\frac{\partial^2\varphi}{\partial\theta\partial r}+\cos\theta\frac{\partial\varphi}{\partial r}+\frac{\cos\theta}{r}\frac{\partial^2\varphi}{\partial\theta^2}-\frac{\sin\theta}{r}\frac{\partial\varphi}{\partial\theta}\right\},$$

which becomes, when $\theta=0$,

$$\frac{\partial^2\varphi}{\partial x\partial y}=\frac{1}{r}\frac{\partial^2\varphi}{\partial r\partial\theta}-\frac{1}{r^2}\frac{\partial\varphi}{\partial\theta},\quad\text{. . . . . (13.56)}$$

whence

$$\mathrm{F}=\mathrm{E}\left(\frac{1}{r^2}\frac{\partial\varphi}{\partial\theta}-\frac{1}{r}\frac{\partial^2\varphi}{\partial r\partial\theta}\right)\quad\text{. . . . . (13.57)}$$

228. The strains in terms of polar coordinates.

Let the particle which was at (r, θ) in the unstrained state be displaced to $(r+\mathrm{U}, \theta+\eta)$. Then the radial strain is clearly

$$a_1=\frac{\partial \mathrm{U}}{\partial r}\quad\text{. . . . . . . , (13.58)}$$

The element $rd\theta$ on the circle of radius r is extended to $(r+\mathrm{U})\times d(\theta+\eta)$ on a circle of radius $(r+\mathrm{U})$. Consequently the circumferential strain is

$$\beta_1=\frac{(r+\mathrm{U})(d\theta+d\eta)-rd\theta}{rd\theta}$$

$$=\frac{rd\eta+\mathrm{U}d\theta}{rd\theta}$$

$$=\frac{\partial\eta}{\partial\theta}+\frac{\mathrm{U}}{r}. \qquad (13.59)$$

Again to get the shear strain consider the relative displacements of the corners of a small rectangle bounded in the unstrained state by r, $r+dr$, θ, $\theta+d\theta$. Thus, in fig. 117, A′B′C′D′ is the displaced and distorted rectangle, circles being shown dotted for reference. The radial displacements of A′ and D′ are U and $\mathrm{U}+\frac{\partial \mathrm{U}}{\partial\theta}d\theta$. Therefore the relative radial displacement, which is represented by KD′ in the figure, is

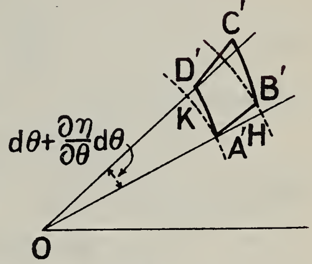


Fig. 117

$$\frac{\partial \mathrm{U}}{\partial\theta}d\theta,$$

whence the small angle KA′D′ is

$$\frac{1}{\mathrm{A'K}}\frac{\partial \mathrm{U}}{\partial\theta}d\theta=\frac{1}{rd\theta}\frac{\partial \mathrm{U}}{\partial\theta}d\theta=\frac{1}{r}\frac{\partial \mathrm{U}}{\partial\theta}.$$

Again η and $\eta+\frac{\partial\eta}{\partial r}dr$ are the angular displacements of OA′ and OB′. Therefore the angle A′OB′ is $\frac{\partial\eta}{\partial r}dr$, and consequently the angle HA′B′ is approximately

$$\frac{\mathrm{HB'}}{\mathrm{A'H}}=\frac{1}{dr}\left(r+dr\right)\frac{\partial\eta}{\partial r}dr=r\frac{\partial\eta}{\partial r}.$$

Now the sum of the angles KA′D′ and HA′B′ is the shear strain of the element; thus the shear strain is

$$\frac{1}{r}\frac{\partial \mathrm{U}}{\partial\theta}+r\frac{\partial\eta}{\partial r} \qquad (13.60)$$

229. Expression for the displacements when the accelerations and body forces are zero.

When the accelerations and body forces are zero the quantities P'_1 and P'_2 in (13.7) may be taken to be zero. Then (13.9), (13.10), (13.11), become

$$\left.\begin{aligned}\frac{\partial u}{\partial x}&=\frac{\partial^2\varphi}{\partial y^2}-\sigma\frac{\partial^2\varphi}{\partial x^2}\\ \frac{\partial v}{\partial y}&=\frac{\partial^2\varphi}{\partial x^2}-\sigma\frac{\partial^2\varphi}{\partial y^2}\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (13.61)$$

$$\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}=-2(1+\sigma)\frac{\partial^2\varphi}{\partial x\partial y} \quad . \quad . \quad . \quad . \quad . \quad . \quad (13.62)$$

Morover the differential equation for φ is

$$\nabla_1{}^4\varphi=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13.63)$$

Now take, as in (13.31),

$$\nabla_1{}^2\varphi=f(x+iy); \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13.64)$$

then
$$\frac{\partial^2\varphi}{\partial y^2}=-\frac{\partial^2\varphi}{\partial x^2}+f(x+iy) \quad . \quad . \quad . \quad . \quad . \quad . \quad (13.65)$$

Consequently the first of equations (13.61) becomes

$$\frac{\partial u}{\partial x}=-(1+\sigma)\frac{\partial^2\varphi}{\partial x^2}+f(x+iy).$$

Integrating this we find

$$u=-(1+\sigma)\frac{\partial\varphi}{\partial x}+f_1(x+iy)+F(y) \quad . \quad . \quad (13.66)$$

where
$$f_1(z)=\int f(z)\,dz$$

and $F(y)$ is an arbitrary function of y.

In the same way we get

$$\frac{\partial v}{\partial y}=-(1+\sigma)\frac{\partial^2\varphi}{\partial y^2}+f(x+iy),$$

whence
$$v=-(1+\sigma)\frac{\partial\varphi}{\partial y}+\int f(x+iy)dy+G(x), \quad . \quad . \quad (13.67)$$

$G(x)$ being an arbitrary function of x.

Now let
$$z=x+iy;$$

then, since x is constant in the integral in equation (13.67),

$$dz=idy;$$

consequently

$$\int f(z)dy=\frac{1}{i}\int f(z)\,dz=-if_1(z).$$

Therefore

$$v=-(1+\sigma)\frac{\partial\varphi}{\partial y}-if_1(x+iy)+G(x) \quad . \quad . \quad (13.68)$$

We have now found values of u and v; but these contain two new arbitrary functions. We can now show that these functions, since they satisfy equation (13.62), add nothing to the strains. Thus, using the values of u and v from (13.66) and (13.68), we get

$$\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}=-2(1+\sigma)\frac{\partial^2\varphi}{\partial x\partial y}+if(x+iy)-if(x+iy)$$
$$+F'(y)+G'(x)\,. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13.69)$$

By comparing this with (13.62) we find that

$$F'(y)+G'(x)=0\,.$$

Since a function of x cannot by identically equal to a function of y, the only possible values for the functions in the last equation are

$$F'(y)=-A\,,$$
$$G'(x)=A\,,$$

A being a constant. Therefore

$$F(y)=-Ay+B\,,$$
$$G(x)=Ax+C\,.$$

Thus the complete expressions for the displacements are

$$u=-(1+\sigma)\frac{\partial\varphi}{\partial x}+f_1(x+iy)-Ay+B\,, \quad . \quad . \quad (13.70)$$

$$v=-(1+\sigma)\frac{\partial\varphi}{\partial y}-if_1(x+iy)+Ax+C\,. \quad . \quad . \quad (13.71)$$

The terms involving A, B, C, add nothing to the strains. The terms B and C represent merely components of a displacement without rotation or strain, and the terms containing A represent a rigid-body rotation of the whole plate through an angle A, which must naturally be a small fraction.

The polar displacements can easily be got from u and v. Thus it is easy to see from a figure that, assuming u and v are small,

$$\left.\begin{aligned}U&=u\cos\theta+v\sin\theta\\ r\eta&=v\cos\theta-u\sin\theta\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (13.72)$$

With the values of $\frac{\partial\varphi}{\partial x}$ and $\frac{\partial\varphi}{\partial y}$ from (13.50) and (13.51) these become

$$U=-(1+\sigma)\frac{\partial\varphi}{\partial r}+(\cos\theta-i\sin\theta)f_1(x+iy)$$
$$+Ax\sin\theta-Ay\cos\theta+B\cos\theta+C\sin\theta$$
$$=-(1+\sigma)\frac{\partial\varphi}{\partial r}+e^{-i\theta}f_1(re^{i\theta})+B\cos\theta+C\sin\theta\,. \quad . \quad . \quad (13.73)$$

$$r\eta=-(1+\sigma)\frac{1}{r}\frac{\partial\varphi}{\partial\theta}-(i\cos\theta+\sin\theta)f_1(re^{i\theta}$$
$$+A(x\cos\theta+y\sin\theta)+C\cos\theta-B\sin\theta$$
$$=-(1+\sigma)\frac{1}{r}\frac{\partial\varphi}{\partial\theta}-ie^{-i\theta}f_1(re^{i\theta})+Ar+C\cos\theta-B\sin\theta \quad (13.74)$$

The last two equations show clearly that the only effect of A is to add a constant to η, and this indicates a rotation such as a rigid body could have.

It is to be understood, of course, that the actual displacements are the real parts of the expressions on the right hand sides of equations (13.70), (13.71), (13.73), (13.74).

230. The strain energy in a stretched plate.

We shall find the work done in straining a small element of dimensions $dx \times dy$ by the forces on the edges of the element. The extensional strains being α and β it follows that the actual extensions of the element are αdx and βdy. The forces in the directions of these extensions are $2hdyP_1$ and $2hdxP_2$. Consequently the work done by these forces is

$$\tfrac{1}{2}(2hdyP_1\alpha dx + 2hdxP_2\beta dy)$$
$$= hdxdy\{P_1\alpha + P_2\beta\}.$$

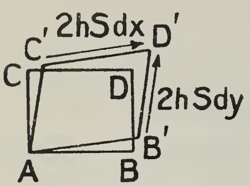


Fig 118

Again the shear forces on the faces are $2hdyS$ and $2hdxS$. Since, in fig. 118, the component of CC′ in the direction of the x-axis is $\frac{\partial u}{\partial y}dy$ and the component of BB′ in the direction of the y-axis is $\frac{\partial v}{\partial x}dx$ the work done by these forces in the relative displacements is

$$\frac{1}{2}\left\{2hdxS\left(\frac{\partial u}{\partial y}dy\right) + 2hdyS\left(\frac{\partial v}{\partial x}dx\right)\right\}$$
$$= hS\left\{\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right\}dxdy = hSc\,dxdy,$$

c being the shear strain of the element. Thus the total work done on the element by the faces on its edges is

$$h\{P_1\alpha + P_2\beta + Sc\}dxdy;$$

therefore the work per unit area is

$$W = h(P_1\alpha + P_2\beta + Sc), \quad . \quad . \quad . \quad . \quad . \quad (13.75)$$

and when the strains are expressed in terms of the stresses this work becomes

$$W = hE^{-1}\{P_1{}^2 + P_2{}^2 - 2\sigma P_1P_2 + 2(1+\sigma)S^2\}$$
$$= hE^{-1}\{(P_1 + P_2)^2 \pm 2(1+\sigma)(S^2 - P_1P_2)\}$$
$$= Eh\left[(\nabla_1{}^2\varphi)^2 + 2(1+\sigma)\left\{\left(\frac{\partial^2\varphi}{\partial x\partial y}\right)^2 - \frac{\partial^2\varphi}{\partial x^2}\frac{\partial^2\varphi}{\partial y^2}\right\}\right] \quad (13.76)$$

The total strain energy in any plate is

$$V = \int W dA, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13.77)$$

dA being an element of area of the plate and the integral being taken over the whole plate.

231. The effect of a small circular hole in a strained plate. The presence of a small hole in a plate will make the state of stress very different in its immediate neighbourhood from the state that would exist if the hole were not there and if other conditions were the same. Moreover, the effect of the hole is not likely to be appreciable at a distance of a few diameters from the edge of the hole. Consequently a distance of only a few diameters from the centre of the hole may reasonably be regarded as an infinite distance. We shall solve therefore the following exact problem and from the result it will be clear what distance may be regarded as infinite.

Let O be the centre of a small hole of radius a in an infinite plate, and let the principal stresses in the plate at an infinite distance from O in any direction be P and Q, these being constant and in the same direction at all infinitely distant points. Let the axes OX, OY, be parallel to the principal stresses. Thus our conditions are

$$\left.\begin{array}{l} P_1 = P, \\ P_2 = Q, \\ S = 0, \end{array}\right\} \text{ where } r = \infty \quad . \quad . \quad . \quad (13.78)$$

r being the distance from the centre of the hole. Moreover, the conditions at the circular boundary are

$$R = 0, \ F = 0, \text{ where } r = a \quad . \quad . \quad . \quad . \quad . \quad (13.79)$$

Let us put

$$\begin{aligned} E\varphi &= E\varphi_1 + \tfrac{1}{2}Py^2 + \tfrac{1}{2}Qx^2 \\ &= E\varphi_1 + \tfrac{1}{2}Pr^2\sin^2\theta + \tfrac{1}{2}Qr^2\cos^2\theta \quad . \quad . \quad . \quad (13.80) \end{aligned}$$

Then the conditions at infinity are

$$E\frac{\partial^2\varphi_1}{\partial y^2} = 0, \ E\frac{\partial^2\varphi_1}{\partial x^2} = 0, \ E\frac{\partial^2\varphi_1}{\partial x\partial y} = 0, \quad . \quad . \quad . \quad (13.81)$$

which means that the stress system due to φ_1 is zero where $r = \infty$.

The conditions at the circular boundary are

$$\frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2} + \frac{1}{r}\frac{\partial\varphi}{\partial r} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (13.82)$$

$$\frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{\partial\varphi}{\partial r} - \frac{\varphi}{r}\right) = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (13.83)$$

which become, when expressed in terms of φ_1,

$$\left.\begin{array}{l} E\left\{\dfrac{1}{r^2}\dfrac{\partial^2\varphi_1}{\partial\theta^2} + \dfrac{1}{r}\dfrac{\partial\varphi_1}{\partial r}\right\} = -P\cos^2\theta - Q\sin^2\theta \\ E\dfrac{1}{r}\dfrac{\partial}{\partial\theta}\left\{\dfrac{\partial\varphi_1}{\partial r} - \dfrac{\varphi_1}{r}\right\} = -\dfrac{1}{2}(P-Q)\sin 2\theta \end{array}\right\} \begin{array}{l}\text{where} \\ r = a\end{array} \quad . \quad (13.84)$$

Now the differential equation for φ, namely,

$$E\nabla_1^4\varphi = 0$$

becomes

$$E\nabla_1^4\varphi_1+\nabla_1^4(\tfrac{1}{2}Py^2+\tfrac{1}{2}Qx^2)=0,$$

which reduces to

$$E\nabla_1^4\varphi_1=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13.85)$$

The solution of this is

$$E\varphi_1=F(re^{i\theta})+\tfrac{1}{2}xf_1(re^{i\theta}) \quad . \quad . \quad . \quad . \quad . \quad (13.86)$$

Now the conditions at infinity suggest that only negative powers of r can appear in the stresses. Moreover, the conditions at the circular boundary suggest that only periodic functions of 2θ occur in φ_1. If we take

$$F(re^{i\theta})=A\log(re^{i\theta})+Br^{-2}e^{-2i\theta},$$

$$f_1(re^{i\theta})=2Cr^{-1}e^{-i\theta},$$

and put φ_1 equal to the real part of the right hand side of (13.86) it will be found that all the conditions can be satisfied. Thus

$$E\varphi_1=A\log_e r+Br^{-2}\cos 2\theta+C\cos^2\theta \quad . \quad . \quad . \quad (13.87)$$

Then

$$E\left\{\frac{1}{r^2}\frac{\partial^2\varphi_1}{\partial\theta^2}+\frac{1}{r}\frac{\partial\varphi_1}{\partial r}\right\}=\frac{A}{r^2}-\frac{6B}{r^4}\cos 2\theta-\frac{2C}{r^2}\cos 2\theta \quad . \quad (13.88)$$

$$\frac{E}{r}\frac{\partial}{\partial\theta}\left(\frac{\partial\varphi_1}{\partial r}-\frac{\varphi_1}{r}\right)=\frac{6B}{r^4}\sin 2\theta+\frac{C}{r^2}\sin 2\theta \quad . \quad . \quad . \quad (13.89)$$

The stress T involves similar powers of r, and it is clear therefore that the conditions (13.81) at the outer boundary are satisfied.

In order that the conditions at the circular boundary may be satisfied the following equations must hold:—

$$\frac{A}{a^2}-\frac{6B}{a^4}\cos 2\theta-\frac{2C}{a^2}\cos 2\theta=-P\cos^2\theta-Q\sin^2\theta$$

$$=-\tfrac{1}{2}(P+Q)-\tfrac{1}{2}(P-Q)\cos 2\theta$$

$$\left(\frac{6B}{a^4}+\frac{C}{a^2}\right)\sin 2\theta=-\tfrac{1}{2}(P-Q)\sin 2\theta$$

Since these are identities the coefficients of $\cos 2\theta$ and $\sin 2\theta$ must be equal on each side of the equations. Therefore

$$A=-\tfrac{1}{2}(P+Q)a^2,$$

$$\frac{6B}{a^4}+\frac{2C}{a^2}=\tfrac{1}{2}(P-Q),$$

$$\frac{6B}{a^4}+\frac{C}{a^2}=-\tfrac{1}{2}(P-Q);$$

whence

$$C=(P-Q)a^2$$

$$6B=-\tfrac{3}{2}(P-Q)a^4.$$

The circumferential stress is

$$T = E \frac{\partial^2 \varphi}{\partial r^2}$$

$$= P \sin^2 \theta + Q \cos^2 \theta + E \frac{\partial^2 \varphi_1}{\partial r^2}$$

$$= P \sin^2 \theta + Q \cos^2 \theta - \frac{A}{r^2} + 6 \frac{B}{r^4} \cos 2\theta \quad . \quad . \quad . \quad (13.90)$$

At the edge of the hole where $r = a$, this becomes

$$T = P \sin^2 \theta + Q \cos^2 \theta + \tfrac{1}{2}(P + Q) - \tfrac{3}{2}(P - Q) \cos 2\theta,$$
$$= P + Q - 2(P - Q) \cos 2\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13.91)$$

Suppose P is greater than Q in magnitude. The maximum value of T at the edge of the hole is the greatest stress in the plate. This maximum occurs where $\theta = \pm \frac{\pi}{2}$ and its magnitude is

$$T = P + Q + 2(P - Q)$$
$$= 3P - Q \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13.92)$$

Thus the ratio of the maximum stress to the greatest stress at an infinite distance from the hole is

$$\frac{T}{P} = 3 - \frac{Q}{P}.$$

When Q is negative and equal to P this has its greatest value, and when Q is positive and equal to P it has its least value, since it has been postulated that the magnitude of Q must be less than the magnitude of P. Thus

$$\left.\begin{aligned} \frac{T}{P} &= 4 \text{ when } Q = -P \\ &= 2 \text{ when } Q = +P \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (13.93)$$

These are the maximum and minimum ratios of the greatest stress at the hole to the greatest stress at infinity. This is true whether the greatest stress is a tension or a thrust, since the result for a thrust can be got from the result for a tension by a change of signs throughout the equations. In the particular case where Q is zero the maximum stress is 3 P.

The final value of T at any point of an infinite plate is

$$T = P \sin^2 \theta + Q \cos^2 \theta + \tfrac{1}{2}(P + Q) \frac{a^2}{r^2} - \frac{3}{2}(P - Q) \frac{a^4}{r^4} \cos 2\theta \quad (13.94)$$

At a distance of 10 a from the centre of the hole the terms containing r in the value of T, which are the stresses due to the hole, have only about $\frac{1}{200}$ of their value at the edge of the hole. Thus we may quite reasonably regard a distance of five diameters from the centre of a hole as practically an infinite distance away.

232. A wrench, or couple, applied to a small element of plate, about the normal to the plate.

An interesting problem arises out of the solution

$$E\varphi = -iH\log re^{i\theta}$$
$$= H\theta - iH\log r.$$

Taking the real part we get

$$E\varphi = H\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13.95)$$

The three polar stresses in this case are

$$\left.\begin{aligned} R &= E\left(\frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2} + \frac{1}{r}\frac{\partial\varphi}{\partial r}\right) = 0, \\ T &= E\frac{\partial^2\varphi}{\partial r^2} = 0, \\ F &= -\frac{E}{r}\frac{\partial}{\partial\theta}\left(\frac{\varphi}{r} - \frac{\partial\varphi}{\partial r}\right) = \frac{H}{r^2}, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (13.96)$$

This stress system is indicated in figure 119.

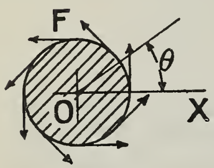


Fig. 119

This is the stress set up by forces gripping the plate over a circular area and giving it a twist about the normal. The stress would be produced by tightening up a nut on a bolt passing through the plate if the plate resisted the applied couple. The moment of the twisting couple is

$$C = r \times (2\pi r F)$$
$$= 2\pi H \quad . \quad . \quad . \quad . \quad . \quad (13.97)$$

233. Strains symmetrical about a normal to the plate.

The most general value of φ which is a function of r only is, by (13.45),

$$\varphi = (A + Br^2)\log r + Kr^2 + H \quad . \quad . \quad . \quad . \quad . \quad (13.98)$$

We can show however, that the term $Br^2\log r$ is not an admissible value of φ for strains symmetrical about the pole because it does not satisfy the stress-strain relations

$$\frac{dU}{dr} = \frac{R - \sigma T}{E} = \frac{1}{r}\frac{d\varphi}{dr} - \sigma\frac{d^2\varphi}{dr^2},$$
$$\frac{U}{r} = \frac{T - \sigma R}{E} = \frac{d^2\varphi}{dr^2} - \frac{\sigma}{r}\frac{d\varphi}{dr}.$$

Although this particular term gives stresses that are symmetrical about the pole it nevertheless requires that the angular displacement η should not be zero. Then neglecting this term, as well as the constant H, which gives no stresses, we get

$$\varphi = A\log r + Kr^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13.99)$$

The stresses due to this value of φ are

$$R = \frac{E}{r}\frac{d\varphi}{dr} = E\left(\frac{A}{r^2} + 2K\right) \quad . \quad . \quad . \quad . \quad . \quad (13.100)$$

$$T = E\frac{d^2\varphi}{dr^2} = E\left(-\frac{A}{r^2} + 2K\right) \quad . \quad . \quad . \quad . \quad (13.101)$$

$$F = 0.$$

Suppose the radial stress is given over the edges of two concentric circles. Thus suppose

$$R = R_1 \text{ where } r = a,$$
$$R = R_2 \text{ where } r = b.$$

Then

$$R_1 = E\left(\frac{A}{a^2} + 2K\right),$$

$$R_2 = E\left(\frac{A}{b^2} + 2K\right),$$

whence

$$EA = -\frac{a^2b^2}{b^2-a^2}(R_2 - R_1),$$

$$2EK = \frac{b^2R_2 - a^2R_1}{b^2-a^2}.$$

Therefore, in general,

$$R = \frac{1}{b^2-a^2}\left\{\frac{a^2b^2}{r^2}(R_1 - R_2) + b^2R_2 - a^2R_1\right\} \quad . \quad . \quad (13.102)$$

$$T = \frac{1}{b^2-a^2}\left\{\frac{a^2b^2}{r^2}(R_2 - R_1) + b^2R_2 - a^2R_1\right\} \quad . \quad . \quad (13.103)$$

The stresses are exactly the same as we got in the last chapter in the problem of the thick cylinder. The present problem is, in fact, identical with the problem of the thick cylinder with no resultant force across any section perpendicular to the axis of the cylinder.

234. Circular strip under the action of couples at its ends. The problem we now propose to solve is to find the state of stress in a sector of a hollow cylinder when couples are applied at the ends of the sector. It is practically the same problem as that of a rod of uniform rectangular section, whose middle line is a circle, bent by couples at its ends, the outer and inner radii having any ratio whatever.

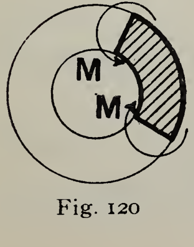


Fig. 120

Let r_1, r_2, be the inner and outer radii in the unstrained state. The conditions of the problem are that

$$R = 0 \text{ where } r = r_1 \text{ and } r = r_2, \quad (13.104)$$

and

$$\int_{r_1}^{r_2} T\,dr = 0 \quad . \quad . \quad . \quad (13.105)$$

These reduce the forces on the body to couples at the ends.

The stresses, but not necessarily the strains, must be functions of r only. Therefore, taking the most general value of φ that is a function of r only, we have

$$E\varphi = (A + Br^2)\log r + Kr^2 + H; \quad . \quad . \quad . \quad (13.106)$$

whence

$$R = \frac{E}{r}\frac{d\varphi}{dr}$$

$$= \frac{A}{r^2} + 2B\log r + C, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13.107)$$

$$T = E\frac{d^2\varphi}{dr^2}$$

$$= -\frac{A}{r^2} + 2B\log r + C + 2B, \quad . \quad . \quad . \quad (13.108)$$

where $C = B + 2K.$

Now the conditions that R should be zero at $r = r_1$ and $r = r_2$ give

$$\left.\begin{aligned} 0 &= \frac{A}{r_1{}^2} + 2B\log r_1 + C \\ 0 &= \frac{A}{r_2{}^2} + 2B\log r_2 + C \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (13.109)$$

whence

$$A = 2B\frac{r_1{}^2 r_2{}^2}{r_2{}^2 - r_1{}^2}\log\frac{r_2}{r_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13.110)$$

$$C = -\frac{2B}{r_2{}^2 - r_1{}^2}\{r_2{}^2\log r_2 - r_1{}^2\log r_1\} \quad . \quad . \quad . \quad (13.111)$$

Substituting these values of A and C in the expression for T we get

$$T = \frac{2B}{r_2{}^2 - r_1{}^2}\left\{r_2{}^2 - r_1{}^2 + r_2{}^2\log\frac{r}{r_2} - r_1{}^2\log\frac{r}{r_1} - \frac{r_1{}^2 r_2{}^2}{r^2}\log\frac{r_2}{r_1}\right\} \quad (13.112)$$

If we write mr_1 for r_2 this becomes

$$T = \frac{2B}{m^2 - 1}\left\{m^2 - 1 + m^2\log\frac{r}{mr_1} - \log\frac{r}{r_1} - m^2\frac{r_1{}^2}{r^2}\log m\right\}$$

$$= 2B\left\{1 + \log\frac{r}{r_1} - \frac{m^2\log m}{m^2 - 1}\left(1 + \frac{r_1{}^2}{r^2}\right)\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (13.113)$$

The condition expressed by (13.105) is satisfied as a consequence of the two conditions (13.104); for

$$\int T dr = \int E\frac{d^2\varphi}{dr^2}dr = E\frac{d\varphi}{dr} = rR \quad . \quad . \quad . \quad (13.114)$$

which is zero at both limits because R is zero at both limits

To find the neutral axis, that is, the radius at which the stress T is zero, write s for $\frac{r_1}{r}$. Then we have to find s from the equation

$$0 = 1 - \log s - \frac{m^2 \log m}{m^2 - 1}(1 + s^2)$$

or

$$\log_e s = 1 - \frac{m^2 \log_e m}{m^2 - 1}(1 + s^2) \quad . \quad . \quad . \quad . \quad . \quad (13.115)$$

When m is given this can be solved by plotting the two curves

$$\left.\begin{aligned} y &= \log_e s \\ y &= 1 - \frac{m^2 \log_e m}{m^2 - 1}(1 + s^2) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (13.116)$$

and finding their point of intersection.
When $m = 2$ the equation for s is

$$\begin{aligned} \log_e s &= 1 - 0{\cdot}9242(1 + s^2) \\ &= 0{\cdot}0758 - 0{\cdot}9242 s^2 \quad . \quad . \quad . \quad . \quad (13.117) \end{aligned}$$

Since s is a proper fraction it is convenient for calculations to add $\log_e 10$ to both sides of the last equation. Then

$$\log_e 10s = 2{\cdot}3784 - 0{\cdot}9242 s^2 \quad . \quad . \quad . \quad . \quad (13{\cdot}118)$$

The approximate value of the root of this equation is

$$s = 0{\cdot}693,$$

whence

$$r = \frac{r_1}{0{\cdot}693} = 1{\cdot}443 r_1.$$

Thus

$$\frac{r - r_1}{r_2 - r_1} = \frac{r - r_1}{r_1} = 0{\cdot}443 \quad . \quad . \quad . \quad . \quad . \quad (13{\cdot}119)$$

Therefore when the thickness is equal to the inner radius the neutral axis divides the thickness into two parts having the ratio 0·443 to 0·557 to each other, this neutral axis being nearer the concave side.

It is easy to show that, if m is nearly equal to unity, the ratio in equation (13.119) would be nearly a half. The position of the neutral axis depends, as the preceding equations show, on the ratio of the thickness to r_1.

The displacements must satisfy the equations

$$E\frac{\partial U}{\partial r} = R - \sigma T \quad . \quad . \quad . \quad . \quad . \quad (13.120)$$

$$E\left(\frac{U}{r} + \frac{\partial \eta}{\partial \theta}\right) = T - \sigma R \quad . \quad . \quad . \quad . \quad . \quad (13.121)$$

$$\frac{E}{2(1 + \sigma)}\left\{\frac{\partial U}{r\partial \theta} + r\frac{\partial \eta}{\partial r}\right\} = F = 0 \quad . \quad . \quad . \quad . \quad . \quad (13.122)$$

Integrating the first of these we get

$$\mathrm{EU} = -(1+\sigma)\frac{\mathrm{A}}{r} + 2(1-\sigma)\mathrm{B}(r\log r - r) + (\mathrm{C} - \sigma\mathrm{C} - 2\sigma\mathrm{B})r + f'(\theta).$$

Consequently equation (13.121) gives

$$\mathrm{E}\frac{\partial\eta}{\partial\theta} = \mathrm{T} - \sigma\mathrm{R} - \frac{\mathrm{EU}}{r}$$

$$= -(1+\sigma)\frac{\mathrm{A}}{r^2} + 2(1-\sigma)\mathrm{B}\log r + \mathrm{C} + 2\mathrm{B} - \sigma\mathrm{C} + (1+\sigma)\frac{\mathrm{A}}{r^2} - 2(1-\sigma)\mathrm{B}(\log r - 1) - (\mathrm{C} - \sigma\mathrm{C} - 2\sigma\mathrm{B}) - \frac{f'(\theta)}{r}$$

$$= 4\mathrm{B} - \frac{f'(\theta)}{r},$$

whence

$$\mathrm{E}\eta = 4\mathrm{B}\theta - \frac{f(\theta)}{r} + \mathrm{F}(r).$$

Substituting these values of U and η in (13.122) we get

$$\frac{f''(\theta)}{r} + \frac{f(\theta)}{r} + r\mathrm{F}'(r) = 0 \quad . \quad . \quad . \quad . \quad . \quad (13.123)$$

which is satisfied if

$$f(\theta) = 0, \quad \mathrm{F}(r) = 0.$$

Equation (13.123) is also satisfied if

$$f''(\theta) + f(\theta) = \mathrm{N}\ (a\ \text{constant}),$$
$$r^2\mathrm{F}'(r) = -\mathrm{N},$$

from which

$$\left.\begin{aligned} f(\theta) &= \mathrm{L}\cos\theta + \mathrm{M}\sin\theta + \mathrm{N} \\ \mathrm{F}(r) &= \frac{\mathrm{N}}{r} + \mathrm{G} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (13.124)$$

The displacements due to these last values of $f(\theta)$ and $\mathrm{F}(r)$ are, however, only rigid-body displacements, and can therefore be neglected. We may thus take

$$\left.\begin{aligned} \mathrm{EU} &= -(1+\sigma)\frac{\mathrm{A}}{r} + 2(1-\sigma)\mathrm{B}r\log r + 2\mathrm{B}r + (1-\sigma)\mathrm{C}r, \\ \mathrm{E}\eta &= 4\mathrm{B}\theta + \mathrm{G}. \end{aligned}\right\} \quad . \quad (13.125)$$

Since B ist not zero the angular displacement η ist not zero, from which it follows that the displacement is not purely radial.

235. Pressure applied at the edge of an infinite plate.

One possible value of φ for a plate with no accelerations and no body forces is given by

$$\mathrm{E}\varphi = \mathrm{H}y\theta = \mathrm{H}r\theta\sin\theta \quad . \quad . \quad . \quad . \quad . \quad (13.126)$$

The corresponding polar stresses are

$$R = E\left\{\frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2} + \frac{1}{r}\frac{\partial\varphi}{\partial r}\right\}$$

$$= H\left\{\frac{1}{r}(-\theta\sin\theta + 2\cos\theta) + \frac{1}{r}\theta\sin\theta\right\}$$

$$= \frac{2H}{r}\cos\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13.127)$$

$$T = E\frac{\partial^2\varphi}{\partial r^2} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13.128)$$

$$F = \frac{E}{r}\frac{\partial}{\partial\theta}\left(\frac{\varphi}{r} - \frac{\partial\varphi}{\partial r}\right) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13.129)$$

The displacements must satisfy the three equations

$$E\frac{\partial U}{\partial r} = R - \sigma T = \frac{2H}{r}\cos\theta \quad . \quad . \quad . \quad (13.130)$$

$$E\left(\frac{\partial\eta}{\partial\theta} + \frac{U}{r}\right) = T - \sigma R = -\frac{2\sigma H}{r}\cos\theta \quad . \quad . \quad (13.131)$$

$$E\left(\frac{1}{r}\frac{\partial U}{\partial\theta} + r\frac{\partial\eta}{\partial r}\right) = 2(1+\sigma)F = 0. \quad . \quad . \quad . \quad . \quad (13.132)$$

Integrating (13.130) we get

$$EU = 2H\cos\theta\log r + f'(\theta).$$

Now (13.131) becomes

$$E\frac{\partial\eta}{\partial\theta} = -\frac{2\sigma H}{r}\cos\theta - 2H\cos\theta\frac{\log r}{r} - \frac{f'(\theta)}{r},$$

whence

$$E\eta = -\frac{2H}{r}\sin\theta(\log r + \sigma) - \frac{f(\theta)}{r} + F(r).$$

Substituting for U and η in (13.132) we get

$$-2(1-\sigma)\frac{H}{r}\sin\theta + \frac{1}{r}\{f''(\theta) + f(\theta)\} + rF'(r) = 0,$$

which is satisfied only if

$$\left.\begin{aligned} f''(\theta) + f(\theta) &= 2(1-\sigma)H\sin\theta + C, \\ r^2F'(r) &= -C \end{aligned}\right\} \quad . \quad . \quad . \quad (13.133)$$

The solutions of these two equations are

$$\left.\begin{aligned} f(\theta) &= \quad (1-\sigma)H\theta\cos\theta + A\cos\theta + B\sin\theta + C, \\ F(r) &= \frac{C}{r} + G \end{aligned}\right\}. \quad (13.134)$$

If we omit the terms involving A, B, G, which represent the rigid-body displacements, we get

$$\left.\begin{aligned} EU &= 2H\cos\theta\log r + (1-\sigma)H(\theta\sin\theta - \cos\theta), \\ Er\eta &= -2H\sin\theta(\log r + \sigma) + (1-\sigma)H\theta\cos\theta \end{aligned}\right\} \quad . \quad (13.135)$$

The results would look more rational if they contained $\log\frac{r}{a}$ instead of $\log r$. It is only necessary to put $B = -2H\log a$ in order to get

$$\left.\begin{aligned} EU &= 2H\cos\theta\log\frac{r}{a} + (1-\sigma)H(\theta\sin\theta - \cos\theta), \\ Er\eta &= -2H\sin\theta\left(\log\frac{r}{a} + \sigma\right) + (1-\sigma)H\theta\cos\theta. \end{aligned}\right\} \quad . \quad (13.136)$$

Now since the expressions representing the displacements are not single-valued functions of the coordinates, that is, since their values are not repeated when θ increases by 2π, it must mean that these displacements are not possible for a plate which is continuous from $\theta = 0$ to $\theta = 2\pi$. We see, however, that the stresses are zero on the edge where $\theta = -\frac{\pi}{2}$ or $\frac{\pi}{2}$. Then clearly the equations could all apply to the half-infinite plate extending from $x = 0$ to $x = \infty$, and from $y = -\infty$ to $y = +\infty$. The edge $x = 0$ is free from stress, but equations (13.136) show that its particles are displaced.

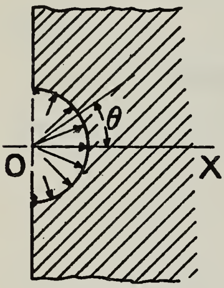


Fig. 121

The state of stress represented by equations (13.127), (13.128), (13.129), is shown on a semicircle in fig. 121, H being assumed to be negative so as to make R into a thrust.

The resultant force acting on the semicircle in the direction OX is

$$W = 2h\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} -R\cos\theta r d\theta = -\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 4hH\cos^2\theta d\theta$$

$$= -2\pi hH \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13.137)$$

Thus the force is constant whatever be the radius of the semicircle. Since there is no stress on the straight part of the edge $x = 0$, and since the stress is zero at infinity, it follows that the stress given by (13.127) is the stress in the semi-infinite plate due to forces over a semicircular notch such as are shown in fig. 121. Moreover, since the radius of this semicircle may be as small as we please, we may regard the force distributed over the semicircle as a concentrated force of magnitude W applied at O. We conclude then that equations (13.127), (13.128), (13.129), give the state of stress in a semi-infinite plate due to a concentrated force of magnitude $2\pi hH$ applied to the edge in the direction perpendicular to the edge.

If the length of edge over which W is distributed is very small compared with the dimensions of the plate we may regard the force as concentrated at a point. Although the actual distribution of stress very near the place where W is applied will depend on the way in which W is distributed, yet the state of stress at a short distance away will be that given by (13.127).

In terms of W the radial stress is

$$R = -\frac{W}{\pi hr}\cos\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad (13.138)$$

Since the axis of x makes an angle $-\theta$ with the radius vector r equations (1.22), (1.23), (1.24), give, as the stresses on sections perpendicular to OX, OY,

$$\left.\begin{aligned} P_1 &= R\cos^2(-\theta) = -\frac{W}{\pi hr}\cos^3\theta \\ P_2 &= R\sin^2(-\theta) = -\frac{W}{\pi hr}\cos\theta\sin^2\theta, \\ S &= F\cos(-2\theta) - \tfrac{1}{2}R\sin(-2\theta) \\ &= -\frac{W}{\pi hr}\cos^2\theta\sin\theta. \end{aligned}\right\} \quad . \quad . \quad (13.139)$$

If the force W were applied at the point of the edge where $x = 0$, $y = y_1$, and if

$$r_1^2 = x^2 + (y - y_1)^2,$$

then the expressions for the stresses at (x, y) become

$$\left.\begin{aligned} P_1 &= -\frac{W}{\pi h}\frac{x^3}{r_1^4} \\ P_2 &= -\frac{W}{\pi h}\frac{x(y-y_1)^2}{r_1^4} \\ S &= \frac{W}{\pi h}\frac{x^2(y-y_1)}{r_1^4} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (13.140)$$

236. Stresses in semi-infinite plate due to any distribution of pressure on one edge.

Suppose there is a distribution of pressure on the edge $x = 0$ of the semi-infinite plate we have just been dealing with, the thrust on a length dy_1 being $w\,dy_1$. Then the stresses due to $w\,dy_1$ are got by substituting this expression for W in equations (13.140). Then, because the total stress due to the sum of several forces is the sum of the stresses due to the separate forces, we get, as the stresses due to the distributed force,

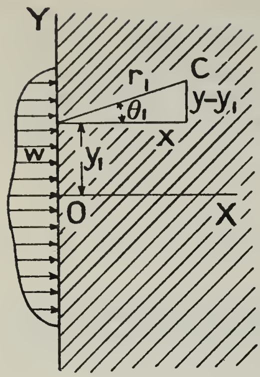


Fig. 122

$$P_1 = -\frac{1}{\pi h}\int \frac{wx^3 dy_1}{r_1^{\,4}}$$

$$= -\frac{x^3}{\pi h}\int \frac{w dy_1}{\{x^2 + (y - y_1)^2\}^2} \quad . \quad . \quad . \quad (13.141)$$

$$P_2 = -\frac{x}{\pi h}\int \frac{w(y-y_1)^2 dy_1}{\{x^2 + (y-y_1)^2\}^2} \quad . \quad . \quad (13.142)$$

$$S = \frac{x^2}{\pi h}\int \frac{w(y-y_1) dy_1}{\{x^2 + (y-y_1)^2\}^2} \quad . \quad . \quad . \quad (13.143)$$

Since x is constant in these integrals we may put, as in fig. 122,

$$\left.\begin{aligned} y - y_1 &= x\tan\theta_1; \\ \text{whence} \quad dy_1 &= -x\sec^2\theta_1 d\theta_1, \\ r_1 &= x\sec\theta_1. \end{aligned}\right\} \quad . \quad (13.144)$$

Then the stresses become

$$P_1 = \frac{1}{\pi h}\int w\cos^2\theta_1\, d\theta_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13.145)$$

$$P_2 = \frac{1}{\pi h}\int w\sin^2\theta_1\, d\theta_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13.146)$$

$$S = -\frac{1}{\pi h}\int w\sin\theta_1\cos\theta_1\, d\theta_1. \quad . \quad . \quad . \quad . \quad (13.147)$$

wherein w must be expressed as a function of θ_1.

As a particular case suppose w is constant from $y_1 = -a$ to $y_1 = +a$, and zero over the rest of the edge. Then the stresses at any point C are (fig. 123)

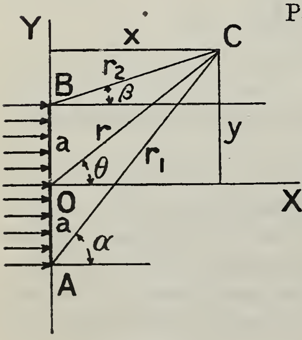


Fig. 123

$$P_1 = \frac{1}{\pi h}\int_\alpha^\beta \tfrac{1}{2}w(1 + \cos 2\theta_1)d\theta_1$$

$$= \frac{w}{2\pi h}\Big[\theta_1 + \tfrac{1}{2}\sin 2\theta_1\Big]_\alpha^\beta$$

$$= \frac{w}{2\pi h}\{\beta - \alpha + \tfrac{1}{2}\sin 2\beta - \tfrac{1}{2}\sin 2\alpha\}, \quad (13.148)$$

α and β being the values of θ_1 at the limits where $y = -a$ and $y = +a$. These values are functions of x and y satisfying the equations

$$\left.\begin{aligned} y - a &= x\tan\beta, \\ y + a &= x\tan\alpha. \end{aligned}\right\} \quad . \quad . \quad (13.149)$$

Also
$$P_2 = \frac{w}{2\pi h}\int_\alpha^\beta (1-\cos 2\theta_1)d\theta_1$$
$$= \frac{w}{2\pi h}\left\{\beta-\alpha-\tfrac{1}{2}\sin 2\beta+\tfrac{1}{2}\sin 2\alpha\right\}, \quad . \quad . \quad . \quad (13.150)$$
$$S = -\frac{w}{4\pi h}\left\{\cos 2\alpha-\cos 2\beta\right\}. \quad . \quad . \quad . \quad (13.151)$$

These results can be written thus
$$\left.\begin{aligned} P_1 &= -\frac{w}{2\pi h}\{\alpha-\beta+\sin(\alpha-\beta)\cos(\alpha+\beta)\} \\ P_2 &= -\frac{w}{2\pi h}\{\alpha-\beta-\sin(\alpha-\beta)\cos(\alpha+\beta)\} \\ S &= \frac{w}{2\pi h}\sin(\alpha-\beta)\sin(\alpha+\beta). \end{aligned}\right\} \quad . \quad . \quad (13.152)$$

237. Circular plate with two concentrated forces at the opposite ends of a diameter.

Let poles O_1, O_2, be taken at the two points where the concentrated forces, each of magnitude W, are applied.

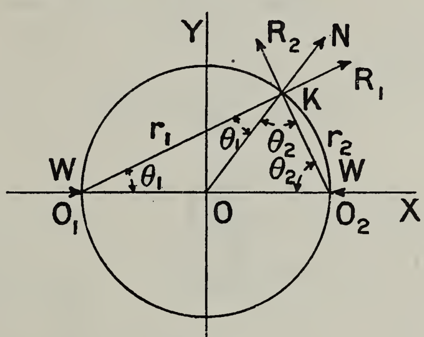


Fig. 124

We have to find a stress system that gives no surface forces on the boundary of the circle. If we take
$$\varphi = \varphi_1 + \varphi_2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13.153)$$
where
$$E\varphi_1 = -\frac{W}{2\pi h}r_1\theta_1\sin\theta_1, \quad . \quad . \quad . \quad . \quad . \quad (13.154)$$
$$E\varphi_2 = -\frac{W}{2\pi h}r_2\theta_2\sin\theta_2, \quad . \quad . \quad . \quad . \quad . \quad (13.155)$$
the only stress due to φ_1 is
$$R_1 = -\frac{W}{\pi h r_1}\cos\theta_1. \quad . \quad . \quad . \quad . \quad . \quad . \quad (13.156)$$

Now let d denote the diameter of the circular plate. Then, from fig. 124

$$\cos\theta_1 = \frac{r_1}{d}. \qquad (13.157)$$

Therefore at the edge of the plate

$$R_1 = -\frac{W}{\pi h d}. \qquad (13.158)$$

Likewise the stress at the same point due to W at θ_2 is

$$R_2 = -\frac{W}{\pi h d}. \qquad (13.159)$$

Thus across sections perpendicular to r_1 and r_2 at a point K on the edge of the plate there are equal normal stresses, both of which are thrusts, and no shear stresses. Since r_1 and r_2 are at right angles it follows that R_1 and R_2 are principal stresses at K, and these being equal the stress on any other section at K is also purely normal and has the same magnitude as R_1 or R_2. Thus the normal stress across the edge itself is

$$N = -\frac{W}{\pi h d}. \qquad (13.160)$$

Now the stress function

$$E\varphi_3 = \frac{W}{2\pi h d}(x^2 + y^2) \qquad (13.161)$$

gives

$$\left.\begin{aligned} P_1 &= \frac{W}{\pi h d}, \\ P_2 &= \frac{W}{\pi h d}, \\ S &= 0, \end{aligned}\right\} \qquad (13.162)$$

and the origin for x and y may be anywhere in the plane of the middle surface. This system of stresses exactly neutralises the stresses R_1 and R_2 at any point on the edge of the plate.

If then we take

$$\varphi = \varphi_1 + \varphi_2 + \varphi_3 \qquad (13.163)$$

it follows that the normal stress and the shear stress on the edge of the circular plate are both zero. Thus the edge of the plate is free from stress except at the points O_1 and O_2 and here the applied forces are each W. It follows then that equation (13.163) gives the solution to the problem we set out to solve.

With reference to rectangular axes OO_2X, OY, through the centre of the circle, the stresses are, by (13.139), or by differentiating φ,

$$P_1 = -\frac{W}{\pi h}\left\{\frac{\cos^3\theta_1}{r_1} + \frac{\cos^3\theta_2}{r_2}\right\} + \frac{W}{\pi h d} \quad . \quad . \quad . \quad . \quad . \quad (13.164)$$

$$P_2 = -\frac{W}{\pi h}\left\{\frac{\cos\theta_1 \sin^2\theta_1}{r_1} + \frac{\cos\theta_2 \sin^2\theta_2}{r_2}\right\} + \frac{W}{\pi h d} \quad . \quad (13.165)$$

$$S = -\frac{W}{\pi h}\left\{\frac{\cos^2\theta_1 \sin\theta_1}{r_1} - \frac{\cos^2\theta_2 \sin\theta_2}{r_2}\right\} \quad . \quad . \quad . \quad . \quad (13.166)$$

The difference of signs in the brackets in the last equation is due to the fact that the x-axis is in contrary directions relative to the two concentrated forces, and therefore the shear force due to R_2, which would be negative if the positive x-axis were taken in the direction from O_2 to O_1, is positive when this axis is taken in the contrary direction. It comes to the same thing if we regard the stress R_2 as being inclined at $-\theta_2$ to the x-axis, and therefore the x-axis inclined at $+\theta_2$ to R_2. Then the shear stress due to R_2 is

$$-R_2 \sin\theta_2 \cos\theta_2 = +\frac{W}{\pi h r_2}\sin\theta_2 \cos^2\theta_2 .$$

238. Finite force applied to a point in a plate at a great distance from any part of the boundary.

The problem, stated in mathematical terms, is to find the stresses in an infinite plate due to a force applied, in the plane of the middle surface, at a point at an infinite distance from any part of the boundary.

We have already found that the function

$$E\varphi_1 = Hr\theta \sin\theta$$

gives a system of stresses due to a force at a point, but it fails to apply to a plate completely surrounding the point because the displacements due to these stresses are not single valued functions of the coordinates. To get over this difficulty we need to add another value of φ which will remove the troublesome terms from the displacements. For this purpose consider the function

$$E\varphi_2 = Ar\cos\theta \log r \quad . \quad . \quad . \quad . \quad . \quad (13.167)$$

From this we get, using the suffix 2 to indicate quantities derived from φ_2,

$$\left.\begin{aligned} R_2 &= E\left\{\frac{1}{r^2}\frac{\partial^2\varphi_2}{\partial\theta^2} + \frac{1}{r}\frac{\partial\varphi_2}{\partial r}\right\} = \frac{A}{r}\cos\theta \\ T_2 &= E\frac{\partial^2\varphi_2}{\partial r^2} \cdot \qquad\qquad\qquad = \frac{A}{r}\cos\theta \\ F_2 &= \frac{E}{r}\frac{\partial}{\partial\theta}\left(\frac{\varphi_2}{r} - \frac{\partial\varphi_2}{\partial r}\right) \qquad = \frac{A}{r}\sin\theta \end{aligned}\right\} \quad . \quad . \quad . \quad (13.168)$$

The equations for the displacements are

$$\frac{\partial U_2}{\partial r} = \frac{R - \sigma T}{E} = (1 - \sigma)\frac{A}{r}\cos\theta \quad . \quad . \quad . \quad (13.169)$$

$$\frac{U_2}{r} + \frac{\partial \eta_2}{\partial \theta} = \frac{T - \sigma R}{E} = (1 - \sigma)\frac{A}{r}\cos\theta \quad . \quad . \quad . \quad (13.170)$$

$$\frac{1}{r}\frac{\partial U_2}{\partial \theta} + r\frac{\partial \eta_2}{\partial r} = 2(1 + \sigma)\frac{F}{E} = 2(1 + \sigma)\frac{A}{r}\sin\theta. \quad . \quad (13.171)$$

The first of these gives, on integration,

$$U_2 = (1 - \sigma)A\cos\theta\log r + f'(\theta),$$

$f'(\theta)$ being any arbitrary function of θ.

Then (13.170) becomes

$$\frac{\partial \eta_2}{\partial \theta} = (1 - \sigma)A\left(\frac{1}{r} - \frac{\log r}{r}\right)\cos\theta - \frac{f'(\theta)}{r},$$

whence

$$\eta_2 = (1 - \sigma)A\left(\frac{1}{r} - \frac{\log r}{r}\right)\sin\theta - \frac{f(\theta)}{r} + F(r).$$

Substituting these values of U and η in (13.171) we get

$$2(1 + \sigma)\frac{A}{r}\sin\theta = -(1 - \sigma)A\sin\theta\frac{\log r}{r} + \frac{1}{r}f''(\theta)$$

$$-(1 - \sigma)A\sin\theta\left(\frac{2}{r} - \frac{\log r}{r}\right) + \frac{1}{r}f(\theta) + rF'(r).$$

This is satisfied if

$$rF'(r) = 0,$$

and

$$f''(\theta) + f(\theta) = 4A\sin\theta.$$

Particular integrals of these are

$$F(r) = 0,$$

$$f(\theta) = -2A\theta\cos\theta.$$

Thus the stress-system in equations (13.168) gives rise to displacements

$$\left.\begin{aligned} U_2 &= (1 - \sigma)A\cos\theta\log r + 2A(\theta\sin\theta - \cos\theta),\\ r\eta_2 &= (1 - \sigma)A\sin\theta(1 - \log r) + 2A\theta\cos\theta.\end{aligned}\right\} \quad . \quad (13.172)$$

Now the function $E\varphi_1$ gives displacements

$$\left.\begin{aligned} U_1 &= 2H\cos\theta\log r + (1 - \sigma)H(\theta\sin\theta - \cos\theta),\\ r\eta_1 &= -2H\sin\theta(\log r + \sigma) + (1 - \sigma)H\theta\cos\theta),\end{aligned}\right\} \quad . \quad (13.173)$$

and stresses

$$R_1 = \frac{2H}{r}\cos\theta,\ T_1 = 0,\ F_1 = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (13.174)$$

Let us now put

$$2A = -(1 - \sigma)H, \quad . \quad . \quad . \quad . \quad . \quad . \quad (13.175)$$

and take

$$E\varphi = E\varphi_1 + E\varphi_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13.176)$$

Then the displacements due to φ are

$$U = \{2 - \tfrac{1}{2}(1-\sigma)^2\}H\cos\theta\log r$$
$$= \tfrac{1}{2}(1+\sigma)(3-\sigma)H\cos\theta\log r \quad . \quad . \quad . \quad . \quad . \quad . \quad (13.177)$$
$$r\eta = -\{2(\log r + \sigma) - \tfrac{1}{2}(1-\sigma)^2(\log r - 1)\}H\sin\theta$$
$$= -\tfrac{1}{2}(1+\sigma)H\sin\theta\{(3-\sigma)\log r + (1+\sigma)\} \quad . \quad . \quad (13.178)$$

Also the stresses are

$$\left.\begin{aligned} R &= R_1 + R_2 = \tfrac{1}{2}(3+\sigma)\frac{H}{r}\cos\theta \\ T &= T_1 + T_2 = -\tfrac{1}{2}(1-\sigma)\frac{H}{r}\cos\theta \\ F &= F_1 + F_2 = -\tfrac{1}{2}(1-\sigma)\frac{H}{r}\sin\theta. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (13.179)$$

It will be noticed that the displacements due to φ are single-valued functions of the coordinates x and y, the troublesome terms of the form $\theta \sin\theta$ and $\theta\cos\theta$ which occur in the displacements due to φ_1 and φ_2 having disappeared. It follows that φ is a possible function at any point inside a plate. It is, in fact, the correct function for the stresses due to a force at the origin in the direction of the axis of x.

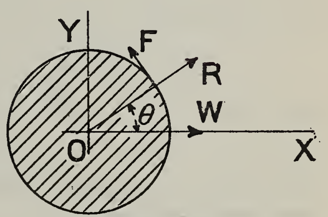


Fig. 125

We shall now find the force at O to which the system of stresses due to φ are equivalent by finding the resultant of the stresses on the edge of a circle of radius r.

The sum of the components in the x-direction of the forces due to R and F on a length $rd\theta$ of the circle is

$$(R\cos\theta - F\sin\theta)rd\theta.$$

Hence the force at the centre of the circle which balances the total x-component force on the edge is

$$W = -2h\int_0^{2\pi}(R\cos\theta - F\sin\theta)rd\theta$$
$$= -2h\int_0^{2\pi}\tfrac{1}{2}H\{(3+\sigma)\cos^2\theta + (1-\sigma)\sin^2\theta\}d\theta$$
$$= -\pi hH\{(3+\sigma)+(1-\sigma)\} = -4\pi hH \quad . \quad . \quad . \quad (13.180)$$

The component force at the centre in the y-direction is

$$-2h\int_0^{2\pi}(\mathrm{R}\sin\theta+\mathrm{F}\cos\theta)rd\theta$$

$$=-h\mathrm{H}\int_0^{2\pi}\{(3+\sigma)-(1-\sigma)\}\sin\theta\cos\theta d\theta$$

$$=0.$$

It is worth while to notice that φ_2 contributes nothing to the force W, this force being wholly due to φ_1; the part played by φ_2 is to modify the displacements due to φ_1 so that they are possible when the origin is inside the plate instead of being restricted to the edge. Of course φ_2 modifies the stresses at the same time.

Since the force W has the same magnitude whatever be the radius of the circle over which we find the resultant it follows that the whole force W is concentrated at the origin O.

239. Shear stress zero at the edge of a circle.

It is possible to modify the stresses in (13.179) so as to reduce F to zero at the rim of a circle of given radius a, without affecting the magnitude of the force W. For this purpose we need the stress function

$$\mathrm{E}\varphi_3=\frac{\mathrm{A}}{r}\cos\theta. \quad \ldots \quad (13.181)$$

The stresses due to this are

$$\mathrm{R}_3=-\frac{2\mathrm{A}}{r^3}\cos\theta,\ \mathrm{T}_3=\frac{2\mathrm{A}}{r^3}\cos\theta,\ \mathrm{F}_3=-\frac{2\mathrm{A}}{r^3}\sin\theta. \quad (13.182)$$

Now let

$$2\mathrm{A}=-\tfrac{1}{2}(1-\sigma)\mathrm{H}a^2 \quad \ldots \quad (13.183)$$

and let these stresses be added to the stresses in (13.179). Then the new total stresses are

$$\left.\begin{aligned}\mathrm{R}&=\tfrac{1}{2}\mathrm{H}\left\{\frac{3+\sigma}{r}+\frac{(1-\sigma)a^2}{r^3}\right\}\cos\theta\\ \mathrm{T}&=-\tfrac{1}{2}\mathrm{H}(1-\sigma)\left(\frac{1}{r}+\frac{a^2}{r^3}\right)\cos\theta\\ \mathrm{F}&=-\tfrac{1}{2}\mathrm{H}(1-\sigma)\left(\frac{1}{r}-\frac{a^2}{r^3}\right)\sin\theta\end{aligned}\right\} \quad \ldots \quad (13.184)$$

Over the rim of the circle $r=a$ the stress F is zero.

It is easy to show that the additional displacements due to φ_3 are given by

$$\left.\begin{aligned}\mathrm{EU}_3&=(1+\sigma)\frac{\mathrm{A}}{r^2}\cos\theta=-\tfrac{1}{4}(1-\sigma^2)\frac{\mathrm{H}}{r^2}\cos\theta\\ \mathrm{E}r\eta_3&=(1+\sigma)\frac{\mathrm{A}}{r^2}\sin\theta=-\tfrac{1}{4}(1-\sigma^2)\frac{\mathrm{H}}{r^2}\sin\theta.\end{aligned}\right\} \quad . \quad (13.185)$$

240. The resultant force on any piece of a plate bounded by a closed curve.

Consider the forces on the edges of a rectangle ABCD (fig. 126) whose sides are parallel to the coordinate axes, the body force and acceleration being supposed to be zero along every side.

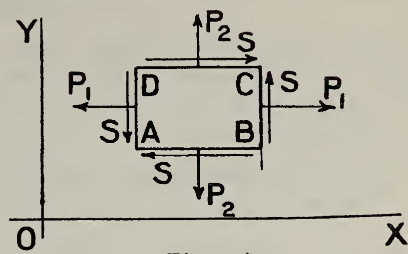


Fig. 126

It is to be understood that the stress represented by P_1 in the figure is not constant but variable along each side and from one side to the opposite side. Similar remarks apply to the other stresses.

Now the whole x-component force acting on the edge AB is

$$X_1 = -\int_A^B S dx = \int_A^B E \frac{\partial^2 \varphi}{\partial x \partial y} dx$$

$$= E\left\{\left(\frac{\partial \varphi}{\partial y}\right)_B - \left(\frac{\partial \varphi}{\partial y}\right)_A\right\}$$

Again the x-component force on BC is

$$X_2 = \int_B^C P_1 dy = \int_B^C E \frac{\partial^2 \varphi}{\partial y^2} dy$$

$$= E\left\{\left(\frac{\partial \varphi}{\partial y}\right)_C - \left(\frac{\partial \varphi}{\partial x}\right)_B\right\} \quad . \quad . \quad . \quad . \quad . \quad (13.186)$$

By proceeding along the sides CD and DA in the same way and adding all the four forces we get, as the total x-component force on the rectangle,

$$X = E\left\{\left(\frac{\partial \varphi}{\partial y}\right)_{A_1} - \left(\frac{\partial \varphi}{\partial y}\right)_A\right\}, \quad . \quad . \quad . \quad . \quad . \quad (13.187)$$

the quantity in large brackets $\{\ \}$ being understood to mean the increase in $\frac{\partial \varphi}{\partial y}$ in tracing out the contour of the figure ABCDA. Likewise the y-component force is

$$Y = -E\left\{\left(\frac{\partial \varphi}{\partial x}\right)_{A_1} - \left(\frac{\partial \varphi}{\partial x}\right)_A\right\} \quad . \quad . \quad . \quad . \quad (13.188)$$

the contour being supposed traced out in the same direction as before

Now the only possible value of φ, at a point inside a plate where no body force acts, which has a different value after tracing a contour and yet gives single valued functions for the stresses and displacements is one involving log $(re^{i\theta})$; that is, such terms as

$$(A + Bx + Cy)\log re^{i\theta} \quad . \quad . \quad . \quad . \quad . \quad (13.189)$$

Such functions have already been dealt with, and the stresses are given in (13.96) and (13.179). These can be extended only by taking the origin at any point, and the x-axis in any direction; and also by summing the stresses due to any number of poles, or centres of force.

241. Stresses proportional to cos $n\theta$.

The stress function having a factor $\cos n\theta$ is

$$E\varphi_n = (Ar^{-n} + Br^{2-n} + Cr^n + Dr^{n+2})\cos n\theta.$$

Taking only those terms that decrease as the distance from the origin increases we get

$$E\varphi_n = (A_n r^{-n} + B_n r^{2-n})\cos n\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13.190)$$

The stresses due to φ_n are

$$R_n = E\left\{\frac{1}{r^2}\frac{\partial^2\varphi_n}{\partial\theta^2} + \frac{1}{r}\frac{\partial\varphi_n}{\partial r}\right\}$$

$$= -\left\{n(n+1)A_n + (n-1)(n+2)B_n r^2\right\} r^{-n-2}\cos n\theta \quad . \quad (13.191)$$

$$T_n = E\frac{\partial^2\varphi_n}{\partial r^2}$$

$$= \left\{n(n+1)A_n + (n-1)(n-2)B_n r^2\right\} r^{-n-2}\cos n\theta \quad . \quad , \quad (13.192)$$

$$F_n = \frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{\varphi_n}{r} - \frac{\partial\varphi_n}{\partial r}\right)$$

$$= -\left\{n(n+1)A_n + n(n-1)B_n r^2\right\} r^{-n-2}\sin n\theta. \quad . \quad . \quad . \quad (13.193)$$

The corresponding displacements are given by

$$\left.\begin{aligned} EU_n &= \left\{(1+\sigma)nA_n + (1+\sigma)nB_n r^2 + 2(1-\sigma)B_n r^2\right\} r^{-n-1}\cos n\theta \\ Er\eta_n &= \left\{(1+\sigma)nA_n + (n+\sigma n-4)B_n r^2\right\} r^{-n-1}\sin n\theta. \end{aligned}\right\} \quad (13.194)$$

The resultant of the stresses R_n and F_n over the circumference of a circle of radius r with pole at O is zero.

242. Stresses vanishing at infinity and satisfying any possible conditions over the edge of a given circle.

If all powers of r higher than the first are omitted from the value of φ given in (13.48) the resulting stresses will all contain only negative powers of r, and will therefore vanish at infinity. Moreover it is possible to satisfy given boundary conditions over any circle $r = a$ by means of this restricted value of φ, provided that the plate is continuous round the circle. In order that the displacements shall be continuous functions of θ it is necessary to take a particular combination of terms involving θ and log r such as we got in equation (13.176). There is a similar combination to the one in (13.176) with

$\sin\theta$ and $-\cos\theta$ interchanged. The stress system which is symmetrical about the x-axis and involves $\cos\theta$ and $\sin\theta$ is derived by adding the values of φ in (13.176) and (13.182). Thus

$$E\varphi_1 = B_1\{r\theta\sin\theta - \tfrac{1}{2}(1-\sigma)r\cos\theta\log r\} + \frac{A_1}{r}\cos\theta . \quad (13.195)$$

Now let

$$E\varphi = E\varphi_0 + E\varphi_1 + E\sum_{n=2}^{n=\infty}\varphi_n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13.196)$$

where

$$\varphi_0 = A\log r + C\theta, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13.197)$$

φ_1 is the function given by (13.195), and φ_n is given by (13.190). Then this function φ gives stresses which are symmetrical about the x-axis, vanish at infinity, and can be made to satisfy any possible boundary conditions over the edge of a given circle $r=a$.

If a pull W were applied to a plate by means of a rivet it is quite possible that the whole of the force exerted by the rivet may be applied by pressure on one side only of the hole through which it passes. The stress system near the hole must therefore differ from the stresses in (13.179) because the stress R in these equations is applied partly as a thrust on one semicircle and a tension on the other; and also because F is not zero.

Let us then find the stress-system which satisfies the following conditions

$$R = 0, \quad T = 0, \quad F = 0, \quad \text{where } r = \infty; \quad . \quad . \quad (13.198)$$

$$\left.\begin{aligned} F &= 0, \\ R &= -K\cos\theta \quad \text{where } \cos\theta \text{ is positive,} \\ &= 0 \qquad\qquad \text{where } \cos\theta \text{ is negative.} \end{aligned}\right\} \text{where } r = a \,. \quad (13.199)$$

The term $C\theta$ in φ_0 can be omitted in this case because there is clearly no wrench about the normal at O. Then the stress system given by (13.196) is

$$R = \frac{A}{r^2} + \tfrac{1}{2}(3+\sigma)\frac{B_1}{r}\cos\theta - \frac{2A_1}{r^3}\cos\theta$$
$$- \sum\{n(n+1)A_n + (n-1)(n+2)B_n r^2\}r^{-n-2}\cos n\theta . \quad (13.200)$$

$$T = -\frac{A}{r^2} - \tfrac{1}{2}(1-\sigma)\frac{B_1}{r}\cos\theta + \frac{2A_1}{r^3}\cos\theta$$
$$+ \sum\{n(n+1)A_n + (n-1)(n-2)B_n r^2\}r^{-n-2}\cos n\theta . \quad (13.201)$$

$$F = -\tfrac{1}{2}(1-\sigma)\frac{B_1}{r}\sin\theta - \frac{2A_1}{r^3}\sin\theta$$
$$- \sum\{n(n+1)A_n + n(n-1)B_n r^2\}r^{-n-2}\sin n\theta . \quad . \quad (13.202)$$

Now when $r = a$ the stresses are each represented by a simple Fourier series with constant coefficients. We have to determine these coefficients so as to make R and F satisfy the conditions (13.199).

Now the problem before us is to represent the function shown in fig. 127 as a series of cosines of θ.

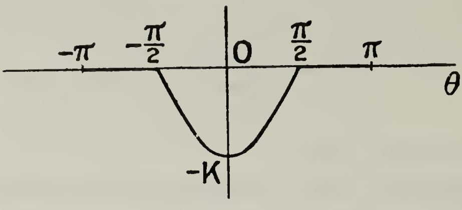


Fig. 127

The given function of θ is defined between $-\pi$ and π by the equations

$$\left.\begin{aligned} f(\theta) &= -\text{K}\cos\theta \text{ when } \frac{\pi}{2} > \theta > -\frac{\pi}{2} \\ f(\theta) &= 0 \qquad \text{when } \theta > \frac{\pi}{2} \text{ and when } \theta < -\frac{\pi}{2} \end{aligned}\right\} \quad (13.203)$$

Now assuming that

$$f(\theta) = b_0 + b_1\cos\theta + b_2\cos 2\theta + \quad + b_n\cos n\theta +$$

we get, if $n > 1$,

$$\int_{-\pi}^{\pi} f(\theta)\cos n\theta d\theta = b_n\int_{-\pi}^{\pi}\cos^2 n\theta d\theta,$$

whence

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} -\text{K}\cos\theta\cos n\theta d\theta = \pi b_n,$$

or

$$\pi b_n = -\tfrac{1}{2}\text{K}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\{\cos(n-1)\theta + \cos(n+1)\theta\}\,d\theta$$

$$= -\text{K}\left\{\frac{1}{n-1}\sin(n-1)\frac{\pi}{2} + \frac{1}{n+1}\sin(n+1)\frac{\pi}{2}\right\}$$

$$= -\text{K}\left\{\frac{1}{n-1} - \frac{1}{n+1}\right\}\sin(n-1)\frac{\pi}{2}$$

$$= -\frac{2}{n^2-1}\text{K}\sin(n-1)\frac{\pi}{2}$$

If n is odd b_n is zero, whereas if n is even

$$\pi b_n = \mp\frac{2}{n^2-1}\text{K}$$

The particular case $n=1$ gives

$$b_1\int_{-\pi}^{\pi}\cos^2\theta\,d\theta=\int_{-\pi}^{\pi}f(\theta)\cos\theta\,d\theta,$$

whence

$$\pi b_1=-K\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^2\theta\,d\theta=-\frac{\pi}{2}K.$$

Also

$$\int_{-\pi}^{\pi}b_0\,d\theta=-K\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos\theta\,d\theta=-2K,$$

that is,

$$2\pi b_0=-2K.$$

Thus the series is

$$f(\theta)=-\frac{K}{\pi}\left\{1+\frac{\pi}{2}\cos\theta+\frac{2}{2^2-1}\cos 2\theta-\frac{2}{4^2-1}\cos 4\theta+\ldots\right\} \quad (13.204)$$

This series must be identical with (13.200) when $r=a$. Therefore, when n is odd and greater than 2,

$$n(n+1)A_n+(n-1)(n+2)B_na^2=\mp\frac{2}{n^2-1}\frac{K}{\pi}a^{n+2}. \quad (13.205)$$

Also

$$2A_1-\tfrac{1}{2}(3+\sigma)B_1a^2=\tfrac{1}{2}Ka^3, \quad . \quad . \quad . \quad . \quad (13.206)$$

$$A=-\frac{K}{\pi}a^2. \quad . \quad . \quad . \quad (13.207)$$

Moreover, because F is zero over the whole circle $r=a$ the coefficents of the sines in the expression for F must be all zero. That is,

$$n(n+1)A_n+n(n-1)B_na^2=0 \quad (n>1) \quad . \quad . \quad (13.208)$$

$$2A_1+\tfrac{1}{2}(1-\sigma)B_1a^2=0 \quad . \quad . \quad . \quad . \quad . \quad (13.209)$$

Solving equations (13.205) and (13.208) we find, when $n>1$,

$$\left.\begin{aligned} n(n+1)A_n&=\mp\frac{1}{n^2-1}\frac{K}{\pi}a^{n+2},\\ (n-1)B_n&=\pm\frac{1}{n^2-1}\frac{K}{\pi}a^n.\end{aligned}\right\} \quad . \quad . \quad . \quad (13.210)$$

Also, from (13.206) and (13.207),

$$\left.\begin{aligned} B_1&=-\tfrac{1}{4}Ka,\\ A_1&=\tfrac{1}{16}(1-\sigma)Ka^3.\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (13.211)$$

The substitution of these values in equations (13.200), (13.201), (13.202), gives the complete expression for the stresses.

If we denote the resultant force on the plate by W we can express K in terms of W. Thus, since W is the resultant of the forces on the edge of the hole $r=a$, we get

$$W = 2h\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} - R\cos\theta a d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2haK\cos^2\theta d\theta$$

$$= h\pi aK. \quad (13.212)$$

In terms of W the stress R is

$$R = -\frac{W}{\pi^2 ah}\frac{a^2}{r^2} - \frac{W\cos\theta}{8\pi ah}\left\{(3+\sigma)\frac{a}{r} + (1-\sigma)\frac{a^3}{r^3}\right\}$$

$$+\frac{W}{\pi^2 ah}\sum_{m=1}^{m=\infty}\frac{(-1)^m}{4m^2-1}\left\{2m\left(\frac{a}{r}\right)^{2m+2} - (2m+2)\left(\frac{a}{r}\right)^{2m}\right\}\cos 2m\theta. \quad (13.213)$$

243. Rotating disk.

We propose here to solve again the problem of the rotating disk, which has already been solved in Chapter 12, Art. 220

When the origin is on the axis of rotation the component accelerations at x, y, are $-\omega^2 x$, $-\omega^2 y$. Therefore equations (13.2) become

$$\left.\begin{aligned}\frac{\partial P_1}{\partial x} + \frac{\partial S}{\partial y} &= -\varrho\omega^2 x\\ \frac{\partial P_2}{\partial y} + \frac{\partial S}{\partial x} &= -\varrho\omega^2 y\end{aligned}\right\} \quad (13.214)$$

Since the strains are symmetrical about the axis of rotation we shall first find a particular solution of these equations satisfying the condition of symmetry.

With the notation of (13.7) we find

$$\left.\begin{aligned}P'_1 &= \int -\varrho\omega^2 x dx = -\tfrac{1}{2}\varrho\omega^2 x^2\\ P'_2 &= -\tfrac{1}{2}\varrho\omega^2 y^2\end{aligned}\right\} \quad (13.215)$$

Therefore (13.15) becomes

$$E\nabla_1{}^4\varphi_1 = \tfrac{1}{2}\varrho\omega^2\left\{\frac{\partial^2}{\partial y^2}(x^2 - \sigma y^2) + \frac{\partial^2}{\partial x^2}(y^2 - \sigma x^2)\right\}$$

$$= -2\sigma\varrho\omega^2. \quad (13.216)$$

Since the value of φ_1 we are seeking is clearly an even function of x and y, and since it contains terms of the fourth degree in x and y, we may assume

$$E\varphi_1 = -\varrho\omega^2(cx^4 + cy^4 + dx^2y^2). \quad (13.217)$$

Substituting this value in (13.216) we get

$$48c + 8d = 2\sigma. \quad (13.218)$$

The stresses corresponding to φ_1 are

$$P_1 = E\frac{\partial^2\varphi_1}{\partial y^2} + P'_1.$$

$$= -\tfrac{1}{2}\varrho\omega^2\{(4d+1)x^2 + 24cy^2\} \quad . \quad . \quad . \quad (13.219)$$

$$P_2 = -\tfrac{1}{2}\varrho\omega^2\{(4d+1)y^2 + 24cx^2\} \quad . \quad . \quad . \quad (13.220)$$

$$S = -E\frac{\partial^2\varphi_1}{\partial x\partial y} = 4d\varrho\omega^2 xy. \quad . \quad . \quad . \quad . \quad . \quad (13.221)$$

The radial stress derived from these is, by (1.22),

$$R = P_1\cos^2\theta + P_2\sin^2\theta + 2S\sin\theta\cos\theta$$

$$= \frac{1}{r^2}\{P_1x^2 + P_2y^2 + 2Sxy\}$$

$$= -\frac{\varrho\omega^2}{2r^2}\{(4d+1)(x^4+y^4) + (48c-16d)x^2y^2\} \qquad (13.222)$$

The condition of symmetry about the axis of rotation requires that this last should be a function of (x^2+y^2). Therefore

$$(4d+1)(x^4+y^4) + (48c-16d)x^2y^2 = (4d+1)(x^2+y^2)^2, \quad . \quad (13.223)$$

whence

$$48c - 16d = 2(4d+1) \quad . \quad . \quad . \quad . \quad . \quad (13.224)$$

Equations (13.218) and (13.224) determine c and d, their values being

$$c = \tfrac{1}{96}(1+3\sigma), \quad d = -\tfrac{1}{16}(1-\sigma); \quad . \quad . \quad . \quad . \quad (13.225)$$

whence

$$P_1 = -\tfrac{1}{8}\varrho\omega^2\{(1+3\sigma)y^2 + (3+\sigma)x^2\},$$
$$P_2 = -\tfrac{1}{8}\varrho\omega^2\{(1+3\sigma)x^2 + (3+\sigma)y^2\},$$
$$S = -\tfrac{1}{4}(1-\sigma)\varrho\omega^2 xy,$$
$$R = -\tfrac{1}{8}(3+\sigma)\varrho\omega^2 r^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13.226)$$

Also, writing θ' for $\dfrac{\pi}{2} + \theta$ and T for R in (13.222), we get

$$T = P_1\cos^2\theta' + P_2\sin^2\theta' + 2S\sin\theta'\cos\theta'$$
$$= P_1\sin^2\theta + P_2\cos^2\theta - 2S\sin\theta\cos\theta$$
$$= -\tfrac{1}{8}(1+3\sigma)\varrho\omega^2 r^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13.227)$$

It remains to find φ_2 satisfying the equation

$$E\nabla_1{}^4\varphi_2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13.228)$$

and to add the stresses due to φ_2 to those already found It is clear that φ_2 must be a function of r only. Thus, we get, as in (13.99),

$$E\varphi_2 = A\log r + Kr^2, \quad . \quad . \quad . \quad . \quad . \quad (13.229)$$

which gives stresses

$$\left.\begin{aligned} R_2 &= \frac{A}{r^2} + 2K \\ T_2 &= -\frac{A}{r^2} + 2K \\ F_2 &= 0. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13.230)$$

Adding these to the stresses already found we get the complete stress system, which is given by the equations:—

$$\left.\begin{aligned} R &= \frac{A}{r^2} + 2K - \tfrac{1}{8}(3+\sigma)\varrho\omega^2 r^2, \\ T &= -\frac{A}{r^2} + 2K - \tfrac{1}{8}(1+3\sigma)\varrho\omega^2 r^2 \\ F &= 0. \end{aligned}\right\} \quad . \quad . \quad . \quad (13.231)$$

These are the same as in (12.107) and (12.108), and the rest of the procedure is exactly as in Chapter 12.

CHAPTER XIV

THE BENDING OF THIN PLATES UNDER NORMAL PRESSURES.

244. Statement of the problem.

The problem of the loaded plate with which we are about to deal in this chapter is exactly analogous to the problem of the loaded beam. The final equation (14.21) for the deflexion of the middle surface has generally been considered to be as widely applicable to plates as equation (6.14) is to beams. There is, however, a very big difference between the ranges within which the two equations may be applied. The equation for the deflexion of the plate is approximately true only so long as the deflexion of the middle surface of the plate, measured either from a plane or from some developable surface, is small in comparison with the thickness of the plate, whereas the equation for the deflexion of a beam is approximately true for deflexions such that the slope of the beam is everywhere small in comparison with unity. While it is impossible to bend a thin plate into any form but that of a developable surface without stretching or contracting some part of the middle surface yet the middle line of a thin rod may remain unstretched whatever curvature it has.

Thomson and Tait, in their "*Natural Philosophy*", and Föppl, in his "*Mechanik*", appear to be the only writers who have hitherto pointed out this limitation of the usual theory of thin plates. In the next chapter equations are worked out which can be used so long as the tangent planes at any two points of the bent middle surface make small angles with each other. These equations have as wide application to the plate as equation (6.14) has to beams. Unfortunately, however, these improved equations are usually too difficult to solve exactly for w when the pressure p is given. They can, however, be used to find p from a given value of w, as the problems worked out in the next chapter will show. Moreover, very accurate approximate methods are explained in that chapter.

245. Development of the theory.

We shall now develop the usually accepted theory in the form in which Poisson and Kirchhoff left it. This theory, it is necessary to repeat, is valid only when the deflexion w is a small fraction of the thickness.

In order to deduce the necessary equations for thin plates we shall have to make use of the fundamental theory of this subject which is given in Chapter II.

Let the thickness of the plate be constant and be denoted by $2h$. By the *middle surface* of the bent plate we mean the surface containing the particles which, in the unstrained state of the plate, lay in the plane at distance h from each face of the plate.

Let the origin of coordinates be taken at some point in the middle surface of the bent plate, and let the z-axis be the normal to this surface at the origin. Let the component displacements of any particle of the plate be u, v, w, as in Chapter II. Then we shall show that the following expressions are sufficiently good approximations to these displacements for a plate bent without any appreciable stretching of the middle surface.

$$\left.\begin{aligned} w &= f + z^2\varphi \\ u &= -z\left(\frac{\partial f}{\partial x} + \frac{\partial \xi}{\partial x}\right) + z^3\frac{\partial \psi}{\partial x} \\ v &= -z\left(\frac{\partial f}{\partial y} + \frac{\partial \xi}{\partial y}\right) + z^3\frac{\partial \psi}{\partial y} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (14.1)$$

where f, φ, ξ, ψ, are functions of x and y or constants, but not functions of z. From the above equations we find that the three longitudinal and the three shear strains are

$$\left.\begin{aligned} a &= \frac{\partial u}{\partial x} = -z\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 \xi}{\partial x^2}\right) + z^3\frac{\partial^2 \psi}{\partial x^2} = \frac{\partial^2}{\partial x^2}\{-z(f+\xi) + z^3\psi\} \\ \beta &= \frac{\partial v}{\partial y} = -z\left(\frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 \xi}{\partial y^2}\right) + z^3\frac{\partial^2 \psi}{\partial y^2} = \frac{\partial^2}{\partial y^2}\{-z(f+\xi) + z^3\psi\} \\ \gamma &= \frac{\partial w}{\partial z} = 2z\varphi \end{aligned}\right\} \quad (14.2)$$

$$\left.\begin{aligned} a &= \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = \frac{\partial}{\partial y}\{-\xi + 3z^2\psi + z^2\varphi\} \\ b &= \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} = \frac{\partial}{\partial x}\{-\xi + 3z^2\psi + z^2\varphi\} \\ c &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{\partial^2}{\partial x \partial y}\{-2z(f+\xi) + z^3\psi\} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (14.3)$$

The two shear strains a and b have to be zero at the faces of the plate where $z = \pm h$. These conditions will be satisfied provided

$$\xi = h^2(3\psi + \varphi) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14.4)$$

Therefore
$$a=\frac{\partial}{\partial y}\left\{-\xi+\frac{z^2}{h^2}\xi\right\}$$
$$=-\frac{\partial}{\partial y}\left\{\left(1-\frac{z^2}{h^2}\right)\xi\right\} \quad . \quad . \quad . \quad . \quad . \quad (14.5)$$
$$b=-\frac{\partial}{\partial x}\left\{\left(1-\frac{z^2}{h^2}\right)\xi\right\} \quad . \quad . \quad . \quad . \quad . \quad (14.6)$$

The shear stresses to which a und b are due act on sections of the plate perpendicular to OY and OX respectively.

Let the resultant of these shear stresses acting on the sections of lengths dx and dy be denoted by $F_2\,dx$ and $F_1\,dy$ respectively. Thus since the shear stresses are na and nb, the shear forces per unit length are

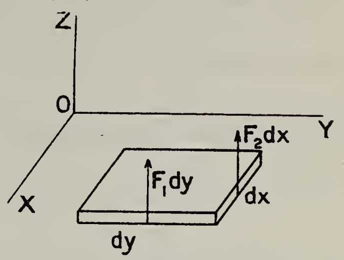


Fig. 128

$$F_1=\int_{-h}^{h} nb\,dz=-n\int_{-h}^{h}\frac{\partial}{\partial x}\left\{\left(1-\frac{z^2}{h^2}\right)\xi\right\}dz$$
$$=-n\frac{\partial}{\partial x}\int_{-h}^{h}\xi\left(1-\frac{z^2}{h^2}\right)dz$$
$$=-\frac{4}{3}n\frac{\partial(h\xi)}{\partial x}, \quad . \quad . \quad . \quad . \quad . \quad (14.7)$$

and
$$F_2=-\frac{4}{3}n\frac{\partial(h\xi)}{\partial y} \quad . \quad . \quad . \quad . \quad . \quad (14.8)$$

Again, assuming that the plate is in equilibrium, and writing
$$\nabla_1^2 \text{ for } \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right),$$
equations (2.28), (2.29), (2.30), give
$$-\varrho X=m\frac{\partial\Delta}{\partial x}+n\nabla^2 u$$
$$=\frac{\partial}{\partial x}\left[-(m+n)z\nabla_1^2(f+\xi)+(m+n)z^3\nabla_1^2\psi + 2mz\varphi+6nz\psi\right]$$
$$=\frac{\partial\eta}{\partial x} \text{ say} \quad . \quad . \quad . \quad . \quad . \quad (14.9)$$
$$-\varrho Y=\frac{\partial\eta}{\partial y} \quad . \quad . \quad . \quad . \quad . \quad (14.10)$$
$$-\varrho Z=m\frac{\partial\Delta}{\partial z}+n\nabla^2 w$$
$$=-(m-n)\nabla_1^2 f-m\nabla_1^2\xi+2(m+n)\varphi + z^2\nabla_1^2(3m\psi+n\varphi) \quad . \quad . \quad (14.11)$$

Now X and Y will each be zero or negligible for all values of z provided that

$$(m+n)\nabla_1^2(f+\xi) = 2m\varphi + 6n\psi, \quad . \quad . \quad . \quad (14.12)$$

and provided that $z^3\nabla_1\psi$ is either zero or negligible. We shall assume that this is negligible on account of the factor z^3. We shall also assume that equation (14.12) holds. Let us next assume that Z is zero at the middle surface of the plate. This gives

$$(m-n)\nabla_1^2 f = 2(m+n)\varphi - m\nabla_1^2\xi \quad . \quad . \quad . \quad (14.13)$$

and therefore

$$-\varrho Z = z^2\nabla_1^2(3m\psi + n\varphi) \quad . \quad . \quad . \quad . \quad . \quad (14.14)$$

The tensional stress parallel to the z-axis is, by (2.22),

$$P_3 = (m-n)\triangle + 2n\frac{\partial w}{\partial z}$$
$$= -(m-n)\nabla_1^2\{z(f+\xi) - z^3\psi\} + 2(m+n)z\varphi.$$

The values of P_3 at the two faces of the plate are

$$(P_3)_h = -(m-n)\{h\nabla_1^2(f+\xi) - h^3\nabla_1^2\psi\} + 2(m+n)h\varphi$$
$$(P_3)_{-h} = +(m-n)\{h\nabla_1^2(f+\xi) - h^3\nabla_1^2\psi\} - 2(m+n)h\varphi.$$

The whole external force on the plate in the z-direction, reckoned per unit area of the middle surface, is

$$p = (P_3)_h - (P_3)_{-h} + \int_{-h}^{h}\varrho Z dz$$
$$= -2(m-n)\{h\nabla_1^2(f+\xi) - h^3\nabla_1^2\psi\} + 4(m+n)h\varphi$$
$$-\tfrac{2}{3}h^3\nabla_1^2(3m\psi + n\varphi) \quad . \quad . \quad . \quad . \quad . \quad (14.15)$$

The elimination of f between (14.13) and (14.15) gives

$$p = 2nh\nabla_1^2\xi - \tfrac{2}{3}nh^3\nabla_1^2(3\psi+\varphi) \quad . \quad . \quad . \quad (14.16)$$

Equation (14.4) further reduces this to

$$p = \tfrac{4}{3}nh\nabla_1^2\xi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14.17)$$

Now by eliminating ψ from equation (14.4) and (14.12) we get

$$(m+n)\nabla_1^2 f = 2(m-n)\varphi + 2n\frac{\xi}{h^2} - (m+n)\nabla_1^2\xi.$$

Again, on eliminating φ from (14.13) and this last equation we get

$$4mn\nabla_1^2 f = \frac{2n(m+n)}{h^2}\xi - n(3m+n)\nabla_1^2\xi \quad . \quad . \quad (14.18)$$

Now we may neglect $h^2\nabla_1^2\xi$ in comparison with ξ, since this amounts to neglecting h^2 in comparison with x^2 or y^2, that is, neglecting the square of the thickness in comparison with the squares of the lateral dimensions of the plate. Then equation (14.18) becomes

$$n\xi = \frac{2mn}{m+n}h^2\nabla_1^2 f \quad . \quad . \quad . \quad . \quad . \quad . \quad (14.19)$$

Therefore, by (14.17),

$$p = \frac{2}{3}\frac{4mn}{m+n} h\nabla_1^2(h^2\nabla_1^2 f)$$

$$= \frac{2}{3}\frac{E}{1-\sigma^2} h^3 \nabla_1^4 f$$

$$= \frac{EI}{1-\sigma^2}\nabla_1^4 f$$

$$= \frac{EI}{1-\sigma^2}\left(\frac{\partial^4 f}{\partial x^4} + 2\frac{\partial^4 f}{\partial x^2 \partial y^2} + \frac{\partial^4 f}{\partial y^4}\right), \quad . \quad . \quad . \quad (14.20)$$

where I denotes the moment of inertia of unit length of a normal section of the plate about the line where the middle surface meets this section; that is, I is the moment of inertia of a rectangle of depth $2h$ and length unity, about the line midway between the sides of unit length.

The displacement, in the z-direction, of a particle in the middle surface is

$$w = f.$$

We shall, in the rest of this chapter, use w for this displacement. Then the equation for p can be written

$$p = \frac{EI}{1-\sigma^2}\nabla_1^4 w \quad . \quad . \quad . \quad . \quad . \quad (14.21)$$

This corresponds to the equation for the load per unit length on a beam, namely,

$$w = EI\frac{d^4\eta}{dx^4} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14.22)$$

It is important, however, to notice that equation (14.21) does not reduce exactly to (14.22) when the displacement w is a function of x only. If the plate has a small width b, and if its displacement is independent of y, equation (14.21) becomes

$$p = \frac{EI}{1-\sigma^2}\frac{d^4 w}{dx^4}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14.23)$$

from which

$$pb = \frac{EIb}{1-\sigma^2}\frac{d^4 w}{dx^4} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14.24)$$

Here pb is the load per unit length, and is therefore the same quantity as w in (14.22). Also Ib in this equation is the same as I in (14.22), and w is the same as η. Thus the difference between the two equations (14.23) and (14.24) is the factor $\frac{1}{1-\sigma^2}$. This is due to the fact that the displacement in the beam is not really a function of x only. It was shown in Art. 39 that the cross-section of a rectangular beam is distorted, the sides perpendicular to the load becoming circles with curvature $\sigma \frac{d^2\eta}{dx^2}$. Thus the beam theory is not derived from the plate theory merely by assuming that the displacement is a function of x only.

We have yet to get expressions for the remaining stresses in the plate. The most important stresses are P_1, P_2, and S_3.

It is now evident from the preceding work that the functions $\nabla_1{}^2 f, \varphi, \psi$, are quantities of the same order, and that $\nabla_1{}^2 \xi$ is a quantity of smaller order. We shall therefore neglect the last in comparison with the former in the expressions for the stresses. We shall also neglect terms containing powers of z beyond the first in the three stresses we are seeking.

Now

$$\begin{aligned} P_1 &= (m-n)\Delta + 2n\frac{\partial u}{\partial x} \\ &= -(m-n)\{z\nabla_1{}^2(f+\xi) - 2z\varphi - z^3\nabla_1{}^2\psi\} \\ &\quad - 2nz\frac{\partial^2}{\partial x^2}(f+\xi) + 2nz^3\frac{\partial^2\psi}{\partial x^2} \\ &= -z\left\{(m-n)\nabla_1{}^2 f + 2n\frac{\partial^2 f}{\partial x^2} - 2(m-n)\varphi\right\} \qquad (14.25) \end{aligned}$$

approximately.

By means of equations (14.13) and (14.19) this last equation becomes, when the small term with coefficient h^2 is neglected,

$$\begin{aligned} P_1 &= -z\left\{(m-n)\nabla_1{}^2 f + 2n\frac{\partial^2 f}{\partial x^2} - \frac{(m-n)^2}{m+n}\nabla_1{}^2 f\right\} \\ &= -2nz\left\{\frac{2m}{m+n}\frac{\partial^2 f}{\partial x^2} + \frac{m-n}{m+n}\frac{\partial^2 f}{\partial y^2}\right\} \\ &= -\frac{Ez}{1+\sigma}\left\{\frac{1}{1-\sigma}\frac{\partial^2 f}{\partial x^2} + \frac{\sigma}{1-\sigma}\frac{\partial^2 f}{\partial y^2}\right\} \\ &= -\frac{Ez}{1-\sigma^2}\left\{\frac{\partial^2 f}{\partial x^2} + \sigma\frac{\partial^2 f}{\partial y^2}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (14.26) \end{aligned}$$

Likewise

$$P_2 = -\frac{Ez}{1-\sigma^2}\left\{\frac{\partial^2 f}{\partial y^2} + \sigma\frac{\partial^2 f}{\partial x^2}\right\} \quad . \; . \; . \; . \; . \; . \; . \quad (14.27)$$

Again

$$\begin{aligned} S_3 &= n\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \\ &= n\frac{\partial^2}{\partial x \partial y}\{-2z(f+\xi) + z^3\psi\} \\ &= -2nz\frac{\partial^2 f}{\partial x \partial y} \quad . \; . \; . \; . \; . \; . \; , \; . \; . \; . \; . \; . \quad (14.28) \end{aligned}$$

approximately.

The factor $\frac{E}{1-\sigma^2}$ occurs so often that we shall write E′ for it in the rest of this chapter.

In terms of the displacement w of a point in the middle surface we can now write

$$\left.\begin{aligned} P_1 &= -E'z\left(\frac{\partial^2 w}{\partial x^2}+\sigma\frac{\partial^2 w}{\partial y^2}\right), \\ P_2 &= -E'z\left(\frac{\partial^2 w}{\partial y^2}+\sigma\frac{\partial^2 w}{\partial x^2}\right), \end{aligned}\right\} \quad \ldots \ldots \quad (14.29)$$

$$S_3 = -2nz\frac{\partial^2 w}{\partial x \partial y} = -\frac{E}{1+\sigma}z\frac{\partial^2 w}{\partial x \partial y}, \quad \ldots \quad (14.30)$$

$$\begin{aligned} F_1 &= -\frac{4}{3}nh\frac{\partial \xi}{\partial x} = -\frac{8}{3}\frac{mn}{m+n}h^3\frac{\partial}{\partial x}(\nabla_1{}^2 w) \\ &= -\frac{2}{3}E'h^3\frac{\partial}{\partial x}(\nabla_1{}^2 w) \end{aligned}$$

$$\left.\begin{aligned} &= -E'I\frac{\partial}{\partial x}(\nabla_1{}^2 w), \\ F_2 &= -E'I\frac{\partial}{\partial y}(\nabla_1{}^2 w). \end{aligned}\right\} \quad \ldots \ldots \quad (14.31)$$

Each of the first three of these stresses is zero at the middle surface and has different signs on opposite sides of this surface. The stresses P_1, like the tensional stresses in a bent beam, are equivalent to a couple M_1 per unit length of a section perpendicular to the x-axis, the moment of this couple being given by

$$\begin{aligned} M_1 &= -\int_{-h}^{h} P_1 z\,dz \\ &= \frac{2}{3}E'h^3\left(\frac{\partial^2 w}{\partial x^2}+\sigma\frac{\partial^2 w}{\partial y^2}\right) \\ &= E'I\left(\frac{\partial^2 w}{\partial x^2}+\sigma\frac{\partial^2 w}{\partial y^2}\right) \end{aligned} \quad \ldots \ldots \quad (14.32)$$

The stresses P_2 are equivalent to a similar couple M_2 on the sections perpendicular to the y-axis. Thus

$$M_2 = E'I\left(\frac{\partial^2 w}{\partial y^2}+\sigma\frac{\partial^2 w}{\partial x^2}\right) \quad \ldots \ldots \quad (14.33)$$

Again the shear stresses S_3 are equivalent to couples acting on the same two sections, the axes of the couples being the normals to the sections.

The shear forces are shown in fig. 130 on the assumption that $\frac{\partial^2 w}{\partial x \partial y}$ is positive. The couple, Q per unit length, due to these shear stresses, is given by

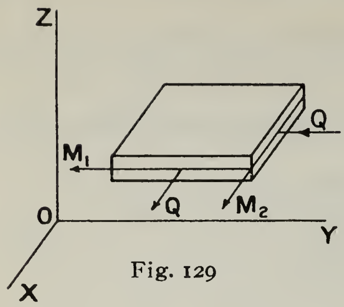
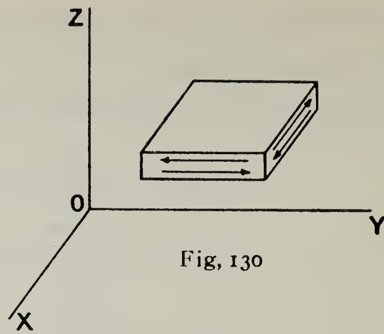


Fig. 129



Fig, 130

$$Q = -\int_{-h}^{h} z S_3 \, dz$$

$$= \tfrac{2}{3}(1-\sigma)E'h^3 \frac{\partial^2 w}{\partial x \partial y}$$

$$= (1-\sigma)E'I \frac{\partial^2 w}{\partial x \partial y} \quad . \; . \; . \; . \; . \; . \; (14.34)$$

The couples M_1, M_2, Q, are represented in fig. 129 by vectors on the right-handed screw system. The couples M_1 and M_2 are the elastic resistances to bending, and the couples Q are the resistances to torsion. We shall call them bending moments and torques.

246. Plate of variable thickness.

It is not easy to adapt the preceding theory to a plate in which the thickness is variable, but if we make some reasonable assumptions it is not difficult to deduce the correct results for plates in which the thickness varies so slowly that the faces of the plate are everywhere nearly parallel. Suppose we borrow from the preceding results the equations

$$F_1 = -\frac{4}{3} n \frac{\partial (h\xi)}{\partial x},$$

$$F_2 = -\frac{4}{3} n \frac{\partial (h\xi)}{\partial y},$$

and

$$n\xi = \frac{2mn}{m+n} h^2 \nabla_1{}^2 w,$$

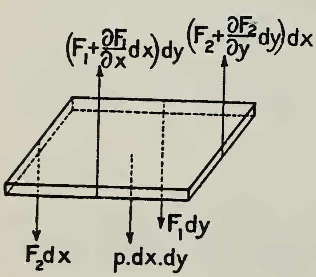


Fig. 131

h being now a function of x and y.

Then, by resolving in the direction of the z-axis the forces acting on the element of area $dxdy$ of the plate, we get

$$\left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}\right) dxdy + p\,dxdy = 0;$$

that is,
$$p = \tfrac{4}{3} n \nabla_1{}^2 (h\xi)$$
$$= \frac{8}{3} \frac{mn}{m+n} \nabla_1{}^2 (h^3 \nabla_1{}^2 f)$$
$$= \mathrm{E}' \nabla_1{}^2 (\mathrm{I} \nabla_1{}^2 w) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14.35)$$

In terms of w the equations for F_1, F_2, are

$$\left.\begin{aligned} \mathrm{F}_1 &= \mathrm{E}' \frac{\partial}{\partial x} (\mathrm{I} \nabla_1{}^2 w) \\ \mathrm{F}_2 &= \mathrm{E}' \frac{\partial}{\partial y} (\mathrm{I} \nabla_1{}^2 w) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (14.36)$$

Equations (14.32), (14.33), and (14.34), giving the couples M_1, M_2, and Q, remain unaltered.

247. The boundary conditions.

Suppose w_1 is any particular integral of (14.21); that is, suppose that w_1 is a function of x and y such that

$$p = \mathrm{E}' \mathrm{I} \nabla_1{}^4 w_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14.37)$$

Then, by subtraction of (14.37) from (14.21), we get

$$\mathrm{E}' \mathrm{I} (\nabla_1{}^4 w - \nabla_1{}^4 w_1) = 0,$$

that is,

$$\nabla_1{}^4 (w - w_1) = 0,$$

for it is easy to see, from the meaning of $\nabla_1{}^4$, that

$$\nabla_1{}^4 w - \nabla_1{}^4 w_1 = \nabla_1{}^4 (w - w_1).$$

Thus, writing φ for $(w - w_1)$, we find that the complete solution of (14.21) is

$$w = w_1 + \varphi,$$

where φ is the complete solution of

$$\nabla_1{}^4 \varphi = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14.38)$$

We have already shown (Chap. 13) that the complete solution of (14.38) contains two arbitrary functions. By means of two such functions we can satisfy two, and only two, independent conditions at the boundary of a plate. But in order to satisfy completely the boundary conditions at the edge of a plate where no external forces are applied it would be necessary to make the bending moment M, the shearing force F, and the torque Q, all vanish along the edge. That is, if the boundary is perpendicular to the axis of x, it would be necessary to satisfy the following conditions along the edge:

$$\frac{\partial^2 w}{\partial x^2} + \sigma \frac{\partial^2 w}{\partial y^2} = 0,$$
$$\frac{\partial}{\partial x}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) = 0,$$
$$\frac{\partial^2 w}{\partial x \partial y} = 0.$$

But to satisfy three such conditions all round the boundary of a plate we should require three arbitrary functions in w and we have only two at our disposal. Thus there is something incomplete in our solution. Poisson, who attacked the problem of the thin plate by a method resembling the one used in this chapter, imagined that it was possible to satisfy all three boundary conditions. He did not discover the difficulty because it did not arise in the particular problems he worked out. Kirchhoff, however, by attacking the problem in a different way, arrived at the correct number of boundary conditions. Kirchhoff's method was to make the potential energy of the bent plate and the applied forces a minimum, and his boundary conditions, as well as the differential equation (14.21), arise from the conditions for a minimum. Thus his boundary conditions were at least consistent with his initial assumptions concerning the energy in the bent plate.

One great weakness in the foregoing theory is the assumption that, in dealing with strains and stresses, z is negligible in comparison with the lateral dimensions of the plate; for this is clearly not valid at points near the edge of the plate. In order to correct our equations we should need one more arbitrary function in the expression for the stresses, but the addition that this function would make to the stresses would be negligible at all points not near the edge of the plate. If, then, a fictitious boundary be taken parallel to the real boundary and quite near it, the stresses at this fictitious boundary are expressed accurately enough by our equations. But the conditions at the two boundaries are not the same.

We shall deduce the boundary conditions at this fictitious boundary by the method given by Professor Lamb in a paper read before the London Mathematical Society in 1889.

Let AB be an element of the edge of length ds; and let AA′, BB′ be short lengths along the normals to the edge such that A′B′ is a portion of the fictitious boundary which is parallel to the real boundary.

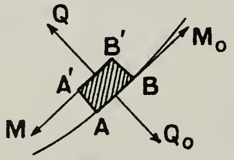


Fig. 132

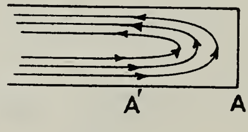


Fig. 133

Let F_0 be the shear force on unit length of the real edge, and let M_0, Q_0, denote the component couples applied to unit length of the real edge about the tangent and the outward normal respectively, the vector representing M_0 being in the direction in which ds is positive. Now the shear lines on such a section as AA′ are as indicated in fig. 133, just as in a thin strip under pure torsion. The shear lines which, in the body of the plate, are nearly parallel to the faces, turn through 180^0 in the neighbourhood of A′A. The component shear stress perpendicular to the faces of the plate on the element of area A′A is of a higher order of magnitude than the shear stress in the same direction in the body of the plate. Consequently the total shear force perpendicular to the middle surface on the small section A′A (which may be only two or three times the thickness of the plate) is a force of the same order as the shear force on unit length in the body of the plate

Let R denote the total shear force in the z direction on A′A, $R+\frac{\partial R}{\partial s}ds$ the corresponding shear force on B′B, and let the length of A′A be denoted by b. Let F, M, Q, be the actions at the fictitious boundary corresponding to F_0, M_0, Q_0 on the real boundary.

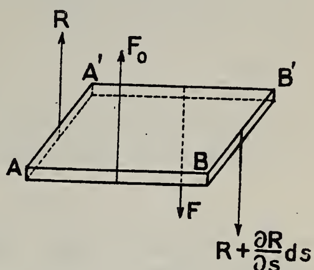


Fig. 134

Resolving in the direction of z for the equilibrium of the element A′ABB′ (fig. 134) we get

$$ds(F_0-F)-\frac{\partial R}{\partial s}ds=0$$

whence

$$F_0-F-\frac{\partial R}{\partial s}=0. \qquad (14.39)$$

Again, by taking moments about AB and A′A respectively, we get

$$(M_0-M)ds-Fdsb=0,$$
$$(Q_0-Q)ds-Rds=0,$$

that is, since b is very small,

$$M_0-M=0. \qquad (14.40)$$
$$Q_0-Q=R. \qquad (14.41)$$

Eliminating R from (14.39) and (14.41) we get

$$\frac{\partial Q_0}{\partial s}-\frac{\partial Q}{\partial s}=\frac{\partial R}{\partial s}$$
$$=F_0-F;$$

whence

$$\frac{\partial Q_0}{\partial s}-F_0=\frac{\partial Q}{\partial s}-F \qquad (14.42)$$

Thus when the real boundary is not fixed in any way the conditions at the fictitious boundary, within which all our earlier equations are valid, are the conditions expressed by (14.40) and (14.42).

The conditions at the boundary of a plate which rests on supports or which is clamped at the edge are similar to the conditions in a beam similarly supported.

In order to be able to specify distinctly the boundary conditions we shall make use of a pair of elements of length at the boundary of the middle surface of the plate, these elements being in the tangent plane to the middle surface at that point; one element dr is drawn along the outward normal to the boundary (or edge-line) of the middle surface, and the second ds touches the edge, and its direction is so chosen that dr, ds, and the axis OZ form a right handed screw system of axes.

When the plate is clamped so that the whole of the edge line is in the xy plane, and this plane is the tangent plane to the middle surface at every point of the edge line, then the boundary conditions are

$$w = 0, \quad \frac{\partial w}{\partial r} = 0,$$

at all points of the edge. These two conditions are equivalent to the following three

$$w = 0, \quad \frac{\partial w}{\partial x} = 0, \quad \frac{\partial w}{\partial y} = 0, \quad . \quad . \quad . \quad . \quad . \quad (14.43)$$

the first of which is hardly necessary since the last two make w constant along the boundary.

If the plate is merely supported with the whole of the edge line in the xy plane the boundary conditions are

$$w = 0, \quad \frac{\partial^2 w}{\partial r^2} + \sigma \frac{\partial^2 w}{\partial s^2} = 0, \quad . \quad . \quad . \quad . \quad . \quad (14.44)$$

the second of these conditions expressing the fact that the bending moment about the edge is zero.

If the boundary is subject to a given shearing force F_0 per unit length, and given couples $M_0\, ds$ and $Q_0\, dr$ about ds and dr respectively, the boundary conditions are

$$\left.\begin{aligned} M &= M_0 \\ F - \frac{\partial Q}{\partial s} &= F_0 - \frac{\partial Q_0}{\partial s} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (14.45)$$

In case there are no external forces on the edge the boundary conditions are, of course, obtained by putting M_0, Q_0, F_0, all zero. Then

$$M = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14.46)$$

$$F - \frac{\partial Q}{\partial s} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14.47)$$

248. Formulae for circular plates.

In dealing with circular plates we need to express all the stresses and strains in terms of polar coordinates. We shall now make the necessary transformations.

Let $$x = r\cos\theta, \quad y = r\sin\theta \quad . \quad . \quad . \quad . \quad . \quad (14.48)$$

and suppose

$$w = f(x, y). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14.49)$$

Then w is also a function of r and θ. Therefore, by (13.52),

$$\frac{\partial w}{\partial x} = \cos\theta \frac{\partial w}{\partial r} - \frac{\sin\theta}{r}\frac{\partial w}{\partial \theta}, \quad . \quad . \quad . \quad . \quad . \quad (14.50)$$

and

$$\frac{\partial w}{\partial y} = \sin\theta \frac{\partial w}{\partial r} + \frac{\cos\theta}{r}\frac{\partial w}{\partial \theta}. \quad . \quad . \quad . \quad . \quad . \quad (14.51)$$

Again, writing ξ for $\frac{\partial w}{\partial x}$, we find

$$\begin{aligned} \frac{\partial^2 w}{\partial x^2} &= \frac{\partial \xi}{\partial x} \\ &= \cos\theta \frac{\partial \xi}{\partial r} - \frac{\sin\theta}{r}\frac{\partial \xi}{\partial \theta} \\ &= \cos\theta \left\{ \cos\theta \frac{\partial^2 w}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial w}{\partial \theta} - \frac{\sin\theta}{r}\frac{\partial^2 w}{\partial r \partial \theta} \right\} \\ &\quad - \frac{\sin\theta}{r}\left\{ -\sin\theta \frac{\partial w}{\partial r} + \cos\theta \frac{\partial^2 w}{\partial r \partial \theta} - \frac{\cos\theta}{r}\frac{\partial w}{\partial \theta} - \frac{\sin\theta}{r}\frac{\partial^2 w}{\partial \theta^2} \right\} \quad . \quad (14.52) \end{aligned}$$

Let c_1, c_2, denote the curvatures of the sections of the surface represented by (14.49) in the direction of the radius vector r and perpendicular to this direction respectively.

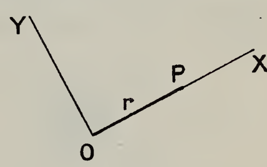


Fig. 135

Then c_1 is the value of $\frac{\partial^2 w}{\partial x^2}$ when the axis of x coincides with the direction of r, as shown in fig. 135; that is, when $\theta = 0$. Thus

$$c_1 = \frac{\partial^2 w}{\partial r^2} \quad . \quad . \quad . \quad (14.53)$$

Likewise c_2 is the value of $\frac{\partial^2 w}{\partial x^2}$ when the x-axis is perpendieular to the radius vector; that is, when $\theta = \frac{\pi}{2}$. Thus

$$c_2 = \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (14.54)$$

Therefore

$$c_1 + c_2 = \frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}. \quad . \quad . \quad . \quad . \quad (14.55)$$

Now $c_1 + c_2$ is the value of $\nabla_1{}^2 w$ for one position of the axes of x and y, and it can be proved that the value of this expression is unaltered by rotating the axes of x and y about OZ, always keeping them perpendicular to each other. In solid geometry this fact is included in the theorem that the sum of the curvatures of two perpendicular normal sections of a surface at any point is constant for all pairs of such perpendicular sections at that point. Therefore

$$\nabla_1{}^2 w = \frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} \quad . \quad . \quad . \quad . \quad (14.56)$$

If M_1 denotes the bending moment on a section perpendicular to r then

$$M_1 = E'I\left\{\frac{\partial^2 w}{\partial r^2} + \sigma\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right)\right\} \quad . \quad . \quad . \quad (14.57)$$

Also the bending moment on the section perpendicular to this last one is

$$M_2 = E'I\left\{\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} + \sigma\frac{\partial^2 w}{\partial r^2}\right\} \quad . \quad . \quad . \quad . \quad (14.58)$$

The shear forces per unit length in the direction of the z-axis on the same two sections are

$$F_1 = -\frac{2}{3}\frac{Eh^3}{1-\sigma^2}\frac{\partial}{\partial r}(\nabla_1{}^2 w) = -E'I\frac{\partial}{\partial r}(\nabla_1{}^2 w), \quad . \quad (14.59)$$

$$F_2 = -E'I\frac{1}{r}\frac{\partial}{\partial \theta}(\nabla_1{}^2 w) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1460)$$

To find the expression for the torque on the same sections we need $\dfrac{\partial^2 w}{\partial x \partial y}$ in polar coordinates. This has already been found in Art. 227.

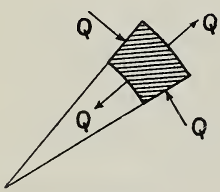


Fig. 136

When $\theta = 0$ we find from (13.56)

$$\frac{\partial^2 w}{\partial x \partial y} = \frac{1}{r}\frac{\partial^2 w}{\partial r \partial \theta} - \frac{1}{r^2}\frac{\partial w}{\partial \theta}$$

$$= \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w}{\partial \theta}\right) \quad . \quad . \quad . \quad (14.61)$$

Therefore the torque on the sections respectively perpendicular and parallel to the radius vector is

$$Q = (1-\sigma)E'I\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w}{\partial \theta}\right) \quad . \quad . \quad . \quad . \quad . \quad (14.62)$$

249. Symmetry about the z-axis.

If everything is symmetrical about the z-axis then w is not a function of θ. In that case

$$\nabla_1{}^2 w = \frac{d^2 w}{dr^2} + \frac{1}{r}\frac{dw}{dr}$$

$$= \frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14.63)$$

$$M_1 = E'I\left\{\frac{d^2w}{dr^2} + \frac{\sigma}{r}\frac{dw}{dr}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14.64)$$

$$M_2 = E'I\left\{\frac{1}{r}\frac{dw}{dr} + \sigma\frac{d^2w}{dr^2}\right\}. \quad . \quad . \quad . \quad . \quad . \quad (14.65)$$

$$F_1 = -E'I\frac{d}{dr}\left\{\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)\right\}. \quad . \quad . \quad . \quad . \quad (14.66)$$

$$F_2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14.67)$$

$$Q = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14.68)$$

Also the equation connecting p and w is

$$p = E'I\nabla_1{}^2(\nabla_1{}^2 w) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14.69)$$

where

$$\nabla_1{}^2 = \frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\right).$$

250. Circular disk under uniform pressure.

Suppose p is uniform and acts downwards and suppose that the forces are such that everything is symmetrical about the axis. Let a denote the radius of the disk. We must measure w downwards since p and w are reckoned positive in the same direction in our theory. When ψ is written for $\nabla_1{}^2 w$ equation (14.69) becomes

$$E'I\frac{1}{r}\frac{d}{dr}\left(r\frac{d\psi}{dr}\right) = p, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14.70)$$

whence

$$E'Ir\frac{d\psi}{dr} = \tfrac{1}{2}pr^2 + A, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14.71)$$

that is,

$$-2hF_1 = \tfrac{1}{2}pr + \frac{A}{r} \quad . \quad . \quad . \quad , \quad . \quad . \quad . \quad (14.72)$$

Integrating again we get

$$E'I\psi = \tfrac{1}{4}pr^2 + A\log r + B \quad . \quad . \quad . \quad (14.73)$$

that is,

$$E'I\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right) = \tfrac{1}{4}pr^2 + A\log r + B, \quad . \quad . \quad . \quad (14.74)$$

The steps in the integration of this are

$$E'Ir\frac{dw}{dr} = \tfrac{1}{16}pr^4 + \tfrac{1}{2}Ar^2(\log r - \tfrac{1}{2}) + \tfrac{1}{2}Br^2 + C \quad . \quad . \quad . \quad (14.75)$$

$$E'Iw = \tfrac{1}{64}pr^4 + \tfrac{1}{4}Ar^2(\log_e r - 1) + \tfrac{1}{4}Br^2 + C\log_e r + D \quad (14.76)$$

If the disk has no central hole C must be zero, for if it were not zero w would be infinite at the centre. Also A must be zero in order that F_1 may not be infinite at the centre.

Thus for a complete disk

$$E'Iw = \tfrac{1}{64}pr^4 + \tfrac{1}{4}Br^2 + D \quad . \quad . \quad . \quad . \quad . \quad (14.77)$$

The constant B depends on the method of supporting the disk and the constant D depends on the level from which w is measured. The constant D is not involved in the stresses, and is therefore unimportant. We shall now complete the solution for different methods of support.

· *Disk with its whole rim supported at the same level.*

Fig. 137

If w is measured from the level of the centre the boundary conditions here are clearly

$$w = 0 \text{ where } r = 0,$$
$$M_1 = 0 \text{ where } r = a.$$

These give

$$D = 0,$$

and

$$\tfrac{3}{16}pa^2 + \tfrac{1}{2}B + \sigma(\tfrac{1}{16}pa^2 + \tfrac{1}{2}B) = 0.$$

Thus

$$B = -\frac{3+\sigma}{8(1+\sigma)}pa^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (14.78)$$

Then finally

$$E'Iw = \tfrac{1}{64}pr^4 - \frac{1}{32}\frac{3+\sigma}{1+\sigma}pa^2r^2$$
$$= \tfrac{1}{64}pr^2\left\{r^2 - \frac{2(3+\sigma)}{1+\sigma}a^2\right\}. \quad . \quad . \quad . \quad . \quad (14.79)$$

Therefore

$$M_1 = -\tfrac{1}{16}p(3+\sigma)(a^2 - r^2), \quad . \quad . \quad . \quad . \quad . \quad (14.80)$$

and

$$M_2 = E'I\left\{\frac{1}{r}\frac{dw}{dr} + \sigma\frac{d^2w}{dr^2}\right\}$$
$$= -\tfrac{1}{16}p\{(3+\sigma)a^2 - (3\sigma+1)r^2\}. \quad . \quad . \quad (14.81)$$

The maximum magnitudes of these bending moments are equal and occur at the centre of the disk. The stresses P_1 and P_2 which result in M_1 and M_2 are connected with the bending moments by the equations

$$P_1 = \frac{z}{I}M_1 = \frac{3z}{2h^3}M_1,$$

$$P_2 = \frac{z}{I}M_2 = \frac{3z}{2h^3}M_2.$$

The maximum tensional stress in the disk occurs at the centre of one face and its value is

$$f = P_1 = P_2 = \frac{3h}{2h^3} \times \tfrac{1}{16}p(3+\sigma)a^2$$
$$= \frac{3a^2}{32h^2}p(3+\sigma)$$
$$= \frac{3a^2}{8t^2}p(3+\sigma) \quad . \quad . \quad . \quad . \quad . \quad . \quad (14.82)$$

t being the thickness of the plate.

Clamped Disk.

If the disk is clamped horizontally all round the edge the conditions are similar to those for a beam. They are

Fig. 138

$$w = 0 \text{ where } r = 0,$$
$$\frac{dw}{dr} = 0 \text{ where } r = a.$$

These give

$$D = 0,$$
$$\tfrac{1}{16}pa^3 + \tfrac{1}{2}Ba = 0,$$

Therefore

$$B = -\tfrac{1}{8}pa^2,$$

and

$$E'Iw = \tfrac{1}{64}p(r^4 - 2a^2r^2). \quad . \quad . \quad . \quad . \quad (14.83)$$

The stresses P_1 and P_2 can be found exactly as for the supported disk. These are always equal at the centre of a complete disk. In this case, however, the maximum stress is the radial stress P_1 at the rim, and its value is

$$P_1 = \frac{3a^2}{4t^2}p \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14.84)$$

251. Uniform pressure over a circle concentric with the disk.

Suppose now that the constant pressure p is applied to a disk of radius a over a circle of radius b concentric with a face of the disk, and that the ring between the two circles is free from pressure. We shall work out the two cases (I) where the disk is supported at the rim, and (II), where the disk is clamped at the rim.

Disk supported at the rim.

The form of the expression for w is different in the two portions of the disk. Let w denote the deflexion for the inner circle, and w_1 the deflexion for the outer ring. For the inner portion, where p acts, we have, measuring w from the level of the middle,

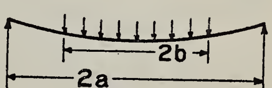


Fig. 139

$$E'Iw = \tfrac{1}{64}pr^4 + \tfrac{1}{4}Br^2. \quad . \quad . \quad . \quad . \quad . \quad (14.85)$$

For the outer ring, where no pressure acts, the equation for w_1 is

$$E'Iw_1 = \tfrac{1}{4}A_1r^2(\log_e r - 1) + \tfrac{1}{4}B_1r^2 + C_1\log_e r + D_1 \quad . \quad . \quad (14.86)$$

The constants are determined from the boundary conditions together with the conditions required for continuity at the junction of the two portions of the disk. The conditions at the junction are that w, $\frac{dw}{dr}$, M_1 and F_1, have the same values at the junction for the two portions. These conditions are clearly satisfied if

$$\left.\begin{aligned} w &= w_1 \\ \frac{dw}{dr} &= \frac{dw_1}{dr} \\ \frac{d^2w}{dr^2} &= \frac{d^2w_1}{dr^2} \\ \frac{d^3w}{dr^3} &= \frac{d^3w_1}{dr^3} \end{aligned}\right\} \text{ where } r = b.$$

The boundary condition at the supported edge is

$$M_1 = 0 \text{ where } r = a,$$

that is,

$$\frac{d^2w_1}{dr^2} + \frac{\sigma}{r}\frac{dw_1}{dr} = 0 \text{ where } r = a.$$

These five conditions determine the five constants. The five conditions lead to the following equations.

$$\tfrac{1}{64}pb^4 + \tfrac{1}{4}Bb^2 = \tfrac{1}{4}A_1b^2(\log b - 1) + \tfrac{1}{4}B_1b^2 + C_1\log b + D_1 \quad . \quad (14.87)$$

$$\tfrac{1}{16}pb^3 + \tfrac{1}{2}Bb = \tfrac{1}{2}A_1b(\log b - \tfrac{1}{2}) + \tfrac{1}{2}B_1b + \frac{C_1}{b}, \quad . \quad . \quad (14.88)$$

$$\tfrac{3}{16}pb^2 + \tfrac{1}{2}B = \tfrac{1}{2}A_1\log b + \tfrac{1}{4}A_1 + \tfrac{1}{2}B_1 - \frac{C_1}{b^2}, \quad . \quad . \quad (14.89)$$

$$\tfrac{3}{8}pb = \frac{1}{2}\frac{A_1}{b} + 2\frac{C_1}{b^3}, \quad . \quad . \quad (14.90)$$

$$\tfrac{1}{2}A_1\log a + \tfrac{1}{4}A_1 + \tfrac{1}{2}B_1 - \frac{C_1}{a^2} + \sigma\left\{\tfrac{1}{2}A_1(\log a - \tfrac{1}{2}) + \tfrac{1}{2}B_1 + \frac{C_1}{a^2}\right\} = 0. \quad (14.91)$$

From (14.88) and (14.89) we get

$$\tfrac{1}{8}pb^3 = \tfrac{1}{2}A_1b - \frac{2C_1}{b}. \quad . \quad . \quad (14.92)$$

Equations (14.90) and (14.92) give

$$A_1 = \tfrac{1}{2}pb^2, \quad . \quad . \quad (14.93)$$

$$C_1 = \tfrac{1}{16}pb^4. \quad . \quad . \quad (14.94)$$

Now (14.89) gives

$$B_1 - B = \tfrac{3}{8}pb^2 - A_1(\log b + \tfrac{1}{2}) + \frac{2C_1}{b^2}$$

$$= \tfrac{1}{2}pb^2(\tfrac{1}{2} - \log b). \quad \text{(14.95)}$$

Also from (14.91)

$$\tfrac{1}{2}(1+\sigma)B_1 = -\tfrac{1}{2}A_1(\log a + \tfrac{1}{2}) - \tfrac{1}{2}\sigma A_1(\log a - \tfrac{1}{2}) + (1-\sigma)\frac{C_1}{a^2}$$

$$= -\tfrac{1}{4}pb^2\{(1+\sigma)\log a + \tfrac{1}{2}(1-\sigma)\} + \tfrac{1}{16}(1-\sigma)p\frac{b^4}{a^2}.$$

Therefore

$$B_1 = -\tfrac{1}{2}pb^2\left\{\log a + \frac{1}{2}\frac{1-\sigma}{1+\sigma}\right\} + \frac{1}{8}\frac{1-\sigma}{1+\sigma}p\frac{b^4}{a^2}. \quad \text{(14.96)}$$

From (14.95) and (14.96)

$$B = -\tfrac{1}{2}pb^2\left\{\log\frac{a}{b} + \frac{1}{1+\sigma} - \frac{1}{4}\frac{1-\sigma}{1+\sigma}\frac{b^2}{a^2}\right\}.$$

Thus

$$E'Iw = \tfrac{1}{64}pr^4 - \tfrac{1}{8}pb^2r^2\left\{\log\frac{a}{b} + \frac{1}{1+\sigma} - \frac{1}{4}\frac{1-\sigma}{1+\sigma}\frac{b^2}{a^2}\right\} \quad \text{(14.97)}$$

and

$$E'Iw_1 = D_1 + \tfrac{1}{8}pb^2r^2(\log r - 1) + \tfrac{1}{16}pb^4\log r$$

$$- \tfrac{1}{8}pb^2r^2\left\{\log a + \frac{1}{2}\frac{1-\sigma}{1+\sigma} - \frac{1}{4}\frac{1-\sigma}{1+\sigma}\frac{b^2}{a^2}\right\}$$

$$= D_1 + \tfrac{1}{8}pb^2r^2\left\{\log\frac{r}{a} - \frac{1}{2}\frac{3+\sigma}{1+\sigma} + \frac{1}{4}\frac{1-\sigma}{1+\sigma}\frac{b^2}{a^2}\right\} + \tfrac{1}{16}pb^4\log r$$

$$= K - \tfrac{1}{8}pb^2r^2\left\{\log_e\frac{a}{r} + \frac{1}{2}\frac{3+\sigma}{1+\sigma} - \frac{1}{4}\frac{1-\sigma}{1+\sigma}\frac{b^2}{a^2}\right\} - \tfrac{1}{16}pb^4\log_e\frac{a}{r} \quad \text{(14.98)}$$

where

$$K = D_1 + \tfrac{1}{16}pb^4\log_e a. \quad \text{(14.99)}$$

Disk clamped at the rim.

The only difference between this case and the last is that the condition at the rim is

$$\frac{dw}{dr} = 0 \text{ where } r = a \quad \text{(14.100)}$$

This replaces (14.91) in the last problem.

Fig. 140

All the equations as far as (14.95) are true for this problem as well as for the last. Now equation (14.100) takes the form

$$0 = \tfrac{1}{2}A_1 a(\log a - \tfrac{1}{2}) + \tfrac{1}{2}B_1 a + \frac{C_1}{a},$$

that is,

$$B_1 = -A_1(\log a - \tfrac{1}{2}) - \frac{2C_1}{a^2}$$

$$= -\tfrac{1}{2}pb^2(\log a - \tfrac{1}{2}) - \tfrac{1}{8}p\frac{b^4}{a^2}. \quad . \quad , \quad . \quad . \quad (14.101)$$

Equations (14.95) and (14.101) give

$$B = -\tfrac{1}{2}pb^2\log\frac{a}{b} - \tfrac{1}{8}p\frac{b^4}{a^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (14.102)$$

Therefore

$$E'Iw = \tfrac{1}{64}pr^4 - \tfrac{1}{32}pb^2r^2\left\{4\log_e\frac{a}{b} + \frac{b^2}{a^2}\right\}, \quad . \quad . \quad . \quad . \quad . \quad (14.103)$$

and

$$E'Iw_1 = D_1 + \tfrac{1}{8}pb^2r^2(\log r - 1) + \tfrac{1}{16}pb^4\log r$$

$$-\tfrac{1}{8}pb^2r^2\left\{\log a - \frac{1}{2} + \frac{1}{4}\frac{b^2}{a^2}\right\}$$

$$= K - \tfrac{1}{8}pb^2r^2\left\{\log_e\frac{a}{r} + \frac{1}{2} + \frac{1}{4}\frac{b^2}{a^2}\right\} - \tfrac{1}{16}pb^4\log_e\frac{a}{r} \quad (14.104)$$

where, again,

$$K = D_1 + \tfrac{1}{16}pb^4\log_e a.$$

252. Load W on a very small area at the centre.

The total load in each of the last two problems is

$$W = \pi pb^2$$

If we write $\frac{W}{\pi}$ for pb^2 in the results we have just got and then make b zero in the rest the new results will be true for the load W concentrated at the middle of the disk. For a concentrated load the deflexion is w_1, since w is the deflexion at one point only.

The results are as follows:—

for the supported disk

$$E'Iw_1 = K - \frac{Wr^2}{8\pi}\left\{\log_e\frac{a}{r} + \frac{1}{2}\frac{3+\sigma}{1+\sigma}\right\}; \quad . \quad . \quad . \quad (14.105)$$

and for the clamped disk

$$E'Iw_1 = K - \frac{Wr^2}{8\pi}\left\{\log_e\frac{a}{r} + \frac{1}{2}\right\}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (14.106)$$

It should be noticed that the bending moments M_1 and M_2 are both infinite at the centre for both the cases of the concentrated load. This, however, is due to the assumption that a finite load can be concentrated at a point. Since the load must always be distributed over an area of some size more correct for values of M_1 and M_2 are obtained from the assumption that the load is distributed over a small circle of radius b.

253. Disk under several loads.

Since the differential equation for w is linear it follows that the deflexion due to several loads acting together is equal to the sum of the deflexions due to each load acting separately. Thus if w_1, w_2 are the deflexions due to pressures p_1, p_2, acting separately,

$$E'I\nabla_1{}^4 w_1 = p_1$$
$$E'I\nabla_1{}^4 w_2 = p_2.$$

By adding corresponding sides of these we get

$$E'I\nabla_1{}^4(w_1 + w_2) = p_1 + p_2. \quad . \quad . \quad . \quad . \quad (14.107)$$

But if w is the deflexion due to p_1 and p_2 acting together

$$E'I\nabla_1{}^4 w = p_1 + p_2. \quad . \quad . \quad . \quad . \quad (14.108)$$

It follows from (14.107) and (14.108), and from the boundary conditions, that

$$w = w_1 + w_2$$

provided that the disk is free, or supported, or clamped at the edge.

254. Load distributed uniformly on a ring between two circles concentric with the disk.

The deflexion for this case can be deduced at once by taking the difference of the deflexions due to uniform loads over complete circles whose circumferences coincide with the inner and outer boundaries of the ring. We shall denote the radii of the inner and outer boundaries by b and c.

Supported disk. By means of equation (14.98) we find that, where r is greater than c, the deflexion in this case is given by the equation

$$E'Iw_1 = K_1 - \tfrac{1}{8}pc^2r^2\left\{\log_e\frac{a}{r} + \frac{1}{2}\,\frac{3+\sigma}{1+\sigma} - \frac{1}{4}\,\frac{1-\sigma}{1+\sigma}\,\frac{c^2}{a^2}\right\} - \tfrac{1}{16}pc^4\log_e\frac{a}{r}$$
$$+ \tfrac{1}{8}pb^2r^2\left\{\log_e\frac{a}{r} + \frac{1}{2}\,\frac{3+\sigma}{1+\sigma} - \frac{1}{4}\,\frac{1-\sigma}{1+\sigma}\,\frac{b^2}{a^2}\right\} + \tfrac{1}{16}pb^4\log_e\frac{a}{r}. \quad (14.109)$$

If we now write W for the total load $\pi(c^2-b^2)p$, the preceding equation becomes

$$E'Iw_1 = K_1 - \frac{Wr^2}{8\pi}\left\{\log_e\frac{a}{r} + \frac{1}{2}\,\frac{3+\sigma}{1+\sigma}\right\}$$
$$+ \frac{W}{32\pi}(b^2+c^2)\left\{\frac{1-\sigma}{1+\sigma}\,\frac{r^2}{a^2} - 2\log_e\frac{a}{r}\right\}. \quad . \quad (14.110)$$

Also by equation (14.97) the deflexion where r is less than b is given by the equation

$$\mathrm{E'I}w = -\tfrac{1}{8}pc^2r^2\left\{\log_e\frac{a}{c} + \frac{1}{1+\sigma} - \frac{1}{4}\frac{1-\sigma}{1+\sigma}\frac{c^2}{a^2}\right\}$$

$$+\tfrac{1}{8}pb^2r^2\left\{\log_e\frac{a}{b} + \frac{1}{1+\sigma} - \frac{1}{4}\frac{1-\sigma}{1+\sigma}\frac{b^2}{a^2}\right\}$$

$$= -\frac{\mathrm{W}r^2}{8\pi}\left\{\frac{1}{1+\sigma} - \frac{1}{4}\frac{1-\sigma}{1+\sigma}\frac{c^2+b^2}{a^2} + \frac{1}{c^2-b^2}\left(c^2\log_e\frac{a}{c} - b^2\log_e\frac{a}{b}\right)\right\} \quad (14.111)$$

The deflexion at a point where the load is applied is a clumsy expression. It is, of course, obtained by combining equations (14.97) and (14.98).

Clamped disk.

For a disk clamped at the edge the following are the results:—
when r is greater than c,

$$\mathrm{E'I}w_1 = \mathrm{K}_1 - \frac{\mathrm{W}r^2}{8\pi}\left\{\log_e\frac{a}{r} + \frac{1}{2}\right\}$$

$$-\frac{\mathrm{W}}{32\pi}(b^2+c^2)\left\{\frac{r^2}{a^2} + 2\log_e\frac{a}{r}\right\}; \quad . \quad . \quad . \quad . \quad (14.112)$$

when r is less than b,

$$\mathrm{E'I}w = -\frac{\mathrm{W}r^2}{8\pi}\left\{\frac{1}{4}\frac{c^2+b^2}{a^2} + \frac{1}{c^2-b^2}\left(c^2\log_e\frac{a}{c} - b^2\log_e\frac{a}{b}\right)\right\} \quad (14.113)$$

255. Load on a very narrow ring.

If $(c-b)$ is very small in comparison with b then the load is practically concentrated over the circumference of the circle of radius b. In this case there is very little error in putting $c=b$ in the results for a load distributed on a ring of finite width. The results are as follows:

supported disk:

where r is greater than b,

$$\mathrm{E'I}w_1 = \mathrm{K}_1 - \frac{\mathrm{W}r^2}{8\pi}\left\{\log_e\frac{a}{r} + \frac{1}{2}\frac{3+\sigma}{1+\sigma}\right\}$$

$$+\frac{\mathrm{W}b^2}{16\pi}\left\{\frac{1-\sigma}{1+\sigma}\frac{r^2}{a^2} - 2\log_e\frac{a}{r}\right\}; \quad . \quad . \quad . \quad . \quad (14.114)$$

and where r is less than b,

$$\mathrm{E'I}w = -\frac{\mathrm{W}r^2}{8\pi}\left\{\log_e\frac{a}{b} - \frac{1}{2} + \frac{1}{1+\sigma} - \frac{1}{2}\frac{1-\sigma}{1+\sigma}\frac{b^2}{a^2}\right\}$$

$$= -\frac{\mathrm{W}r^2}{8\pi}\left\{\log_e\frac{a}{b} + \frac{1}{2}\frac{1-\sigma}{1+\sigma}\left(1 - \frac{b^2}{a^2}\right)\right\}. \quad . \quad . \quad . \quad (14.115)$$

In deducing this last result the following limiting value has been used

$$\lim_{c\to b}\frac{c^2\log c-b^2\log b}{c^2-b^2}=\lim_{c\to b}\frac{\frac{d}{dc}\{c^2\log c-b^2\log b\}}{\frac{d}{dc}(c^2-b^2)}$$

$$=\lim_{c\to b}\frac{2c\log c+c}{2c}$$

$$=\log b+\tfrac{1}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (14.116)$$

Clamped disk. The corresponding results for the clamped disk are: where r is greater than b,

$$E'Iw_1=K_1-\frac{W}{8\pi}\left\{(r^2+b^2)\log_e\frac{a}{r}+\tfrac{1}{2}r^2\left(1+\frac{b^2}{a^2}\right)\right\};$$

and where r is less than b

$$E'Iw=-\frac{Wr^2}{8\pi}\left\{\log_e\frac{a}{b}-\frac{1}{2}+\frac{1}{2}\frac{b^2}{a^2}\right\} \quad . \quad . \quad . \quad . \quad . \quad (14.117)$$

256. Disk with a uniform pressure over the whole of one face and a balancing uniform pressure over a smaller circle on the other face.

Let a be the radius of the disk, and b the radius of the circle on which the supporting pressure acts. Let the pressures be p on the whole face and q on the opposite face. Then

$$\pi a^2 p=\pi b^2 q.$$

There is no need to work this problem out from the beginning because it is possible to get the result from what has already been worked out for a supported disk. The deflexion for this case is the difference of the deflexions for the following two cases of the supported disk.

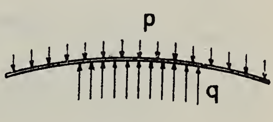


Fig. 141

(i) A supported disk under a uniform pressure p over the whole area.

(ii) A supported disk under a uniform pressure $-q$ over the circle of radius b.

It is clear that the superposition of these two systems of loads gives the same load as in our present problem, because the supporting pressures at the rim neutralise each other. Thus the deflexion for the present problem is the difference of the deflexions given by equations (14.79) and either (14.97) or (14.98), with $\frac{a^2}{b^2}p$ instead of p in the last two equations. Therefore, where r is less than b, the deflexion in the direction of the pressure p in given by

$$\begin{aligned}E'Iw &= \tfrac{1}{64}pr^4 - \tfrac{1}{32}\frac{3+\sigma}{1+\sigma}pa^2r^2 \\ &\quad - \tfrac{1}{64}qr^4 + \tfrac{1}{8}qb^2r^2\left\{\log\frac{a}{b} + \frac{1}{1+\sigma} - \frac{1}{4}\frac{1-\sigma}{1+\sigma}\frac{b^2}{a^2}\right\} \\ &= -\tfrac{1}{64}\frac{pr^4(a^2-b^2)}{b^2} + \tfrac{1}{8}pa^2r^2\left\{\log_e\frac{a}{b} + \frac{1}{4}\frac{1-\sigma}{1+\sigma}\left(1-\frac{b^2}{a^2}\right)\right\}; \quad (14.118)\end{aligned}$$

and where r is greater than b, by

$$\begin{aligned}E'Iw_1 &= \tfrac{1}{64}pr^4 - \tfrac{1}{32}\frac{3+\sigma}{1+\sigma}pa^2r^2 - K + \tfrac{1}{16}qb^4\log\frac{a}{r} \\ &\quad + \tfrac{1}{8}qb^2r^2\left\{\log\frac{a}{r} + \frac{1}{2}\frac{3+\sigma}{1+\sigma} - \frac{1}{4}\frac{1-\sigma}{1+\sigma}\frac{b^2}{a^2}\right\} \\ &= -K + \tfrac{1}{64}pr^4 + \tfrac{1}{8}pa^2r^2\left\{\left(1+\frac{b^2}{2r^2}\right)\log_e\frac{a}{r} + \frac{1}{4}\frac{3+\sigma}{1+\sigma} - \frac{1}{2}\frac{1-\sigma}{1+\sigma}\frac{b^2}{a^2}\right\} \quad (14.119)\end{aligned}$$

If b^2 is infinitely small the equation giving w_1 is approximately the same as

$$E'Iw_1 = -K + \tfrac{1}{64}pr^4 + \tfrac{1}{8}pa^2r^2\left\{\log_e\frac{a}{r} + \frac{1}{4}\frac{3+\sigma}{1+\sigma}\right\} \quad . \; . \; . \; . \; (14.120)$$

In this last case

$$E'I\frac{1}{r}\frac{dw_1}{dr} = \tfrac{1}{16}pr^2 + \tfrac{1}{4}pa^2\left\{\log\frac{a}{r} + \frac{1}{4}\frac{1-\sigma}{1+\sigma}\right\}$$

$$E'I\frac{d^2w_1}{dr^2} = \tfrac{3}{16}pr^2 + \tfrac{1}{4}pa^2\left\{\log\frac{a}{r} - 1 + \frac{1}{4}\frac{1-\sigma}{1+\sigma}\right\}$$

both of which are infinite when $r = 0$. Thus a load concentrated on a point—a physical impossibility of course—gives rise to infinite bending moments. It should be observed, however, that $\frac{dw_1}{dr}$ is zero at the centre in spite of the fact that the curvature is infinite.

257. Bending due to punching.

Suppose a load W acts downwards over a thin ring of radius b and an equal force W acts upwards over another thin ring of radius $b+\varepsilon$, and let us suppose that $\frac{\varepsilon}{b}$ is small. Then the deflexion due to both forces W is the difference of the deflexions due to a load W on each of the two rings when the rim is assumed to be supported.

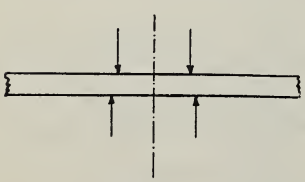


Fig. 142

Where r is less than b this deflexion is given by

$$
\begin{aligned}
E'Iw &= -\frac{Wr^2}{8\pi}\left\{\log_e\frac{b+\varepsilon}{b}+\frac{1}{2}\frac{1-\sigma}{1+\sigma}\frac{(b+\varepsilon)^2-b^2}{a^2}\right\} \\
&= -\frac{Wr^2}{8\pi}\left\{\frac{\varepsilon}{b}+\frac{1}{1+\sigma}\frac{\sigma b\varepsilon}{a^2}\right\} \text{ approximately}, \\
&= -\frac{Wr^2\varepsilon}{8\pi b}\left\{1+\frac{1-\sigma}{1+\sigma}\frac{b^2}{a^2}\right\} \quad \ldots\ldots \quad (14.121)
\end{aligned}
$$

If b is so much smaller than a that $\left(\frac{b}{a}\right)^2$ is negligible in comparison with unity, then the last equation becomes

$$
E'Iw = -\frac{W\varepsilon}{\pi b}r^2. \quad \ldots\ldots \quad (14.122)
$$

Thus the part that is being punched out takes a nearly spherical shape.

If the punch exerted only a small force this force would be distributed as a nearly uniform pressure over a circular area; but when the force is big this circular area bends so much that the pressure is concentrated near the circumference of the circle and approaches the ideal conditions we have assumed. There are, of course, very big shear stresses in the material just beyond the circle where the punch applies its pressure. It must be remembered, however, that there can be no shear stress just inside the surface without an equal shear stress —which would mean friction—on the surface itself. In the punching operation no friction is applied at the surface of the ring between the punch and the hole.

258. Disk with a central hole.

Suppose a disk of radius a, with a central hole of radius b, is subjected to a uniform pressure p, and suppose it is clamped at the outer rim and free at the inner edge. Then, measuring w from the clamped rim, the boundary conditions are

$$
w = 0 \text{ and } \frac{dw}{dr} = 0 \text{ where } r = a,
$$

$$
F_1 = 0 \text{ and } M_1 = 0 \text{ where } r = b.
$$

Now

$$
E'Iw = \tfrac{1}{64}pr^4 + \tfrac{1}{4}Ar^2(\log_e r - 1) + \tfrac{1}{4}Br^2 + C\log_e r + D.
$$

The last three conditions give

$$
\tfrac{1}{16}pa^3 + \tfrac{1}{2}Aa(\log a - \tfrac{1}{2}) + \tfrac{1}{2}Ba + \frac{C}{a} = 0,
$$

$$
\tfrac{1}{2}pb + \frac{A}{b} = 0,
$$

$$
\begin{aligned}
&\tfrac{3}{16}pb^2 + \tfrac{1}{2}A(\log b + \tfrac{1}{2}) + \tfrac{1}{2}B - \frac{C}{b^2} \\
&\quad + \sigma\left\{\tfrac{1}{16}pb^2 + \tfrac{1}{2}A(\log b - \tfrac{1}{2}) + \tfrac{1}{2}B + \frac{C}{b^2}\right\} = 0.
\end{aligned}
$$

From these we get

$$A = -\tfrac{1}{2}pb^2,$$

$$\{(1-\sigma)a^2+(1+\sigma)b^2\}\,B = -\tfrac{1}{8}p\,\{(1-\sigma)(a^4+2a^2b^2)+(1+3\sigma)b^4\}$$
$$+\tfrac{1}{2}pb^2\{(1+\sigma)b^2\log b+(1-\sigma)a^2\log a\}$$

$$\{(1-\sigma)a^2+(1+\sigma)b^2\}\,C = -\tfrac{1}{16}pa^2b^2\{(1+\sigma)a^2+(1-\sigma)b^2\}$$
$$+\tfrac{1}{4}(1+\sigma)pa^2b^4\log\frac{a}{b}\cdot$$

Now suppose the radius of the central hole is very much smaller than a. Then to find the stresses we can neglect powers of b in the preceding constants except in one case. We must not neglect the terms of the order b^2 in the expression for C because

$$\lim_{r\to b}\frac{d^2(b^2\log r)}{dr^2}$$

is finite and not zero. Thus for a small hole, we may take

$$A = 0,$$
$$B = -\tfrac{1}{8}pa^2,$$
$$C = -\frac{1}{16}\,\frac{1+\sigma}{1-\sigma}\,pa^2b^2.$$

Therefore

$$E'Iw = \tfrac{1}{64}p(r^4-2a^2r^2)-\frac{1}{16}\,\frac{1+\sigma}{1-\sigma}\,pa^2b^2\log_e r \quad . \quad . \quad (14.123)$$

The terms not containing $\log_e r$ are, of course, the same as for a clamped disk with no central hole. It follows that

$$M_2 = E'I\left\{\frac{1}{r}\frac{dw}{dr}+\sigma\frac{d^2w}{dr^2}\right\}$$
$$= \tfrac{1}{16}p\{(1+3\sigma)r^2-(1+\sigma)a^2\}-\frac{1+\sigma}{16}\,\frac{pa^2b^2}{r^2} \quad (14.124)$$

On putting $r=b$ in this, and then neglecting b^2, we get

$$M_2 = -\tfrac{1}{16}(1+\sigma)pa^2-\tfrac{1}{16}(1+\sigma)pa^{\cdot}$$
$$= -\tfrac{1}{8}(1+\sigma)pa^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14.125)$$

Thus the moment of the circumferential stresses at the inner edge of a disk with an infinitesimal hole is just twice as great as if C were zero, that is, twice as great as for the clamped disk with no hole. It follows that the stresses of which M_2 is the moment are also twice as great, and since the maximum stress occurs at the edge of the hole we conclude that the maximum is twice as great when there is a small central hole as when there is no hole.

259. Rectangular plate supported along all its edges.

Suppose a rectangular plate is supported at the same level along its four edges. Let the axis of x and y be taken along one pair of edges, and let the other pair be $x = a$, $y = b$. In the first instance we shall assume that the pressure on the plate at (x, y) is

$$p = A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \quad \ldots \ldots \quad (14.126)$$

where m and n are positive integers. Then the equation for w is

$$E'I \nabla_1{}^4 w = A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \quad \ldots \ldots \quad (14.147)$$

One solution of this is

$$E'I w = A_{mn} \frac{a^4 b^4 \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}}{\pi^4 (b^4 m^4 + a^4 n^4 + 2a^2 b^2 m^2 n^2)}$$

$$= A_{mn} \frac{a^4 b^4 \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}}{\pi^4 (m^2 b^2 + n^2 a^2)^2} \quad \ldots \ldots \quad (14.128)$$

With this value of w we find

$$M_1 = E'I \left(\frac{\partial^2 w}{\partial x^2} + \sigma \frac{\partial^2 w}{\partial y^2} \right)$$

$$= -\pi^2 \left(\frac{m^2}{a^2} + \sigma \frac{n^2}{b^2} \right) w \quad \ldots \ldots \quad (14.129)$$

Likewise

$$M_2 = -\pi^2 \left(\frac{n^2}{b^2} + \sigma \frac{m^2}{a^2} \right) w \quad \ldots \ldots \quad (14.130)$$

Thus the value of w that we have found satisfies the boundary conditions of the problem, namely, the conditions

$$\left. \begin{array}{l} w = 0 \\ M_1 = 0 \end{array} \right\} \begin{array}{l} \text{where } x = 0 \\ \text{and where } x = a \end{array}$$

$$\left. \begin{array}{l} w = 0 \\ M_2 = 0 \end{array} \right\} \begin{array}{l} \text{where } y = 0 \\ \text{and where } y = b \end{array}$$

We can use the result just obtained to find the deflexion for any other value of p. For, suppose $p = f(x, y)$;

then it is possible to expand p in a doubly infinite series thus:—

$$p = \sum_{m=1}^{m=\infty} \sum_{n=1}^{n=\infty} A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}.$$

The coefficients are found by a process similar to that used for a simple Fourier series. Thus

$$\int_0^a \int_0^b p \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \, dx dy$$

$$= \int_0^a \int^b A_{mn} \sin^2 \frac{m\pi x}{a} \sin^2 \frac{n\pi y}{b} \, dx dy$$

$$= \frac{ab}{4} A_{mn} \quad \ldots \ldots \quad (14.132)$$

This is obtained by multiplying both sides of equation (14.131) by $\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\,dx\,dy$ and then integrating over the whole plate. Every term in the integral on the right vanishes except the one having the coefficient A_{mn}. Therefore

$$A_{mn}=\frac{4}{ab}\int_0^a\int_0^b p\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\,dx\,dy \quad . \quad . \quad (14.133)$$

The corresponding values of the deflexion and the bending moments are given by the equations

$$E'Iw=\Sigma\Sigma A_{mn}\frac{a^4b^4\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}}{\pi^4(m^2b^2+n^2a^2)^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (14.134)$$

$$\left.\begin{aligned}M_1&=-\pi^2\Sigma\Sigma\left(\frac{m^2}{a^2}+\sigma\frac{n^2}{b^2}\right)\frac{a^4b^4}{\pi^4(m^2b^2+n^2a^2)^2}A_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\\ M_2&=-\pi^2\Sigma\Sigma\left(\frac{n^2}{b^2}+\sigma\frac{m^2}{a^2}\right)\frac{a^4b^4}{\pi^4(m^2b^2+n^2a^2)^2}A_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\end{aligned}\right\}(14.135)$$

260. Uniform pressure over the whole rectangle.

Assuming p to be the constant we find that

$$\begin{aligned}A_{mn}&=\frac{4p}{ab}\int_0^a\int_0^b\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\,dx\,dy\\ &=\frac{4p}{ab}\int_0^a\left[-\frac{b}{n\pi}\cos\frac{n\pi y}{b}\right]_0^b\sin\frac{m\pi x}{a}\,dx\\ &=\frac{4p}{n\pi a}\int_0^a(1-\cos n\pi)\sin\frac{m\pi x}{a}\,dx\\ &=\frac{4p}{mn\pi^2}(1-\cos n\pi)(1-\cos m\pi)\end{aligned}$$

$$\left.\begin{aligned}&=0 \text{ when either } m \text{ or } n \text{ is even}\\ &=\frac{16p}{mn\pi^2}\text{ when both are odd}\end{aligned}\right\} \quad . \quad (14.136)$$

Therefore

$$p=\frac{16p}{\pi^2}\left[\begin{aligned}&\sin\frac{\pi x}{a}\left\{\sin\frac{\pi y}{b}+\tfrac{1}{3}\sin\frac{3\pi y}{b}+\tfrac{1}{5}\sin\frac{5\pi y}{b}+\right\}\\ &+\tfrac{1}{3}\sin\frac{3\pi x}{a}\left\{\sin\frac{\pi y}{b}+\tfrac{1}{3}\sin\frac{3\pi y}{b}+\ldots\right\}\\ &+\text{etc.}\end{aligned}\right] \quad . \quad (13.137)$$

Therefore by (14.134)

$$\frac{\pi^6}{16a^4b^4p}\,E'Iw = \frac{\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}}{(a^2+b^2)^2} + \frac{1}{3}\frac{\sin\frac{\pi x}{a}\sin\frac{3\pi y}{b}}{(3^2a^2+b^2)^2} +$$

$$+\frac{1}{3}\frac{\sin\frac{3\pi x}{a}\sin\frac{\pi y}{b}}{(a^2+3^2b^2)^2} + \frac{1}{3^2}\frac{\sin\frac{3\pi x}{a}\sin\frac{3\pi y}{b}}{(3^2a^2+3^2b^2)^2} +$$

$$+\frac{1}{5}\frac{\sin\frac{5\pi x}{a}\sin\frac{\pi y}{b}}{(a^2+5^2b^2)^2} + \frac{1}{15}\frac{\sin\frac{5\pi x}{a}\sin\frac{3\pi y}{b}}{(3^2a^2+5^2b^2)^2} +$$

$$+ \text{etc.} \qquad (14.138)$$

Putting $b = a$ for a square plate we get

$$\frac{\pi^6}{16a^4p}\,E'Iw = \frac{1}{2^2}\sin\frac{\pi x}{a}\sin\frac{\pi y}{a} + \frac{1}{2^2 3^6}\sin\frac{3\pi x}{a}\sin\frac{3\pi y}{a} +$$

$$+\frac{1}{1.3(3^2+1^2)^2}\left(\sin\frac{3\pi x}{a}\sin\frac{\pi y}{a} + \sin\frac{\pi x}{a}\sin\frac{3\pi y}{a}\right)$$

$$\frac{1}{1.5(5^2+1^2)^2}\left(\sin\frac{5\pi x}{a}\sin\frac{\pi y}{a} + \sin\frac{\pi x}{a}\sin\frac{5\pi y}{a}\right)$$

$$+ \text{etc.}$$

$$\frac{1}{3.5(5^2+3^2)^2}\left(\sin\frac{3\pi x}{a}\sin\frac{5\pi y}{a} + \sin\frac{5\pi x}{a}\sin\frac{3\pi y}{a}\right)$$

$$+ \text{etc.} \qquad (14.139)$$

At the centre of the plate, where $x = \frac{1}{2}a$, $y = \frac{1}{2}a$, we find

$$-\frac{\pi^4}{16a^2p}M_1 = -\frac{\pi^4}{16a^2p}M_2$$

$$= \frac{1}{2^2}(1+\sigma)\left(1+\frac{1}{3^4}+\frac{1}{5^4}+\ldots\right)$$

$$-\frac{(3^2+\sigma)+(3^2\sigma+1)}{1.3(3^2+1^2)^2} + \frac{(5^2+\sigma)+(5^2\sigma+1)}{1.5(5^2+2)^2} -$$

$$-\frac{(3^2+5^2\sigma)+(3^2\sigma+5^2)}{3.5(3^2+5^2)^2} +$$

$$= (1+\sigma)\left\{\begin{array}{l} \frac{1}{2^2}\left(1+\frac{1}{3^4}+\frac{1}{5^4}+\ldots\right) \\ -\frac{1}{1.3(3^2+1^2)} + \frac{1}{1.5(5^2+1^2)} - \ldots \\ -\frac{1}{3.5(5^2+3^2)} + \frac{1}{3.7(7^2+3^2)} - \ldots \\ -\ \text{etc.} \end{array}\right\}$$

$$= 0{\cdot}224\,(1+\sigma)\ \text{approximately.} \qquad (14.140)$$

Thus

$$-M_1 = -M_2 = 0{\cdot}224\,\frac{16(1+\sigma)}{\pi^4}\,a^2p$$

$$= 0{\cdot}0368(1+\sigma)a^2p. \quad . \quad . \quad . \quad . \quad (14.141)$$

Since the curvatures of the normal sections of the middle surface parallel to the axes of x and y at the middle of the square are principal curvatures of the surface, and since these are equal, it follows from a theorem of Euler's that the surface at that point is spherical, and therefore that the bending moment across any section through the middle of the plate is equal to M_1. Moreover, this is the maximum bending moment in the plate.

If $b = 2a$ and if $\sigma = 0{\cdot}25$, we find that the maximum bending moment, which occurs at the centre across a section parallel to the longer sides, is

$$-M_1 = 0{\cdot}604\,\frac{16}{\pi^4}\,a^2p$$

$$= 0{\cdot}0992\,a^2p. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14.142)$$

If $\dfrac{b}{a}$ is very great it is easy to see, without using the preceding analysis, that the influence of the supports at the short sides is negligible at some distance from these sides. The bending moment across the middle of a narrow strip at the centre of the plate, the longer sides of the strip being parallel to the short sides of the plate, may be found by treating this strip as a beam. This bending moment, which is clearly the maximum bending moment in the plate, is

$$-M_1 = \tfrac{1}{8}pa^2.$$

261. The problem of the clamped rectangular plate.

The problem of the clamped rectangular plate under a given normal pressure appears to have no simple analytical solution. At any rate it has not yet been solved. In fact in only a few cases has the exact solution of the problem of a plate bent by normal pressures been discovered. As we have seen, the problem of the rectangular plate is tractable if the four edges are supported at the same level, but not if they are clamped. In the case of an elliptic plate under uniform pressure the solution is very simple if its rim is clamped, but unknown if it is supported. The solution for the clamped elliptic plate is worth giving because it is one more precise solution.

262. Clamped elliptic plate under uniform pressure.

Let the equation to the rim of the plate be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14.143)$$

Now in the equation

$$p = E'I\nabla_1{}^4 w \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14.144)$$

put

$$w = C\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^2. \quad . \quad . \quad . \quad . \quad (14.145)$$

This gives

$$p = E'IC\left(\frac{24}{a^4} + \frac{24}{b^4} + \frac{16}{a^2b^2}\right). \quad . \quad . \quad . \quad . \quad (14.146)$$

Thus p is constant, and if we substitute the value of C in terms of p in (14.145) we get

$$E'Iw = \frac{p}{8}\frac{\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^2}{\frac{3}{a^4} + \frac{3}{b^4} + \frac{2}{a^2b^2}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (14.147)$$

Now it is easy to see that, with the above value of w,

$$w = 0, \quad \frac{\partial w}{\partial x} = 0, \quad \frac{\partial w}{\partial y} = 0,$$

at the rim of the plate. But these are the conditions given in (14.43) as the boundary conditions for a clamped plate. Then (14.147) gives the solution of the clamped elliptic plate under uniform pressure.

The two principal bending moments at the middle of the plate are

$$-M_1 = \frac{p}{8}\frac{4\left(\frac{1}{a^2} + \frac{\sigma}{b^2}\right)}{\frac{3}{a^4} + \frac{3}{b^4} + \frac{2}{a^2b^2}}$$

$$= \frac{p}{2}\frac{\frac{1}{a^2} + \frac{\sigma}{b^2}}{\frac{3}{a^4} + \frac{3}{b^4} + \frac{2}{a^2b^2}}; \quad . \quad . \quad . \quad . \quad . \quad (14.148)$$

and

$$-M_2 = \frac{p}{2}\frac{\frac{1}{b^2} + \frac{\sigma}{a^2}}{\frac{3}{a^4} + \frac{3}{b^4} + \frac{2}{a^2b^2}}. \quad . \quad . \quad . \quad . \quad . \quad (14.149)$$

Moreover, the bending moments at the ends of the axes of lengths $2a$ and $2b$ are respectively

$$M'_1 = \frac{p}{8}\frac{\frac{8}{a^2}}{\frac{3}{a^4} + \frac{3}{b^4} + \frac{2}{a^2b^2}}$$

$$= p\frac{\frac{1}{a^2}}{\frac{3}{a^4}+\frac{3}{b^4}+\frac{2}{a^2b^2}}$$

and . . . . . (14.150)

$$M'_2 = p\frac{\frac{1}{b^2}}{\frac{3}{a^4}+\frac{3}{b^4}+\frac{2}{a^2b^2}}$$

If b is less than a the greatest of these four bending moments is M'_2 and this is the maximum bending moment in the plate. If $b=a$ the value of this maximum bending moment reduces, of course, to what we have already found for the clamped circle, namely

$$M'_1 = \tfrac{1}{8}pa^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14.151)$$

whereas if the ellipse is so long that $\frac{b}{a}$ is negligible the result becomes

$$M'_2 = \tfrac{1}{3}pb^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14.152)$$

which is the same as for a beam of length $2b$ which carries a load p per unit length and is clamped at the ends.

263. The strain-energy in a plate of uniform or variable thickness bent without stretching of the middle surface.

The work done on the element $dx \times dy$ is the work done by the couples M_1, M_2, and the torque Q, acting on the edges of the element. The work done by the shear forces F_1 and F_2 is negligible compared with the work done by the couples exactly as in the case of a beam.

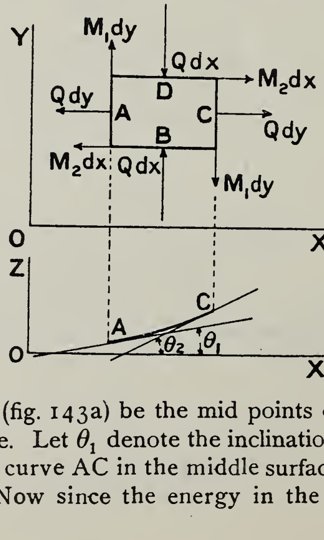


Fig. 143a

Fig. 143b

Let A, B, C, D (fig. 143a) be the mid points of the rectangle $dx \times dy$ in the middle surface. Let θ_1 denote the inclination to the xy plane of the tangent at A to the curve AC in the middle surface, θ_2 the corresponding inclination at C. Now since the energy in the plate depends only on its final state and not on the intermediate states, let us assume that it reaches its final state by a gradual application of the external forces, these forces all starting from zero simultaneously and all keeping the same ratio while they are increasing up to their final values. In that case the stresses and strains and displacements all increase at the same rate as the applied forces. It follows therefore that M_1 is proportional to θ_1 and therefore the work done by $M_1 dy$ at A is $-\frac{1}{2} M_1 \theta_1 dy$, the sign being negative because the couple is turning in the direction contrary to θ_1. Likewise the work done by $M_1 dy$ at B is $\frac{1}{2} M_1 \theta_2 dy$. Thus the total work done by the two couples $M_1\, dy$ is

$$\tfrac{1}{2} M_1\, dy(\theta_2 - \theta_1).$$

Now

$$\theta_1 = \frac{\partial w}{\partial x} \text{ approximately,}$$

and therefore

$$\theta_2 - \theta_1 = d(\theta_1)$$
$$= \frac{\partial \theta_1}{\partial x} dx = \frac{\partial^2 w}{\partial x^2} dx.$$

Thus the work done by the two couples is

$$\tfrac{1}{2} M_1 \frac{\partial^2 w}{\partial x^2} dx\, dy.$$

Likewise the work done by the couples $M_2 dx$ is

$$\tfrac{1}{2} M_2 \frac{\partial^2 w}{\partial y^2} dx\, dy.$$

Again let φ_1 be the mean inclination to the xy plane of the element dy which passes through A, and φ_2 the mean inclination of the element dy that passes through C. Then the work done by the two torques Qdy is

$$\tfrac{1}{2} \varphi_2 Q dy - \tfrac{1}{2} \varphi_1 Q dy = \tfrac{1}{2} Q(\varphi_2 - \varphi_1) dy.$$

But

$$\varphi_1 = \frac{\partial w}{\partial y}$$

and

$$\varphi_2 - \varphi_1 = \frac{\partial \varphi_1}{\partial x} dx = \frac{\partial^2 w}{\partial x \partial y} dx.$$

Therefore the work done by the couples Qdy is

$$\tfrac{1}{2} Q \frac{\partial^2 w}{\partial x \partial y} dx dy.$$

Similarly it may be shown that the work done by the couples Qdx is

$$\tfrac{1}{2} Q \frac{\partial^2 w}{\partial x \partial y} dx\, dy.$$

Thus the total work done by all the couples in straining the element from its equilibrium state is

$$dV = \frac{1}{2}\left\{M_1 \frac{\partial^2 w}{\partial x^2} + M_2 \frac{\partial^2 w}{\partial y^2} + 2Q \frac{\partial^2 w}{\partial x \partial y}\right\} dxdy, \quad . \quad . \quad (14.153)$$

which, by equations (14.32), (14.33), (14.34), becomes

$$dV = \tfrac{1}{2}E'I\left[\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)^2 - 2(1-\sigma)\left\{\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\right\}\right] dxdy$$

$$= \tfrac{1}{2}E'I\left[(\nabla_1{}^2 w)^2 - 2(1-\sigma)\left\{\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\right\}\right] dxdy. \quad . \quad . \quad (14.154)$$

The total energy in the plate is the integral of this over the whole area.

If ϱ_1, ϱ_2, are the principal radii of curvature of the middle surface at the point (x, y) it is shown in books on Analytical Solid Geometry that

$$\frac{1}{\varrho_1} + \frac{1}{\varrho_2} = \nabla_1{}^2 w, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14.155)$$

$$\frac{1}{\varrho_1 \varrho_2} = \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 \quad . \quad . \quad . \quad . \quad (14.156)$$

Since the quantities on the left of the last two equations have nothing to do with the directions of the axes of x and y it follows that the expressions on the right are also independent of the directions of the axes, provided only that they are rectangular. Thus the expressions on the right are *invariants* for all rectangular axes.

It is further shown that the integral

$$\iint \frac{dxdy}{\varrho_1 \varrho_2}$$

taken over any closed surface is equal to what is called the *whole curvature* of the closed surface; this whole curvature is the area cut off a sphere of unit radius by radii of the sphere drawn parallel to the normals to the surface at all points of its boundary. The whole curvature is thus measured by a solid angle the magnitude of which depends only on the condition of the surface at its boundary. For a plate clamped all round the boundary, so that all the normals round the edge are parallel, the solid angle is zero. For such a clamped plate the total strain energy in the plate is

$$V = \iint \tfrac{1}{2}E'I(\nabla_1{}^2 w)^2 dxdy \quad . \quad . \quad . \quad . \quad . \quad (14.157)$$

In terms of polar coordinates r, θ, we know from equations (14.53) (14.54), (14.61), that

$$\nabla_1{}^2 w = \frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}. \quad . \quad . \quad . \quad . \quad (14.158)$$

$$\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}-\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2=\frac{\partial^2 w}{\partial r^2}\left\{\frac{1}{r}\frac{\partial w}{\partial r}+\frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right\}-\left\{\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w}{\partial \theta}\right)\right\}^2, \quad (14.159)$$

and the element of area which replaces $dx\,dy$ is $r\,dr\,d\theta$.

If the bent plate is symmetrical about the z-axis then w is not a function of θ, and therefore

$$\nabla_1{}^2 w=\frac{d^2w}{dr^2}+\frac{1}{r}\frac{dw}{dr}, \quad . \quad . \quad . \quad . \quad . \quad (14.160)$$

$$\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}-\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2=\frac{1}{r}\frac{d^2w}{dr^2}\frac{dw}{dr} \quad . \quad . \quad . \quad . \quad (14.161)$$

If the thickness is not uniform the factor I must be treated as a function of x and y in (14.154) and (14.157).

264. Work done by the pressure on the plate.

The final state of the bent plate can be produced by a gradual application of the pressure in such a way that the pressures everywhere start simultaneously from zero and increase up to the maximum, maintaining the same ratios throughout the process. In this case the pressure on every element is proportional to the deflexion at that point; therefore the work done by the pressure p on an element of area dA is $\frac{1}{2}pw\,dA$. Therefore the whole work done on the plate, assuming that the forces at the boundary do no work, is

$$W=\tfrac{1}{2}\iint pw\,dx\,dy \quad . \quad . \quad . \quad . \quad . \quad . \quad (14.162)$$

Since the work done by the pressure is stored as strain energy in the plate it follows that W must be the same as V, which is obtained by integrating (14.154). Thus

$$\int pw\,dA=\int E'I\left\{\left(\frac{1}{\varrho_1}+\frac{1}{\varrho_2}\right)^2-\frac{2(1-\sigma)}{\varrho_1\,\varrho_2}\right\}dA, \quad . \quad . \quad (14.163)$$

the integrals being taken over the whole area of the plate.

This last equation can be deduced from the differential equation (14.21) by integrating both sides and carrying out some transformations of the integrals.

265. Approximate methods.

In cases where the exact solution of the plate problem is not known quite good approximate solutions can be got by means of equation (14.163). The way to use this equation is to assume a reasonable expression for w of the form

$$w=kf(x,\,y)$$

and then to determine k from equation (14.163). The most reasonable sort of value for w is a value that satisfies both the boundary conditions, and one, moreover, that gives the sort of form to the plate that seems likely under the given pressures. In many cases there are no simple expressions for w that satisfy both the boundary conditions and then we have to be content with an expression that satisfies only one of these conditions. A certain amount of intuition helps in selecting good values of w for a particular problem, but there should be no great difficulty in testing, by comparison with known accurate results, whether any selected value is a suitable one or not.

A second approximate method, of which the one just described is a particular case, is to assume an expression for w involving more than one unknown constant or parameter, then to write down the total potential energy of the system, including the potential energy of the loads applied to the plate, and finally to determine these parameters bv making the potential energy a minimum. For this purpose the work done by the pressures or loads is

$$\iint pw\,dx\,dy$$

wherein p must be regarded as having its final value at every point of the plate. The factor $\frac{1}{2}$ which occurs in (14.162) does not appear in our new expression. The potential energy of the loads must be taken as the negative of this last expression for the work. Thus the present method is to take the potential energy V, given by

$$V = \tfrac{1}{2}\int E'I\left\{\left(\frac{1}{\varrho_1} + \frac{1}{\varrho_2}\right)^2 - \frac{2(1-\sigma)}{\varrho_1\,\varrho_2}\right\}dA - \int pw\,dA, \quad (14.164)$$

and determine the parameters, m, n, k, etc., that occur in w, from the equations

$$\frac{\partial V}{\partial m} = 0, \quad \frac{\partial V}{\partial n} = 0, \quad \frac{\partial V}{\partial k} = 0, \text{ etc.} \quad (14.165)$$

266. Examples of the first approximate method.

(a) *Rectangular plate, supported at the rim.*

As our first example we shall take the square plate of uniform thickness, with sides of length a, under uniform pressure, supported without clamping at the rim. Let us assume that

$$w = k \sin\frac{\pi x}{a}\,\sin\frac{\pi y}{a}, \quad (14.166)$$

a pair of sides being taken as axes.

This value of w satisfies both the conditions given in (14.44) for a supported plate.

Now with this value of w

$$\frac{\partial^2 w}{\partial x^2} = -\frac{\pi^2}{a^2}w$$

$$\frac{\partial^2 w}{\partial y^2} = -\frac{\pi^2}{a^2}w$$

$$\nabla_1^2 w = -\frac{2\pi^2}{a^2}w$$

$$\frac{\partial^2 w}{\partial x \partial y} = \frac{\pi^2}{a^2} k \cos\frac{\pi x}{a} \cos\frac{\pi y}{a}$$

$$\frac{1}{\varrho_1 \varrho_2} = \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2$$

$$= \frac{\pi^4}{a^4} k^2 \left\{ \sin^2\frac{\pi x}{a} \sin^2\frac{\pi y}{a} - \cos^2\frac{\pi x}{a}\cos^2\frac{\pi y}{a} \right\}.$$

Also

$$\int_0^a \int_0^a \sin^2\frac{\pi x}{a} \sin^2\frac{\pi y}{a}\, dx dy = \int_0^a \sin^2\frac{\pi x}{a}\, dx \times \int_0^a \sin^2\frac{\pi y}{a}\, dy$$

$$= \tfrac{1}{2}a \times \tfrac{1}{2}a = \tfrac{1}{4}a^2,$$

and

$$\int_0^a \int_0^a \cos^2\frac{\pi x}{a} \cos^2\frac{\pi y}{a}\, dx dy = \tfrac{1}{4}a^2.$$

Therefore equation (14.163) gives

$$kp \int_0^a \int_0^a \sin\frac{\pi x}{a} \sin\frac{\pi y}{a}\, dx dy = \text{E}'\text{I}\, \frac{4\pi^4 k^2}{a^4} \times \tfrac{1}{4}a^2$$

or

$$\frac{4a^2}{\pi^2} kp = \frac{\pi^4}{a^2} \text{E}'\text{I} k^2,$$

whence

$$\text{E}'\text{I}k = \frac{4a^4 p}{\pi^6}.$$

Thus our approximate value of w is given by

$$\text{E}'\text{I}w = \frac{4a^4 p}{\pi^6} \sin\frac{\pi x}{a} \sin\frac{\pi y}{a}. \quad . \quad . \quad . \quad . \quad (14.167)$$

The maximum bending moment occurs at the middle of the plate and its value, on our present assumption, is

$$-\text{M}_1 = -\text{M}_2 = -\text{E}'\text{I}\left(\frac{\partial^2 w}{\partial x^2} + \sigma \frac{\partial^2 w}{\partial y^2}\right)$$

$$= \text{E}'\text{I}(1 + \sigma)\frac{\pi^2}{a^2} w$$

$$= \frac{4a^2 p}{\pi^4}(1 + \sigma) \sin\frac{\pi}{2} \sin\frac{\pi}{2}$$

$$= \frac{4a^2 p}{\pi^4}(1 + \sigma). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14.168)$$

The accurate solution to this problem gave [equation (14.141)],

$$-\text{M}_1 = 3{\cdot}584 \frac{a^2 p}{\pi^4}(1 + \sigma).$$

The error in the maximum stress by the present method is therefore about 10 %. We must always expect an error of about this magnitude by the present method. Intuition could not have suggested a much better type for w than the one we have used here.

It is worth while to work the same problem with another value of w. Let us take

$$\left.\begin{aligned} w &= kuv \\ \text{where} \quad u &= x^4 - 6x^2c^2 + 5c^4 \\ \text{and} \quad v &= y^4 - 6y^2c^2 + 5c^4, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (14.169)$$

the axes being taken parallel to the sides through the centre of the plate, and c being half the length of a side. Here u and v have not the usual meaning of displacements.

This value of w would also satisfy both the boundary conditions given in (14.44) if σ were a negligible fraction. Now we found in the last case that

$$\iint\left\{\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\right\}dx\,dy = 0,$$

the integral being taken over the whole area of the plate. We can show that this result is always true for a rectangular plate if the assumed value of w is zero all round the boundary and if

$$w = kf(x) \times F(y) = kuv,$$

as in the present and the last case. The following is a proof.

$$\frac{\partial^2 w}{\partial x^2} = kv\frac{d^2u}{dx^2}; \quad \frac{\partial^2 w}{\partial y^2} = ku\frac{d^2v}{dy^2};$$

$$\frac{\partial^2 w}{\partial x \partial y} = k\frac{du}{dx}\frac{dv}{dy}.$$

Now

$$\int_{-c}^{c}\int_{-c}^{c}\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}dx\,dy = k^2\int_{-c}^{c}u\frac{d^2u}{dx^2}dx \times \int_{-c}^{c}v\frac{d^2v}{dy^2}dy.$$

But

$$\int_{-c}^{c}u\frac{d^2u}{dx^2}dx = \left[u\frac{du}{dx}\right]_{-c}^{c} - \int_{-c}^{c}\left(\frac{du}{dx}\right)^2 dx$$

$$= 0 - \int_{-c}^{c}\left(\frac{du}{dx}\right)^2 dx,$$

the integrated term being zero because u contains the factors which make w vanish at the edges $x = c$ and $x = -c$.

It follows that

$$\int_{-c}^{c}\int_{-c}^{c}\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}dx\,dy = k^2\int_{-c}^{c}\left(\frac{du}{dx}\right)^2 dx \times \int_{-c}^{c}\left(\frac{dv}{dy}\right)^2 dy$$

$$= \int_{-c}^{c}\int_{-c}^{c}\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 dx\,dy,$$

which is what we had to prove.

Now $$(\nabla_1{}^2 w)^2 = k^2\left\{v^2\left(\frac{d^2u}{dx^2}\right)^2 + 2uv\frac{d^2u}{dx^2}\frac{d^2v}{dy^2} + u^2\left(\frac{d^2v}{dy^2}\right)^2\right\}$$

With the present values of u and v

$$\frac{du}{dx} = 4x^3 - 12xc^2,$$

$$\frac{d^2u}{dx^2} = 12(x^2 - c^2),$$

$$v^2 = y^8 - 12y^6c^2 + 46y^4c^4 - 60y^2c^6 + 25c^8,$$

$$\int_{-c}^{c} v^2 dy = 2\left(\tfrac{1}{9} - \tfrac{12}{7} + \tfrac{46}{5} - 20 + 25\right)c^9$$

$$= 2 \times 12{\cdot}60c^9,$$

$$\int_{-c}^{c}\left(\frac{d^2u}{dx^2}\right)^2 dx = 288\left(\tfrac{1}{5} - \tfrac{2}{3} + 1\right)c^5$$

$$= 288 \times \tfrac{8}{15}c^5,$$

$$\int_{-c}^{c} u\frac{d^2u}{dx^2}dx = 12\int_{-c}^{c}(x^6 - 7x^4c^2 + 11x^2c^4 - 5c^6)dx$$

$$= -24 \times \tfrac{272}{105}c^7.$$

Therefore

$$\int_{-c}^{c}\int_{-c}^{c}(\nabla_1{}^2w)^2 dx\,dy = 24^2k^2\left\{2 \times 12{\cdot}60 \times \tfrac{8}{15} + 2 \times \left(\tfrac{272}{105}\right)^2\right\}c^{14}$$

$$= 24^2k^2 \times 26{\cdot}86c^{14}.$$

Again

$$\int_{-c}^{c}\int_{-c}^{c} pw\,dx\,dy = kp\int_{-c}^{c} u\,dx \times \int_{-c}^{c} v\,dy$$

$$= p \times \left(\tfrac{32}{5}\right)^2 kc^{10}.$$

Substituting all these quantities in (14.163) we get

$$6{\cdot}4^2pkc^{10} = \text{E}'\text{I}\,24^2 \times 26{\cdot}86\,k^2c^{14},$$

whence

$$\text{E}'\text{I}k = \frac{32p}{9 \times 1343\,c^4}.$$

The bending moment at the centre of the plate, where $x = 0$, $y = 0$, is

$$-\text{M}_1 = -\text{E}'\text{I}\left(\frac{\partial^2 w}{\partial x^2} + \sigma\frac{\partial^2 w}{\partial y^2}\right)$$

$$= 60\,\text{E}'\text{I}k(1 + \sigma)c^6$$

$$= \frac{32 \times 60(1 + \sigma)}{9 \times 1343}pc^2$$

$$= \frac{1 + \sigma}{6{\cdot}30}pc^2. \qquad (14.170)$$

The correct result is

$$-\text{M}_1 = \frac{3{\cdot}584(1 + \sigma)}{\pi^4}(4pc^2)$$

$$= \frac{1 + \sigma}{6{\cdot}79}pc^2.$$

The error in the maximum bending moment in this case is 7·2 per cent, a rather smaller error than the first assumption gave.

(b) *Clamped square plate.*

If the axes are taken through the middle of the plate parallel to the sides, the deflexion given by

$$w = k(c^2 - x^2)^2 (c^2 - y^2)^2$$

satisfies both boundary conditions for a clamped plate.

Putting, in this case,

$$u = (c^2 - x^2)^2,$$
$$v = (c^2 - y^2)^2,$$

we find, as before,

$$(\nabla_1{}^2 w)^2 = k^2 \left\{ v^2 \left(\frac{d^2u}{dx^2}\right)^2 + 2uv \frac{d^2u}{dx^2}\frac{d^2v}{dy^2} + u^2 \left(\frac{d^2v}{dy^2}\right)^2 \right\}.$$

Now

$$\int_{-c}^{c} v^2 dy = 2c^9 \{1 - \tfrac{4}{3} + \tfrac{6}{5} - \tfrac{4}{7} + \tfrac{1}{9}\} = \tfrac{256}{315}c^9,$$

$$\int_{-c}^{c} \left(\frac{d^2u}{dx^2}\right)^2 dx = \tfrac{128}{5}c^5,$$

$$\int_{-c}^{c} u \frac{d^2u}{dx^2} dx = \tfrac{256}{105}c^7.$$

Therefore

$$\int_{-c}^{c}\int_{-c}^{c} (\nabla_1{}^2 w)^2 dx\,dy = 2k^2 \left\{\tfrac{256}{315}\,\tfrac{128}{5}c^{14} + (\tfrac{256}{105})^2 c^{14}\right\}$$

$$= \frac{256^2}{35^2} k^2 c^{14}.$$

Also

$$\int_{-c}^{c}\int_{c}^{c} pw\,dxdy = kp\,\frac{16^2}{15^2}c^{10}.$$

Therefore equation (14.163) gives

$$\frac{16^2}{15^2}kpc^{10} = \frac{256^2}{35^2}E'Ik^2c^{14},$$

whence

$$E'Ikc^4 = \frac{7^2}{48^2}p.$$

Consequently

$$M_1 = E'I\left\{\frac{\partial^2 w}{\partial x^2} + \sigma\frac{\partial^2 w}{\partial y^2}\right\}$$

$$= \frac{7^2}{48^2}\frac{p}{c^4}\{4(3x^2 - c^2)(c^2 - y^2)^2 + 4\sigma(3y^2 - c^2)(c^2 - x^2)^2\}. \quad (14.171)$$

At the middle of the plate, where $x = 0$, $y = 0$, we find that

$$-M_1 = -M_2 = \frac{7^2}{24^2}(1+\sigma)pc^2 = \frac{1+\sigma}{47\cdot 0}pa^2 \quad . \quad . \quad (14.172)$$

where a is written for $2c$, the whole length of a side.

At the middle of one edge of the plate, where $x = c$, $y = 0$, we get

$$M_1 = \frac{2 \times 7^2}{24^2}pc^2 = \frac{2}{47\cdot 0}pa^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14.173)$$

It is worth while to compare these bending moments with the bending moments at the centre and at the rim of a clamped circle of radius c. These latter are

$$-M_1 = \frac{1+\sigma}{16}pc^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14.174)$$

$$M_1 = \tfrac{1}{8}pc^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14.175)$$

The ratio of one of these bending moments to the other is the same as the ratio of the corresponding two bending moments that we have just calculated for the square plate. This is so reasonable that we may safely assume that the approximate results just obtained for the square plate must be accurate enough for practical purposes.

(c) *Elliptic plate subject to uniform pressure and supported at the edge.*

Let $2a$ and $2b$ denote the length of the principal axes of the plate, and let

$$\left.\begin{aligned} m &= \frac{\sigma a^2 + b^2}{\sigma a^2 + 5b^2} \\ n &= \frac{\sigma b^2 + a^2}{\sigma b^2 + 5a^2} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14.176)$$

Then the deflexion given by the equation

$$w = k\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)\left(1 - m\frac{x^2}{a^2} - n\frac{y^2}{b^2}\right) \quad . \quad . \quad . \quad (14.177)$$

satisfies the conditions

$$w = 0$$

at the edge of the plate, and

$$M = 0$$

across sections perpendicular to the principal axes at their ends.

There is no simple expression for w that will satisfy the condition that M should be zero at all points of the edge. When the ellipse becomes a circle the given expression for w is absolutely accurate; and again when one of the axes is very long compared with the other the deflexion is nearly accurate except at points near the ends of the longer axis. The expression then must be a fairly good one for all ellipses, and we may therefore use it in our first approximate method.

Now equation (14.177) can be written in the form

$$w = k\left\{1-(m+1)\frac{x^2}{a^2}-(n+1)\frac{y^2}{b^2}+m\frac{x^4}{a^4}+n\frac{y^4}{b^4}+(m+n)\frac{x^2y^2}{a^2b^2}\right\} \quad (14.178)$$

Therefore

$$\nabla_1{}^2 w = 2k\left\{A\frac{x^2}{a^2}+B\frac{y^2}{b^2}-C\right\} \quad . \quad . \quad . \quad . \quad (14.179)$$

where

$$\left.\begin{aligned} A &= \frac{6m}{a^2}+\frac{m+n}{b^2} \\ B &= \frac{6n}{b^2}+\frac{m+n}{a^2} \\ C &= \frac{m+1}{a^2}+\frac{n+1}{b^2} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (14.180)$$

Therefore

$$(\nabla_1{}^2 w)^2 = 4k^2\left\{A^2\frac{x^4}{a^4}+B^2\frac{y^4}{b^4}+2AB\frac{x^2y^2}{a^2b^2}-2C\left(A\frac{x^2}{a^2}+B\frac{y^2}{b^2}\right)+C^2\right\}$$

Now let

$$\frac{x_1}{a} = +\sqrt{1-\frac{y^2}{b^2}};$$

then the integral of $(\nabla_1{}^2 w)^2$ over the whole area of the plate is

$$\int_{-b}^{b}\int_{-x_1}^{x_1}(\nabla_1{}^2 w)^2\,dy\,dx.$$

Now

$$\int_{-b}^{b}\int_{-x_1}^{x_1}\frac{x^2}{a^2}\,dy\,dx = \int_{-b}^{b}\frac{2}{3}\frac{x_1{}^3}{a^2}\,dy$$

To integrate this put

$$y = b\sin\theta,$$

from which we get

$$x_1 = a\cos\theta,$$

and

$$dy = b\cos\theta\,d\theta.$$

Thus

$$\int_{-b}^{b}\frac{2}{3}\frac{x_1{}^3}{a^2}\,dy = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\tfrac{2}{3}\,ab\cos^4\theta\,d\theta = \tfrac{2}{3}ab\left(\frac{3\pi}{8}\right) = \tfrac{1}{4}\pi ab.$$

It is clear from the symmetry of this last result with respect to the two axes that

$$\iint\frac{y^2}{b^2}\,dx\,dy = \iint\frac{x^2}{a^2}\,dx\,dy = \tfrac{1}{4}\pi ab, \quad . \quad . \quad . \quad (14.181)$$

the integrals being taken over the whole area of the ellipse.

Again

$$\int_{-b}^{b}\int_{-x_1}^{x_1}\frac{x^4}{a^4}\,dy\,dx=\int_{-b}^{b}\frac{2}{5}\,\frac{{x_1}^5}{a^4}\,dy$$

$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\tfrac{2}{5}\,ab\cos^6\theta d\theta$$

$$=\tfrac{1}{8}\pi ab \quad . \quad . \quad . \quad . \quad . \quad . \quad (14.182)$$

Also

$$\int_{-b}^{b}\int_{-x_1}^{x_1}\frac{x^2y^2}{a^2b^2}\,dy\,dx=\int_{-b}^{b}\frac{2}{3}\,\frac{{x_1}^3y^2}{a^2b^2}\,dy$$

$$=\int_{-b}^{b}\tfrac{2}{3}\,ab\cos^4\theta\sin^2\theta d\theta$$

$$=\tfrac{1}{24}\pi ab; \quad . \quad . \quad . \quad . \quad . \quad (14.183)$$

and

$$\iint dx\,dy=\pi ab \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14.184)$$

Therefore

$$\iint(\nabla_1{}^2w)^2dxdy=\pi abk^2\{\tfrac{1}{2}(A^2+B^2)+\tfrac{1}{3}AB-2C(A+B)+4C^2\} \quad (14.185)$$

By means of the preceding integrals it can also be shown that

$$\iint\left\{\frac{\partial^2w}{\partial x^2}\frac{\partial^2w}{\partial y^2}-\left(\frac{\partial^2w}{\partial x\partial y}\right)^2\right\}dx\,dy$$

$$=\frac{\pi k^2}{ab}\{\tfrac{3}{8}(m+n)^2-\tfrac{1}{2}mn-m-n+1\} \quad (14.186)$$

$$\iint pw\,dxdy=\frac{\pi}{8}kpab\{1-\tfrac{1}{6}(m+n)\} \quad . \quad . \quad . \quad . \quad (14.187)$$

Therefore equation (14.163) gives

$$\frac{\pi}{8}kpab\{1-\tfrac{1}{6}(m+n)\}$$

$$=\pi abk^2E'I\{\tfrac{1}{2}(A^2+B^2)+\tfrac{1}{3}AB-2C(A+B)+4C^2\}$$

$$-2(1-\sigma)E'I\frac{\pi k^2}{ab}\{\tfrac{3}{8}(m+n)^2-\tfrac{1}{2}mn-m-n+1\} \quad (14.188)$$

This gives the approximate value of k, and the substitution of this value in (14.177) gives the deflexion.

267. The second approximate method.

To illustrate the second method we shall apply it to a problem that we have already solved, namely, the problem of the circular plate supported, without clamping at the rim, under a uniform pressure.

Let us assume that the deflexion at distance r from the centre is

$$w=k(a^2-r^2)(a^2-nr^2), \quad . \quad . \quad . \quad . \quad . \quad (14.189)$$

a being the radius of the plate, and k and n being constants which have to be determined so as to make the energy a minimum.

Here
$$\nabla_1{}^2 w = \frac{d^2w}{dr^2} + \frac{1}{r}\frac{dw}{dr}$$
$$= 4k\left\{4nr^2 - (n+1)a^2\right\}$$
$$\int \frac{dA}{\varrho_1\varrho_2} = \int_0^a \frac{1}{r}\frac{d^2w}{dr^2}\frac{dw}{dr}\,2\pi r\,dr$$
$$= \left[\pi\left(\frac{dw}{dr}\right)^2\right]_0^a$$
$$= 4\pi k^2a^6(n-1)^2$$
$$\int(\nabla_1{}^2 w)^2 dA = \frac{}{3}\pi k^2a^6(7n^2 - 6n + 3)$$
$$\int pw\,dA = \tfrac{1}{6}\pi kpa^6(3-n)$$

Therefore the expression for the total potential energy is, by (14.164),
$$V = E'I\pi k^2a^6\left\{\tfrac{8}{3}(7n^2-6n+3) - 4(1-\sigma)(n-1)^2\right\} - \tfrac{1}{6}\pi kpa^6(3-n) \quad . \; (14.190)$$

The conditions that V should be a minimum for variations of the parameters k and n are
$$\frac{\partial V}{\partial k} = 0, \qquad \frac{\partial V}{\partial n} = 0;$$
that is,
$$2E'I\pi ka^6\left\{\tfrac{8}{3}(7n^2-6n+3) - 4(1-\sigma)(n-1)^2\right\} = \tfrac{1}{6}\pi pa^6(3-n), \quad (14.191)$$
and
$$E'I\pi k^2a^6\left\{\tfrac{8}{3}(14n-6) - 8(1-\sigma)(n-1)\right\} = -\tfrac{1}{6}\pi pa^6k \quad (14.192)$$

Dividing the sides of the first of these equations by the corresponding sides of the second we get, after removing the common factor $\frac{1}{k}$,
$$\frac{\tfrac{8}{3}(7n^2-6n+3) - 4(1-\sigma)(n-1)^2}{\tfrac{8}{3}(7n-3) - 4(1-\sigma)(n-1)} = n-3,$$
whence
$$\frac{\tfrac{8}{3}(-3n+3) - 4(1-\sigma)(-n+1)}{\tfrac{8}{3}(7n-3) \quad 4(1-\sigma)(n-1)} = -3.$$

The solution of this is
$$n = \frac{1+\sigma}{5+\sigma} \quad . \; . \; . \; . \; . \; . \; . \; (14.193)$$

The substitution of this value for n in (14.192) gives
$$E'Ik = \frac{p}{64n} \quad . \; . \; . \; . \; . \; . \; . \; (14.194)$$

Then the final value of w is
$$w = \frac{p}{64E'I}(a^2-r^2)\left(\frac{a^2}{n} - r^2\right)$$
$$= \frac{p}{64E'I}(a^2-r^2)\left(\frac{5+\sigma}{1+\sigma}a^2 - r^2\right), \quad . \; . \; . \; (14.195)$$

which differs only by a constant from the value of w given by the direct method in equation (14.79). This constant difference is due to the fact that w is measured from the tangent plane at the centre in one case and from the plane of the rim in the other.

In the case that has just been taken the method used has given an absolutely accurate result. This is, of course, a consequence of our having assumed an expression for w which agrees with the accurate expression for w when particular values of the parameters are substituted; that is, the accurate expression for w is one of the possible values of the assumed expression as the parameters are varied; this, however, will not be true in any useful applications of the method, because the method is useful only when the assumed expression for w is a fairly simple one, and if the accurate expression for w is simple it can be found by easier and more direct methods.

268. Example of a disk with variable thickness.

When the thickness of a disk is not constant equation (14.35) must be used to find the deflexion for a given pressure. As an example we shall take

$$h = k + cr^2, \quad . \quad . \quad . \quad (14.196)$$

and assume the disk to be supported at the rim and to carry a uniform load p per unit area in addition to its own weight. Then, if ϱ denotes the weight of unit volume of the material, equation (14.35) can be written in the form

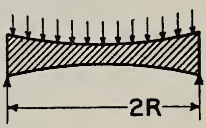


Fig. 144

$$\tfrac{2}{3}E'\nabla_1^2(h^3\nabla_1^2 w) = p + 2\varrho h$$
$$= p + 2\varrho(k + cr^2);$$

that is,

$$\nabla_1^2(h^3\nabla_1^2 w) = a + br^2, \quad . \quad . \quad . \quad . \quad . \quad (14.197)$$

where

$$\left.\begin{aligned} a &= \frac{3}{2}\frac{p + 2\varrho k}{E'} \\ b &= \frac{3\varrho c}{E'} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (14.198)$$

Since w is a function of r only, the equation for w becomes

$$\frac{1}{r}\frac{d}{dr}\left\{r\frac{d}{dr}(h^3\nabla_1^2 w)\right\} = a + br^2,$$

whence

$$r\frac{d}{dr}\{h^3(\nabla_1^2 w)\} = \tfrac{1}{2}ar^2 + \tfrac{1}{4}br^4.$$

No constant of integration is needed in this last equation because both sides clearly vanish at the centre of the plate.

Integrating again we get

$$h^3\nabla_1^2 w = \tfrac{1}{4}ar^2 + \tfrac{1}{16}br^4 + A,$$

that is,

$$(k+cr^2)^3\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)=\tfrac{1}{4}ar^2+\tfrac{1}{16}br^4+\text{A}. \quad . \quad . \quad (14.199)$$

Therefore

$$r\frac{dw}{dr}=\int\frac{\frac{1}{4}ar^2+\frac{1}{16}br^4+\text{A}}{(k+cr^2)^3}rdr.$$

Now

$$r^2=\frac{h-k}{c},$$

$$rdr=\frac{dh}{2c}.$$

Consequently

$$r\frac{dw}{dr}=\int\frac{\frac{1}{4}\frac{a}{c}(h-k)+\frac{1}{16}\frac{b}{c^2}(h-k)^2+\text{A}}{h^3}\frac{dh}{2c}$$

$$=\frac{1}{64c^3}\left\{2b\log h+\frac{4bk-8ac}{h}-\frac{bk^2-4ack+16\text{A}c^2}{h^2}+\text{B}\right\}. \quad (14.200)$$

Finally

$$w=\int\left(r\frac{dw}{dr}\right)\frac{dr}{r}=\int\left(r\frac{dw}{dr}\right)\frac{rdr}{r^2}$$

$$=\frac{1}{2}\int\left(r\frac{dw}{dr}\right)\frac{dh}{h-k}$$

$$=\frac{1}{128c^3}\int\left\{2b\log h+\frac{4bk-8ac}{h}-\frac{bk^2-4ack+16\text{A}c^2}{h^2}+\text{B}\right\}\frac{dh}{h-k}.$$

By resolving into partial fractions the fractions to be integrated we find that

$$w=\frac{1}{128c^3}\int\left\{\begin{array}{l}\dfrac{2b\log h}{h-k}+\dfrac{3bk^2-4ack-16\text{A}c^2}{k^2}\left(\dfrac{1}{h-k}-\dfrac{1}{h}\right)\\[2ex]+\dfrac{bk^2-4ack+16\text{A}c^2}{k}\dfrac{1}{h^2}+\dfrac{\text{B}}{h-k}\end{array}\right\}dh$$

$$=\frac{1}{128c^3}\left\{\begin{array}{l}\dfrac{3bk^2-4ack-16\text{A}c^2}{k^2}\log\dfrac{h-k}{h}\\[2ex]+\text{B}\log(h-k)-\dfrac{bk^2-4ack+16\text{A}c^2}{k}\dfrac{1}{h}\end{array}\right\}$$

$$+\frac{b}{64c^3}\int\frac{\log h}{h-k}dh+\text{C}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14.201)$$

The boundary conditions are that the bending moment about the circumference of the rim is zero, and the deflexion is zero at the rin on the assumption that deflexion is measured from the plane of the rim. Moreover, it is clear from equation (14.200) that B is determined by making the right hand side of that equation zero when $r = 0$ since the left hand side is clearly zero when $r = 0$. Thus, if R denotes the radius of the disk, the equations to determine the constants are

$$\left.\begin{aligned} \frac{d^2w}{dr^2} + \frac{\sigma}{r}\frac{dw}{dr} &= 0 \\ w &= 0 \end{aligned}\right\} \text{where } r = \mathrm{R} \qquad \begin{aligned} &(14.202) \\ &(14.203) \end{aligned}$$

and

$$2b\log k + \frac{4bk - 8ac}{k} - \frac{bk^2 - 4ack + 16\mathrm{A}c^2}{k^2} + \mathrm{B} = 0.$$

The last equation can be written thus

$$\mathrm{B} - \frac{16\mathrm{A}c^2}{k^2} = \frac{4ac}{k} - 3b - 2b\log k. \quad (14.204)$$

Now

$$\nabla_1{}^2 w = \frac{d^2w}{dr^2} + \frac{1}{r}\frac{dw}{dr}.$$

Therefore

$$\frac{d^2w}{dr^2} + \frac{\sigma}{r}\frac{dw}{dr} = \nabla_1{}^2 w - \frac{1-\sigma}{r}\frac{dw}{dr}.$$

It follows then that equation (14.202) is satisfied if

$$(1-\sigma)h^3 r\frac{dw}{dr} = r^2h^3\nabla_1{}^2 w \text{ where } r = \mathrm{R},$$

that is, if

$$\frac{1-\sigma}{64c^3}\{2h_1{}^3 b\log h_1 + (4bk - 8ac)h_1{}^2 - (bk^2 - 4ack + 16\mathrm{A}c^2)h_1 + \mathrm{B}h_1{}^3\}$$
$$= \mathrm{R}^2(\tfrac{1}{4}a\mathrm{R}^2 + \tfrac{1}{16}b\mathrm{R}^4 + \mathrm{A}), \quad (14.205)$$

h_1 being written for the value of h at the rim.

Equations (14.204) and (14.205) determine A and B, and consequently the stresses in the disk. The constant C is unimportant since it depends only on the plane from which w is measured.

If the actual amount of the deflexion is required it will be necessary to find the value of the integral in equation (14.201). This integral is infinite, however, when $h = k$, but it is obvious that the deflexion is not infinite. The infinite value of the integral is, in fact, neutralised by an equal infinity of opposite sign arising from the term $\log(h-k)$ in w. The coefficient of $\log(h-k)$ in the value of w is

$$\frac{1}{128c^3}\left\{\frac{3bk^2 - 4ack - 16\mathrm{A}c^2}{k^2} + \mathrm{B}\right\}$$

which, by (14.204), reduces to

$$-\frac{b\log k}{64c^3}.$$

Grouping together the two terms in (14.201) which may become infinite we get, on omitting the factor $\frac{b}{64c^3}$,

$$\int\frac{\log h}{h-k}\,dh-\log k\log(h-k)$$

$$=\log h\log(h-k)-\int\frac{\log h-k)}{h}\,dh-\log k\log(h-k)$$

$$=\log\frac{h}{k}\log(h-k)-\int\frac{\log(h-k)}{h}\,dh.$$

As h approaches k the first term approaches zero and the integral approaches a finite limit. To evaluate the integral we can expand the logarithm in powers of $\frac{k}{h}$. Thus

$$\int\frac{\log(h-k)}{h}\,dh=\int\frac{\log h+\log\left(1-\frac{h}{k}\right)}{h}\,dh$$

$$=\int\left\{\frac{\log h}{h}-\frac{k}{h^2}-\frac{k^2}{2h^3}-\frac{k^3}{3h^4}\cdots\right\}dh$$

$$=\tfrac{1}{2}(\log h)^2+\frac{k}{h}+\frac{k^2}{2^2h^2}+\frac{k^3}{3^2h^3}+\cdots.$$

When $h=k$ the value of this series is

$$\tfrac{1}{2}(\log k)^2+1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots.$$

$$=\tfrac{1}{2}(\log k)^2+\tfrac{1}{6}\pi^2 \qquad \text{(14.206)}$$

The rest of the work for the solution of this problem consists only in finding the constants A, B, C, from equations (14.203), (14.204), (14.205). The general case is straightforward but laborious. It becomes, however, much easier when numerical values of p, k, c, and ϱ, are given.

CHAPTER XV.

THE BENDING OF THIN PLATES. MORE ACCURATE THEORY.

269. The strains in the middle surface.

We remarked in the last chapter that the theory therein given for the bending of thin plates is accurate enough only on the assumption that the deflexion w, measured from the developable surface which leaves the maximum deflexion as small as possible, is much smaller than the thickness of the plate. When this deflexion is comparable with the thickness the inevitable strains of the middle surface itself are no longer negligible. To complete the theory of the bent plate we shall now take account of these strains of the middle surface, and the equations that we finally obtain are valid wherever the slopes $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ are small fractions.

Let the tangent plane at some point of the bent middle surface of a plate be taken as the xy plane, and let a particle of this middle surface, which would be at x, y, 0, if the plate were not strained, be displaced to $x+u_0$, $y+v_0$, w, where u_0, v_0, w, are functions of x and y only. Let dx, dy, denote the components of the line ds joining two particles in the unstrained state, and let ds_1 be the strained length of the same line. Then

$$(ds)^2 = (dx)^2 + (dy)^2$$
$$(ds_1)^2 = (dx + du_0)^2 + (dy + dv_0)^2 + (dw)^2.$$

Now we may neglect the quantities $(du_0)^2$, $(dv_0)^2$, since we shall be retaining the more important quantities du_0 and dv_0. We have, however, no such reason for neglecting $(dw)^2$. In fact, it is easy to conceive that, for a thin plate, $(dw)^2$ might be a quantity of the same order as $dx du_0$ and $dy dv_0$. It will be clear from the results in the rest of this chapter that the term $(dw)^2$ is, in many cases, quite as important as $dx du_0$ and $dy dv_0$. Then neglecting $(du_0)^2$ and $(dv_0)^2$ we get

$$(ds_1)^2 = (dx)^2 + (dy)^2 + 2 dx\, du_0 + 2 dy\, dv_0 + (dw)^2$$
$$= (ds)^2 + 2\, dx\, du_0 + 2\, dy\, dv_0 + (dw)^2.$$

The extensional strain of ds_1 is

$$\frac{ds_1 - ds}{ds} = \frac{(ds_1)^2 - (ds)^2}{ds(ds_1 + ds)}$$

$$= \frac{2\,dx\,du_0 + 2\,dy\,dv_0 + (dw)^2}{2(ds)^2} \text{ nearly}$$

$$= \frac{dx}{ds}\frac{du_0}{ds} + \frac{dy}{ds}\frac{dv_0}{ds} + \frac{1}{2}\left(\frac{dw}{ds}\right)^2.$$

Now suppose ds makes an angle θ with the x-axis. Then

$$\cos\theta = \frac{dx}{ds}, \quad \sin\theta = \frac{dy}{ds}.$$

Therefore the above strain can be written in the form

$$\cos\theta\frac{du_0}{ds} + \sin\theta\frac{dv_0}{ds} + \frac{1}{2}\left(\frac{dw}{ds}\right)^2.$$

Putting $\theta = 0$ and $dx = ds$ in this we get the strain in the x-direction, namely,

$$\alpha_0 = \frac{\partial u_0}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (15.1)$$

Similarly by putting $\theta = \frac{\pi}{2}$ and $ds = dy$, we find that the strain in the y-direction is

$$\beta_0 = \frac{\partial v_0}{\partial y} + \frac{1}{2}\left(\frac{dw}{\partial y}\right)^2 \quad . \quad . \quad (15.2)$$

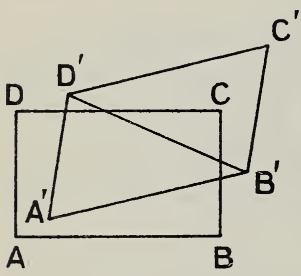


Fig. 145

We have next to get the shear strain of the element which was originally a rectangle with sides dx and dy. Let ABCD (fig. 145) be this rectangle, and A′B′C′D′ the figure into which it is strained.

Now by the definition of shear strain, given in chapter II, the shear strain of A′B′C′D′ is the radian measure of the difference of the angles at A and A′. Let the angle at A′ be $\frac{1}{2}\pi - \theta$. Then the shear strain is θ, which is approximately $\sin\theta$, and

$$\sin\theta = \cos(\tfrac{1}{2}\pi - \theta).$$

But

$$B'D'^2 = A'B'^2 + A'D'^2 - 2A'B'.A'D'\cos(\tfrac{1}{2}\pi - \theta).$$

Therefore

$$\sin\theta = \frac{A'B'^2 + A'D'^2 - B'D'^2}{2A'B'.A'D'}$$

$$= \frac{A'B'^2 + A'D'^2 - B'D'^2}{2\,dx\,dy} \text{ nearly.} \quad . \quad . \quad . \quad . \quad . \quad (15.3)$$

Now the coordinates of B′ relative to A′ are

$$\left(1+\frac{\partial u_0}{\partial x}\right)dx,\ \frac{\partial v_0}{\partial x}dx,\ \frac{\partial w}{\partial x}dx.$$

Also the coordinates of D′ relative to A′ are

$$\frac{\partial u_0}{\partial y}dy,\ \left(1+\frac{\partial v_0}{\partial y}\right)dy,\ \frac{\partial w}{\partial y}dy.$$

Therefore

$$A'B'^2=\left\{\left(1+\frac{\partial u_0}{\partial x}\right)^2+\left(\frac{\partial v_0}{\partial x}\right)^2+\left(\frac{\partial w}{\partial x}\right)^2\right\}(dx)^2,$$

$$A'D'^2=\left\{\left(\frac{\partial u_0}{\partial y}\right)^2+\left(1+\frac{\partial v_0}{\partial y}\right)^2+\left(\frac{\partial w}{\partial y}\right)^2\right\}(dy)^2,$$

Here again we may neglect squares of $\frac{\partial u_0}{\partial x}$, $\frac{\partial v_0}{\partial x}$, $\frac{\partial u_0}{\partial y}$, $\frac{\partial v_0}{\partial y}$. Then we get

$$A'B'^2=\left\{1+2\frac{\partial u_0}{\partial x}+\left(\frac{\partial w}{\partial x}\right)^2\right\}(dx)^2,$$

$$A'D'^2=\left\{1+2\frac{\partial v_0}{\partial y}+\left(\frac{\partial w}{\partial y}\right)^2\right\}(dy)^2.$$

Again by subtracting the coordinates of B′ from those of D′ and squaring and adding the relative coordinates we get

$$B'D'^2=\left(\frac{\partial u_0}{\partial y}dy-\frac{\partial u_0}{\partial x}dx-dx\right)^2+\left(dy+\frac{\partial v_0}{\partial y}dy-\frac{\partial v_0}{\partial x}dx\right)^2$$
$$+\left(\frac{\partial w}{\partial y}dy-\frac{\partial w}{\partial x}dx\right)^2.$$

When squares and products of $\frac{\partial u_0}{\partial x}$, etc., are again neglected this becomes

$$B'D'^2=\left\{1+2\frac{\partial u_0}{\partial x}+\left(\frac{\partial w}{\partial x}\right)^2\right\}(dx)^2+\left\{1+2\frac{\partial v_0}{\partial y}+\left(\frac{\partial w}{\partial y}\right)^2\right\}(dy)^2$$
$$-2dxdy\left\{\frac{\partial u_0}{\partial y}+\frac{\partial v_0}{\partial x}+\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right\}.$$

Now equation (15.3) for the shear strain reduces to

$$\sin\theta=\frac{\partial u_0}{\partial y}+\frac{\partial v_0}{\partial x}+\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}. \quad . \quad . \quad . \quad . \quad . \quad (15.4)$$

We shall write c_0 for this last strain, in agreement with the notation in Chapter II. Thus we have found the following three expressions for the strains in the middle surface

$$a_0=\frac{\partial u_0}{\partial x}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15.5)$$

$$\beta_0=\frac{\partial v_0}{\partial y}+\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15.6)$$

$$c_0=\frac{\partial v_0}{\partial x}+\frac{\partial u_0}{\partial y}+\frac{\partial w}{\partial x}\cdot\frac{\partial w}{\partial y}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (15.7)$$

The assumption in the usual theory is that these strains vanish or are negligible when u_0 and v_0 are zero, but it will be clear from particular examples in this chapter that this assumption is not always correct.

270. Mean stresses in the plate.

When we take account of the strains of the middle surface the equation for u and v in (14.1) must be written

$$u=u_0-z\left(\frac{\partial f}{\partial x}+\frac{\partial \xi}{\partial x}\right)+z^3\frac{\partial \psi}{\partial x},$$

$$v=v_0-z\left(\frac{\partial f}{\partial y}+\frac{\partial \xi}{\partial y}\right)+z^3\frac{\partial \psi}{\partial y},$$

and w need not be altered; that is, w is approximately the same for all the particles in a line perpendicular to the faces of the plate. It follows from what we have proved in this chapter that the strains in equations (14.2) should now be written

$$\begin{aligned}a&=\frac{\partial u}{\partial x}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\\&=\frac{\partial u_0}{\partial x}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2+\frac{\partial^2}{\partial x^2}\{z(f+\xi)+z^3\psi\}\\&=a_0+\frac{\partial^2}{\partial x^2}\{z(f+\xi)+z^3\psi\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15.8)\end{aligned}$$

and

$$\beta=\beta_0+\frac{\partial^2}{\partial y^2}\{z(f+\xi)+z^3\psi\}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15.9)$$

Also the shear strain c becomes

$$\begin{aligned}c&=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}+\frac{\partial w}{\partial x}\cdot\frac{\partial w}{\partial y}\\&=\frac{\partial u_0}{\partial y}+\frac{\partial v_0}{\partial x}+\frac{\partial w}{\partial x}\cdot\frac{\partial w}{\partial y}\\&\quad+\frac{\partial^2}{\partial x\partial y}\{-2z(f+\xi)+z^3\psi\}\\&=c_0+\frac{\partial^2}{\partial x\partial y}\{-2z(f+\xi)+z^3\psi\} \quad . \quad . \quad . \quad . \quad . \quad (15.10)\end{aligned}$$

The additions of a_0, β_0, c_0, to the strains in the last chapter will alter the expressions for the stresses P_1, P_2, S_3, as they are given in (14.29), (14.30), (14.31), to

$$P_1 = E'(a_0 + \sigma\beta_0) - E'z\left(\frac{\partial^2 w}{\partial x^2} + \sigma\frac{\partial^2 w}{\partial y^2}\right) \quad . \quad . \quad . \quad (15.11)$$

$$P_2 = E'(\beta_0 + \sigma a_0) - E'z\left(\frac{\partial^2 w}{\partial y^2} + \sigma\frac{\partial^2 w}{\partial x^2}\right) \quad . \quad . \quad . \quad (15.12)$$

$$S_3 = nc_0 - 2nz\frac{\partial^2 w}{\partial x \partial y} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15.13)$$

The values of these stresses at the middle surface are obtained by omitting the terms which have a coefficient z. Moreover, the mean values of each of these stresses from one face to the opposite face of the plate, that is, from $z = -h$ to $z = +h$, is clearly the same as the value of that stress at the middle surface. Thus

$$\frac{1}{2h}\int_{-h}^{h} P_1 dz = E'(a_0 + \sigma\beta_0),$$

since a_0, β_0, and w, are not functions of z.

In the rest of the chapter we shall use dashed letters to indicate the mean stresses across the thickness of the plate. Thus

$$P'_1 = E'(a_0 + \sigma\beta_0) \quad . \quad . \quad . \quad . \quad . \quad . \quad (15.14)$$

$$P'_2 = E'(\beta_0 + \sigma a_0) \quad . \quad . \quad . \quad . \quad . \quad . \quad (15.15)$$

$$S'_3 = nc_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15.16)$$

271. The pressure supported by the mean stresses.

Another equation that needs modification when we take account of the stresses in the middle surface is equation (14.15). In that equation no allowance is made for the mean stresses P'_1, P'_2, S'_3. We shall now find the additional terms on the right of (14.15) due to these mean stresses.

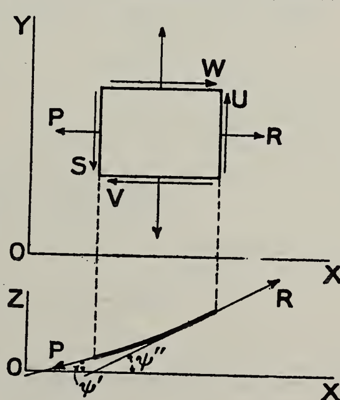


Fig. 146 a

Fig. 146 b

Let us find what pressure on one face of a bent plate can be supported by the mean stresses. We shall deal with the equilibrium of an element of the plate with sides dx, dy.

Let P and R denote the mean values of P'_1 over the two edges of length dy of the element $dx \times dy$, and let these stresses be inclined, as shown in fig. 146, at angles $-\psi'$ and $+\psi''$ with the x-axis. Then the force on the area $2hdy$ due to P is $2Phdy$. The component of this parallel to OZ is $-2P\sin\psi' hdy$ which is approximately $-2P\tan\psi' hdy$. Thus the total force in the direction OZ due to P and R is

$$2hdy\{R\tan\psi''-P\tan\psi'\}.$$

But

$$\tan\psi'=\frac{\partial w}{\partial x},\ \tan\psi''=\tan\psi'+\frac{\partial}{\partial x}(\tan\psi')dx;$$

also

$$R=P+\frac{\partial P}{\partial x}dx,$$

and, in the limit, P becomes P'_1.

Therefore the total force due to P and R in the direction OZ is

$$2hdy\frac{\partial}{\partial x}\left(P'_1\frac{\partial w}{\partial x}\right)dx.$$

Likewise the force in the direction OZ due to the tensions across the other pair of edges is

$$2hdx\frac{\partial}{\partial y}\left(P'_2\frac{\partial w}{\partial y}\right)dy.$$

Again let the mean values of S_3 over the four edges of the element be S, U, V, W; let S be inclined to the y-axis at an angle $-\varphi'$, and U inclined to the same axis at φ''. Then the component in the direction OZ due to the mean stress S is $-2hdy\,S\sin\varphi'$ or approximately $-2hSdy\tan\varphi'$. Thus the total force in the direction OZ due to S and U is

$$2hdy\{S\tan\varphi'-U\tan\varphi''\}$$

But

$$\tan\varphi'=\frac{\partial w}{\partial y},\ \tan\varphi''=\tan\varphi'+\frac{\partial}{\partial x}(\tan\varphi')dx;$$

also

$$U=S+\frac{\partial S}{dx}dx,$$

and, in the limit, S becomes S'_3.

Therefore the above total force due to S and U is

$$2hdy\frac{\partial}{\partial x}\left(S'_3\frac{\partial w}{\partial y}\right)dx.$$

Likewise the corresponding force due to the stresses V and W is

$$2hdx\frac{\partial}{\partial y}\left(S'_3\frac{\partial w}{\partial x}\right)dy.$$

Therefore the total force in the direction OZ due to all the mean stresses P'_1, P'_2, S'_3, acting on the edges of the element $dx \times dy$ is

$$2hdxdy\left\{\frac{\partial}{\partial x}\left(P'_1\frac{\partial w}{\partial x}\right)+\frac{\partial}{\partial y}\left(P'_2\frac{\partial w}{\partial y}\right)+\frac{\partial}{\partial x}\left(S'_3\frac{\partial w}{\partial y}\right)+\frac{\partial}{\partial y}\left(S'_3\frac{\partial w}{\partial x}\right)\right\} \quad (15.17)$$

This expression, divided by the area $dx\,dy$, should be subtracted from the right hand side of equation (14.15).

272. Relations between the stresses in the middle surface. Equations (2.24) and (2.25) are

$$\frac{\partial P_1}{\partial x}+\frac{\partial S_3}{\partial y}+\frac{\partial S_2}{\partial z}+\varrho X=\varrho f_1,$$

$$\frac{\partial P_2}{\partial y}+\frac{\partial S_3}{\partial x}+\frac{\partial S_1}{\partial z}+\varrho Y=\varrho f_2.$$

When a plate is bent by forces perpendicular to its plane the quantities X, Y, f_1, f_2, are all zero. Then

$$\frac{\partial P_1}{\partial x}+\frac{\partial S_3}{\partial y}+\frac{\partial S_2}{\partial z}=0, \quad \ldots\ldots \quad (15.18)$$

$$\frac{\partial P_2}{\partial y}+\frac{\partial S_3}{\partial x}+\frac{\partial S_1}{\partial z}=0. \quad \ldots\ldots \quad (15.19)$$

Now integrating, with respect to z, throughout the first of these equations, between the limits $z=-h$ and $z=+h$, we get

$$\int_{-h}^{h}\left(\frac{\partial P_1}{\partial x}+\frac{\partial S_3}{\partial y}\right)dz+\Big[S_2\Big]_{-h}^{h}=0. \quad \ldots\ldots \quad (15.20)$$

But

$$\int_{-h}^{h}\frac{\partial P_1}{\partial x}\,dz=\frac{\partial}{\partial x}\int_{-h}^{h}P_1\,dz=\frac{\partial}{\partial x}(2hP'_1)$$

since P'_1 is the mean value of P_1 between these limits. In the same way we find

$$\int_{-h}^{h}\frac{\partial S_3}{\partial y}\,dz=\frac{\partial}{\partial y}(2hS'_3).$$

Also S_2 is zero at both surfaces of the plate, and therefore

$$\Big[S_2\Big]_{-h}^{h}=0.$$

Consequently, equation (15.20) becomes, after division by $2h$,

$$\frac{\partial P'_1}{\partial x}+\frac{\partial S'_3}{\partial y}=0. \quad \ldots\ldots \quad (15.21)$$

Likewise equation (15.19) leads to the equation

$$\frac{\partial P'_2}{\partial y}+\frac{\partial S'_3}{\partial x}=0. \quad \ldots\ldots \quad (15.22)$$

Now by means of equations (15.21) and (15.22) the expression in large brackets in (15.17) reduces to

$$P'_1\frac{\partial^2 w}{\partial x^2}+P'_2\frac{\partial^2 w}{\partial y^2}+2S'_3\frac{\partial^2 w}{\partial x\,\partial y}.$$

This, multiplied by $2h$, is the expression which has to be added to p in the last chapter. With this change, equation (14.21) becomes

$$p = \frac{EI}{1 - \sigma^2} \nabla_1^4 w - 2h \left(P'_1 \frac{\partial^2 w}{\partial x^2} + P'_2 \frac{\partial^2 w}{\partial y^2} + 2S'_3 \frac{\partial^2 w}{\partial x \partial y} \right). \quad (15.23)$$

By means of equations (15.21) and (15.22) the three quantities P'_1, P'_2, S'_3, can be expressed in terms of a single function. Let φ be a function such that

$$-E \frac{\partial^2 \varphi}{\partial x \partial y} = S'_3. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15.24)$$

Then equations (15.21) and (15.22) give

$$P'_1 = E \frac{\partial^2 \varphi}{\partial y^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15.25)$$

$$P'_2 = E \frac{\partial^2 \varphi}{\partial x^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15.26)$$

Finally the equation for p is

$$p = \frac{EI}{1 - \sigma^2} \nabla_1^4 w - 2hE \left\{ \frac{\partial^2 \varphi}{\partial y^2} \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 \varphi}{\partial x^2} \frac{\partial^2 w}{\partial y^2} - 2 \frac{\partial^2 \varphi}{\partial x \partial y} \frac{\partial^2 w}{\partial x \partial y} \right\}. \quad (15.27)$$

273. Relation between φ and w.

When p is given this last equation involves two unknown functions, w and φ, which cannot be determined from a single equation. Then we need another equation connecting these two functions, and this other equation we shall now find.

The stress-strain equations in the middle surface are

$$P'_1 - \sigma P'_2 = E\alpha_0 = E \left\{ \frac{\partial u_0}{\partial x} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 \right\}, \quad . \quad . \quad . \quad (15.28)$$

$$P'_2 - \sigma P'_1 = E\beta_0 = E \left\{ \frac{\partial v_0}{\partial y} + \frac{1}{2} \left(\frac{\partial w}{\partial y} \right)^2 \right\}, \quad . \quad . \quad . \quad (15.29)$$

$$S'_3 = nc_0 = n \left\{ \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} + \frac{\partial w}{\partial x} \cdot \frac{\partial w}{\partial y} \right\}. \quad . \quad . \quad (15.30)$$

Writing $\frac{E}{2(1+\sigma)}$ for n, and then eliminating u_0 and v_0 from these three equations, we get

$$\frac{\partial^2}{\partial y^2}(P'_1 - \sigma P'_2) + \frac{\partial^2}{\partial x^2}(P'_2 - \sigma P'_1) - 2(1 + \sigma) \frac{\partial^2 S'_3}{\partial x \partial y}$$
$$= \tfrac{1}{2} E \left\{ \frac{\partial^2}{\partial y^2} \left(\frac{\partial w}{\partial x} \right)^2 + \frac{\partial^2}{\partial x^2} \left(\frac{\partial w}{\partial y} \right)^2 - 2 \frac{\partial^2}{\partial x \partial y} \left(\frac{\partial w}{\partial x} \cdot \frac{\partial w}{\partial y} \right) \right\},$$

which can be written in the form

$$\frac{\partial^4\varphi}{\partial x^4}+2\frac{\partial^4\varphi}{\partial x^2\partial y^2}+\frac{\partial^4\varphi}{\partial y^4}=\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2-\frac{\partial^2 w}{\partial x^2}\cdot\frac{\partial^2 w}{\partial y^2},$$

or

$$\nabla_1{}^4\varphi=\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2-\frac{\partial^2 w}{\partial x^2}\cdot\frac{\partial^2 w}{\partial y^2}. \quad . \quad . \quad . \quad . \quad (15.31)$$

We have now reached the necessary differential equations for the problem we set out to solve. When p is given as a function of x and y, equations (15.27) and (15.31), together with the boundary conditions, determine completely the functions w and φ, and therefore also all the stresses in the plate.

274. Symmetry about the z-axis.

When everything is symmetrical about the z-axis, as when a disk is symmetrically loaded, it was shown in the last chapter that

$$\nabla_1{}^2=\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15.32)$$

where $r=\sqrt{x^2+y^2}$. Also, when we took the x-axis coincident with the radius vector r, we found

$$\left.\begin{aligned}\frac{\partial^2 w}{\partial x^2}&=\frac{d^2 w}{dr^2}\\ \frac{\partial^2 w}{\partial y^2}&=\frac{1}{r}\frac{dw}{dr}\\ \frac{\partial^2 w}{\partial x\partial y}&=\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w}{\partial\theta}\right)=0\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (15.33)$$

The last three equations will, of course, remain correct if we replace w by φ, since, on account of the symmetry, φ is also a function of r only. Therefore equations (15.27) and (15.31) become

$$\begin{aligned}p&=\mathrm{E'I}\nabla_1{}^4 w-2h\mathrm{E}\left\{\frac{1}{r}\frac{d\varphi}{dr}\frac{d^2w}{dr^2}+\frac{1}{r}\frac{dw}{dr}\frac{d^2\varphi}{dr^2}\right\}\\ &=\mathrm{E'I}\nabla_1{}^4 w-\frac{2h\mathrm{E}}{r}\frac{d}{dr}\left(\frac{d\varphi}{dr}\frac{dw}{dr}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad (15.34)\end{aligned}$$

and

$$\nabla_1{}^4\varphi=-\frac{1}{r}\frac{dw}{dr}\frac{d^2w}{dr^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15.35)$$

E′ being written for $\dfrac{\mathrm{E}}{1-\sigma^2}$.

The bending moments, torque, and shearing forces, become

$$\mathrm{M}_1=\mathrm{E'I}\left\{\frac{d^2w}{dr^2}+\frac{\sigma}{r}\frac{dw}{dr}\right\}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (15.36)$$

$$\mathrm{M}_2=\mathrm{E'I}\left\{\frac{1}{r}\frac{dw}{dr}+\sigma\frac{d^2w}{dr^2}\right\}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (15.37)$$

$$Q = 0. \qquad (15.38)$$

$$F_1 = -E'I \frac{d}{dr}(\nabla_1{}^2 w) \qquad (15.39)$$

$$F_2 = 0. \qquad (15.40)$$

The stresses at the middle surface are

$$P'_1 = \frac{E}{r}\frac{d\varphi}{dr}; \; P'_2 = E\frac{d^2\varphi}{dr^2}; \; S'_3 = 0. \qquad (15.41)$$

If u is the radial displacement of a particle of the middle surface then the radial and circumferential strains in the middle surface are

$$\alpha_0 = \frac{du}{dr} + \frac{1}{2}\left(\frac{dw}{dr}\right)^2; \; \beta_0 = \frac{u}{r}. \qquad (15.42)$$

275. First integrals of the equations for w and φ. Since

$$\nabla_1{}^4 = \nabla_1{}^2(\nabla_1{}^2)$$
$$= \frac{1}{r}\frac{d}{dr}\left\{r\frac{d}{dr}(\nabla_1{}^2)\right\}. \qquad (15.43)$$

it is easy to integrate equations (15.34) and (15.35) once. Multiplying (15.34) by r and then integrating from 0 to r we get

$$\int_0^r prdr = E'Ir\frac{d}{dr}(\nabla_1{}^2 w) - 2hE\frac{d\varphi}{dr}\frac{dw}{dr}. \qquad (15.44)$$

No constant need be added to either side of this equation when we are dealing with a disk without a central hole, because, since $\frac{dw}{dr} = 0$ at the centre, all the terms in the equation clearly vanish when $r = 0$.

It is worth while to see the physical aspect of (15.44). If we multiply both sides of the equation by 2π we can write it in the form

$$W = -2\pi rF_1 - 2\pi r(2hP_1)\frac{dw}{dr}. \qquad (15.45)$$

where W denotes the external force on the circle of radius r. The two terms on the right of this last equation are the negative of the total shearing force and the negative of the resultant of the stresses P_1 round the circumference of the circle of radius r. Thus equation (15.45) states the very obvious fact that the forces on any circle are supported by the actions at the rim.

Again, after multiplying both sides of (15.35) by r and integrating with respect to r, we get

$$r\frac{d}{dr}(\nabla_1{}^2\varphi) = -\frac{1}{2}\left(\frac{dw}{dr}\right)^2, \qquad (15.46)$$

no constant being necessary again because both sides vanish when $r = 0$

We cannot easily carry equations (15.44) and (15.46) any further except in special cases. Indeed it is hard to solve the equations for w and φ in the easiest cases, although it is not particularly hard to find p when w is given and the value of P_1 at the rim of a disk is known. We shall now apply our equations to a few problems which have been chosen because they make clearer the meaning of the theory.

276. Circular plate bent into a portion of a spherical surface of small solid angle.

Let a be the radius of the plate, c be the radius of the sphere, and let the z-axis be normal to the middle surface of the plate at its centre. Then the deflexion is approximately

$$w = \frac{r^2}{2c}. \qquad (15.47)$$

Now equation (15.46) gives

$$\frac{d}{dr}(\nabla_1{}^2\varphi) = -\frac{1}{2}\frac{r}{c^2}. \qquad (15.48)$$

Therefore

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\varphi}{dr}\right) = \nabla_1{}^2\varphi = -\frac{1}{4c^2}(r^2-b^2), \qquad (15.49)$$

b^2 being a constant of integration.

Integrating again, we get

$$r\frac{d\varphi}{dr} = -\frac{1}{16c^2}(r^4-2b^2r^2), \qquad (15.50)$$

no constant being necessary because both sides vanish where $r=0$.

Now the mean tensions are

$$\left.\begin{aligned} P'_1 &= \frac{E}{r}\frac{d\varphi}{dr} = \frac{E}{16c^2}(2b^2-r^2) \\ P'_2 &= E\frac{d^2\varphi}{dr^2} = \frac{E}{16c^2}(2b^2-3r^2) \end{aligned}\right\} \qquad (15.51)$$

The radial tension P'_1 at the rim of the plate can have any value whatever, depending on what forces are applied at the rim. Suppose $P'_1 = T$ at the rim. Then

$$T = \frac{E}{16c^2}(2b^2-a^2), \qquad (15.52)$$

and consequently

$$P'_1 - T = \frac{E}{16c^2}(a^2-r^2) \qquad (15.53)$$

$$P'_2 - T = \frac{E}{16c^2}(a^2-3r^2). \qquad (15.54)$$

Since $\nabla_1^4 w$ is zero, equation (15.34) gives

$$p = -\frac{2hE}{r}\frac{d}{dr}\left(\frac{dw}{dr}\frac{d\varphi}{dr}\right)$$

$$= -\frac{2hE}{r}\frac{d}{dr}\left\{-\frac{1}{16c^3}(r^4 - 2b^2r^2)\right\}$$

$$= \frac{hE}{2c^3}(r^2 - b^2)$$

$$= \frac{hE}{2c^3}(r^2 - \tfrac{1}{2}a^2) + \frac{4h}{c}T. \quad . \quad . \quad . \quad . \quad . \quad . \quad (15.55)$$

If T is zero the mean hoop tension P'_2 is positive from the centre of the plate as far as the circle where $r = \frac{1}{\sqrt{3}}a$, and is negative in the rest of the plate. Moreover p is positive, which means in the same direction as w, in the region where r is greater than $\frac{1}{\sqrt{2}}a$, and negative in the other portion.

Equations (15.36), (15.37), (15.39), give

$$M_1 = E'I\frac{1+\sigma}{c}$$

$$= \frac{EI}{(1-\sigma)c}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15.56)$$

$$M_2 = M_1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15.57)$$

$$F_1 = 0.$$

Thus the bending moment is constant everywhere and the shearing forces F_1 and F_2 are zero. It follows that there must be a bending couple of magnitude $M_1 ds$ acting about each element ds of the rim of the plate.

The mean radial stress P'_1 is arbitrary to the extent of an added constant, and this added constant occurs in the value of p with a factor $\frac{4h}{c}$. In fact the term $\frac{4h}{c}$ T in the expression for p is merely the pressure inside a *non-rigid* sphere of radius c and thickness $2h$ when the tensional stress in the material is T.

Fig. 147 shows the way in which the pressure p acts across a diametral section and the direction of the bending moment on the rim. The figure is drawn for the case where the tension T at the rim is zero.

Fig. 147

The usual Poisson-Kirchhoff method, that is, the method of the last chapter, would make p, P'_1, P'_2, all zero, but would give the same values of M_1 and M_2.

If the plate is held so that there is no displacement of the particles at the edge of the middle surface, in which case the radial displacement u is zero where $r=a$, we get

$$u = r\beta_0 = rE(P'_2 - \sigma P'_1) = 0 \text{ where } r = a,$$

whence $$2(1-\sigma^2)b^2 = (3-\sigma)a^2,$$

and $$T = \frac{Ea^2}{8c^2(1-\sigma)}.$$

277. Disk supported without clamping at the rim, with deflexion the same as Poisson's theory gives for uniform pressure.

By equation (14.79) the deflexion for a disk supported at the rim under a uniform pressure p_1 is

$$w = -H(2b^2r^2 - r^4), \quad \ldots \quad (15.58)$$

where $$H = \frac{p_1}{64E'I} = \frac{3}{128}\frac{(1-\sigma^2)p_1}{h^3E}, \quad \ldots \quad (15.59)$$

and $$b^2 = \frac{3+\sigma}{1+\sigma}a^2. \quad \ldots \quad (15.60)$$

Now equation (15.46) gives

$$r\frac{d}{dr}(\nabla_1{}^2\varphi) = -\tfrac{1}{2}H^2(4b^2r - 4r^3)^2$$
$$= -8H^2(b^2r - r^3)^2.$$

Integrating this we get

$$\nabla_1{}^2\varphi = -8H^2(\tfrac{1}{2}b^4r^2 - \tfrac{1}{2}b^2r^4 + \tfrac{1}{6}r^6) + B;$$

that is,

$$\frac{d}{dr}\left(r\frac{d\varphi}{dr}\right) = -4H^2(b^4r^3 - b^2r^5 + \tfrac{1}{3}r^7) + Br.$$

Therefore

$$r\frac{d\varphi}{dr} = -4H^2(\tfrac{1}{4}b^4r^4 - \tfrac{1}{6}b^2r^6 + \tfrac{1}{24}r^8) + \tfrac{1}{2}Br^2 \quad \ldots \quad (15.61)$$

Consequently the mean radial tension is

$$P'_1 = \frac{E}{r}\frac{d\varphi}{dr}$$
$$= -EH^2(b^4r^2 - \tfrac{2}{3}b^2r^4 + \tfrac{1}{6}r^6) + \tfrac{1}{2}B.$$

Let us write P_0 for the mean tension at the centre. Then

$$P_0 = \tfrac{1}{2}B,$$

and therefore

$$P'_1 = P_0 - EH^2(b^4r^2 - \tfrac{2}{3}b^2r^4 + \tfrac{1}{6}r^6) \quad \ldots \quad (15.62)$$

Since P'_1 is zero at the rim of the disk, where $r=a$, we get

$$P_0 = H^2a^2(b^4 - \tfrac{3}{2}b^2a^2 + \tfrac{1}{6}a^4) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15.63)$$

Now equation (15.34) gives

$$\begin{aligned}
p &= E'I\nabla_1{}^4 w - \frac{2hE}{r}\frac{d}{dr}\left(\frac{dw}{dr}\cdot\frac{d\varphi}{dr}\right) \\
&= p_1 - \frac{2hE}{r}\frac{d}{dr}\left(\frac{dw}{dr}\cdot\frac{d\varphi}{dr}\right) \\
&= p_1 + 16\,HEP_0h(b^2 - 2r^2) \\
&\quad - \tfrac{8}{3}hEH^3r^2(12b^6 - 30b^4r^2 + 20b^2r^4 - 5r^6) \\
&= p_1 + \tfrac{8}{3}hEH^3a^2(b^2 - 2r^2)(6b^4 - 4b^2a^2 + a^4) \\
&\quad - \tfrac{8}{3}hEH^3r^2(12b^6 - 30b^4r^2 + 20b^2r^4 - 5r^6) \quad . \quad (15.64)
\end{aligned}$$

Now let w_1 denote the magnitude of the deflexion at the rim. Then

$$w_1 = H(2b^2a^2 - a^4).$$

For convenience let sa^2 be written for b^2; that is,

$$s = \frac{3+\sigma}{1+\sigma}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15.65)$$

Then

$$w_1 = Ha^4(2s-1). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15.66)$$

By means of equations (15.59) and (15.66) we find that

$$\tfrac{8}{3}hEH^3a^8 = \frac{1}{16}\frac{1-\sigma^2}{(2s-1)^2}\left(\frac{w_1}{h}\right)^2 p_1. \quad . \quad . \quad . \quad . \quad (15.67)$$

Now the pressure at the centre of the disk, where $r=0$, is

$$p = p_1 + p_1\left(\frac{w_1}{h}\right)^2 \frac{1-\sigma^2}{16}\;\frac{6s^3 - 4s^2 + s}{(2s-1)^2}$$

If $\sigma = 0{\cdot}25$ this becomes

$$p = p_1\left\{1 + 1{\cdot}08\left(\frac{w_1}{2h}\right)^2\right\}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15.68)$$

Also the pressure at the rim, where $r=a$, is

$$\begin{aligned}
p &= p_1 - p_1\left(\frac{w_1}{h}\right)^2 \frac{1-\sigma^2}{16}\;\frac{6s^3 - 14s^2 + 11s - 3}{(2s-1)^2} \\
&= p_1\left\{1 - 0{\cdot}484\left(\frac{w_1}{2h}\right)^2\right\}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15.69)
\end{aligned}$$

Thus equations (15.68) and (15.69) show that, if w_1 is equal to the thickness of the plate, the pressures at the centre and at the rim are $2{\cdot}08\ p_1$ and $0{\cdot}516\ p_1$, whereas the Poisson-Kirchhoff theory gives p_1 as the pressure at every point of the disk. This plainly shows that the latter theory cannot be used when the greatest deflexion of a plate is comparable with the thickness. But if w_1 is one-tenth of the thickness the pressures at the centre and rim are $1{\cdot}01\, p_1$ and $0{\cdot}995\, p_1$. For this case the errors are certainly negligible. It will probably be safe, in every conceivable case, to use the Poisson-Kirchhoff theory whenever the maximum deflexion is not greater than one-fifth of the thickness, for the error in p will probably not be greater than about five per cent, which is certainly less than the errors due to ignorance of the elastic constants.

When w_1 is several times as great as the thickness the Poisson-Kirchhoff theory does not give a pressure of the right order even. For example, if w_1 is five times the thickness, the pressure at the centre and the rim are respectively $27\, p_1$ and $-11\, p_1$, which are vastly different from a uniform pressure p_1!

It is easy to see that the middle surface of a plate may be bent into a developable surface, a cylindrical form, for example, without the slightest stretching or shrinking of the middle surface. It follows that the maximum deflexion may be many times as great as the thickness and yet the Poisson-Kirchhoff theory may give absolutely accurate pressures and stresses in the plate. In fact if φ is zero the equations of the present chapter are the same as those of the last chapter. But if

$$\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 = 0, \quad . \quad . \quad . \quad . \quad . \quad (15{\cdot}70)$$

and if the mean stresses P'_1, P'_2, S'_3, are all zero at the boundary of the plate, then φ is certainly zero. Now (15.70) is the condition that the middle surface should be a developable surface. Thus our equations tell us, what is quite clear from a physical point of view, namely, that the middle surface may be unstrained if a plate is bent into a developable surface.

The condition expressed by (15.70) is not, of course, a sufficient condition that the value of φ given by (15.31) should be zero. If, however, φ is not zero when (15.70) is satisfied the strain in the middle surface which this value of φ indicates is due to the mean stresses applied at the boundary of the plate and not at all to the deflexion w. When the stresses P'_1, P'_2, S'_3, are due to boundary conditions only, then equation (15.23) is just as consistent with Poisson's theory as with the theory of this chapter.

It has been shown in a particular example that Poisson's theory is accurate enough for all practical purposes when the deflexion w, measured from a plane, is small in comparison with the thickness of the plate. Moreover Poisson's theory is quite accurate when the middle surface is a developable surface. It is not difficult to see then that the same theory will be accurate enough when the deflexion, measured from any developable surface, is small in comparison with the thickness.

278. Rigidity of disk negligible.

Poisson's solution of the plate problem is the solution obtained when the stresses in the middle surface produce a negligible effect on the pressure, that is, when the terms involving φ in equation (15.27) are neglected. We now intend to solve the problem of the plate bent by pressure when the contrary assumption is made, namely that the term due to the stresses in the middle surface are the most important on the right hand of (15.27). This means that the rigidity effect on the pressure is negligible in comparison with the effect of the mean stresses. The term containing $\nabla_1{}^4 w$ in (15.27) represents the rigidity effect, and the other terms on the right of that equation represent the effect of the mean stresses. If S'_3 is zero the mean stresses are like the stresses in a membrane.

The present problem is to find the form of a disk in which the flexural rigidity is negligible when the action at the rim is a uniform radial tension and the pressure is constant over the disk.

Since the flexural rigidity is negligible equation (15.44) becomes

$$\tfrac{1}{2}pr^2 = -2hE\frac{d\varphi}{dr}\cdot\frac{dw}{dr}. \qquad (15.71)$$

It is necessary to use also equation (15.46), which we rewrite here,

$$r\frac{d}{dr}\left\{\frac{1}{r}\frac{d}{dr}\left(r\frac{d\varphi}{dr}\right)\right\} = -\frac{1}{2}\left(\frac{dw}{dr}\right)^2.$$

Now putting

$$\xi = r\frac{d\varphi}{dr},\quad \theta = \frac{1}{r}\frac{dw}{dr},\quad s = r^2, \qquad (15.72)$$

the last two equations become

$$\left.\begin{aligned}\theta\xi &= -\frac{p}{4hE}s\\ \frac{d^2\xi}{ds^2} &= -\tfrac{1}{8}\theta^2.\end{aligned}\right\} \qquad (15.73)$$

When θ is eliminated the equation for ξ is

$$\frac{d^2\xi}{ds^2} = -\frac{1}{128}\frac{p^2}{h^2E^2}\frac{s^2}{\xi^2}. \qquad (15.74)$$

Now let us take two new variables s_1 and ξ_1 such that

$$\left.\begin{aligned}s_1{}^4 &= s^4\frac{p^2}{128h^8E^2}\\ \text{and}\qquad h^2\xi_1 &= \xi.\end{aligned}\right\} \qquad (15.75)$$

These two new variables are mere numbers, that is, quantities with no physical dimensions. In terms of the new variables equation (15.74) becomes

$$\frac{d^2\xi_1}{ds_1^2} = -\frac{s_1{}^2}{\xi_1{}^2}. \qquad (15.76)$$

Although the complete solution of this equation involves two arbitrary constants the solution for the complete disk involves only one such constant. It is easy to see how this constant appears in the solution; for, with the substitutions

$$\xi_1 = c^4 y, \quad s_1 = c^3 x, \qquad (15.77)$$

equation (15.76) gives

$$\frac{d^2y}{dx^2} = -\frac{x^2}{y^2}. \qquad (15.78)$$

Thus c is one of the constants in the solution, and we shall see that it is the only constant necessary for a complete disk.

Equation (15.78) has a solution in the form of an infinite series of positive powers of x. To solve the equation it is best to write it in the form

$$y^2 \frac{d^2y}{dx^2} = -x^2$$

and then assume

$$y = a_0 + a_1 x + a_2 x^2 + \ldots.$$

This gives

$$(a_0 + a_1 x + a_2 x^2 + \ldots)^2 (2a_2 + 6a_3 x + 12 a_4 x^2 + \ldots) = -x^2.$$

Equating coefficients of powers of x and making a_1, which is arbitrary, equal to 1, we get

$$a_0 = 0, \quad a_1 = 1, \quad a_2 = -\tfrac{1}{2}, \quad a_3 = -\tfrac{1}{6}, \quad a_4 = -\tfrac{13}{144}, \text{ etc.},$$

and the expression for y is

$$y = x - \tfrac{1}{2}x^2 - \tfrac{1}{6}x^3 - \tfrac{13}{144}x^4 - \tfrac{17}{288}x^5 - \tfrac{37}{864}x^6 - \tfrac{1205}{36288}x^7 - \qquad (15.79)$$

We have here got a value of y for which $\frac{dy}{dx}$ is unity when x is zero. Now equation (15.78) show that $\frac{dy}{dx}$ decreases as x increases for all values of x and y. Moreover, the curve connecting x and y cannot have an asymptote with a finite slope because, along such an asymptote, $\frac{d^2y}{dx^2}$ approaches zero and $\frac{y}{x}$ approaches a finite quantity, and equation (15.78) shows that both these limits cannot be approached at the same time. Then it is clear that $\frac{dy}{dx}$ goes on decreasing till y becomes zero, and this occurs for a finite value of x.

In the immediate neighbourhood of the point where y vanishes and x is finite we may regard x as a constant. Let this constant value of x be x_1. Then equation (15.78) becomes, in this neighbourhood,

$$\frac{d^2y}{dx^2} = -\frac{{x_1}^2}{y^2},$$

a first integral of which is

$$\frac{1}{2}\left(\frac{dy}{dx}\right)^2 = \frac{x_1{}^2}{y} + \mathrm{A}.$$

This last equation shows that $\frac{dy}{dx}$ approaches infinity as y approaches zero. Therefore the curve connecting y and x meets the x-axis perpendicularly at the point where y vanishes and x is finite.

Now if all the remaining terms of the series in (15.79) are negative it is clear that there is some finite value of x which makes the sum of the negative terms of the series equal in magnitude to the one positive term x. For all values of x up to this value of x, which we have called x_1, the series in (15.79) is certainly convergent, and therefore it represents accurately the value of y.

It is a laborious business to find x_1 by equating to zero the series in (15.79). It is easier to make comparisons with other series. There are two simple functions the expansions for which are very similar to the series in (15.79). These are

$$\begin{aligned} y &= -(1-x)\log_e(1-x) \\ &= x - \tfrac{1}{2}x^2 - \tfrac{1}{6}x^3 - \tfrac{1}{12}x^4 - \tfrac{1}{30}x^5 - \tfrac{1}{42}x^6 - \tfrac{1}{56}x^7 - \quad . \; . \; . \end{aligned} \qquad (15.80)$$

and

$$\begin{aligned} y &= x(1-\tfrac{7}{6}x)^{\frac{3}{7}} \\ &= x - \tfrac{1}{2}x^2 - \tfrac{1}{6}x^3 - \tfrac{11}{108}x^4 - \tfrac{11}{144}x^5 - \tfrac{55}{864}x^6 - \tfrac{55}{1134}x^7 - \ldots \end{aligned} \qquad (15.81)$$

By comparing the terms in the three series we see that the value of the series (15.79), at least as far as the term containing x^7, lies between the two series in (15.80) and (15.81). Moreover the curves represented by these last two equations are similar to the curve (15.79) in that they both have the same value of y and the same slope at $x=0$, and they both meet the x-axis perpendicularly at a point where x is finite. It is safe to conclude that the value of y we are seeking lies between the two values in (15.80) and (15.81), and therefore that it vanishes for some value of x between $x=1$ and $x=\frac{6}{7}$, which are the values for which the series in (15.80) and (15.81) vanish.

By substituting for y, on the right of equation (15.78), the value given by (15.81), we can get a still better approximation to the true y. With this substitution equation (15.78) becomes

$$\frac{d^2y}{dx^2} = -(1-\tfrac{7}{6}x)^{-\frac{6}{7}}. \quad . \; . \; . \; . \; . \; . \; . \qquad (15.82)$$

Integrating this, and adjusting the constants so as to satisfy the conditions

$$y=0 \text{ and } \frac{dy}{dx}=1 \text{ where } x=0,$$

we find

$$y=\frac{9}{2}-\frac{9}{2}\left(1-\frac{7}{6}x\right)^{\frac{8}{7}}-5x$$

$$=x-\frac{1}{2}x^2-\frac{1}{6}x^3-\frac{13}{144}x^4-\frac{17\frac{1}{3}}{288}x^5-\frac{39}{864}x^6-\frac{1326}{36288}x^7\ldots \quad (15.83)$$

This differs so little from (15.79) that there can only be an insignificant error in taking one series for the other. Assuming then that this last equation gives the correct value of y then the value of x for which y vanishes is the root of the equation

$$\tfrac{9}{2}-\tfrac{9}{2}(1-\tfrac{7}{6}x_1)^{\frac{8}{7}}-5x_1=0.$$

This root is approximately

$$x_1=0{\cdot}883. \quad (15.84)$$

The mean radial tension in the disk is

$$P'_1=\frac{E}{r}\frac{d\varphi}{dr}=\frac{E\xi}{r^2}=\frac{E\xi}{s}$$

$$=\tfrac{1}{4}(2p^2E^2)^{\frac{1}{4}}c\frac{y}{x}$$

$$=P_0\left\{1-\tfrac{1}{2}x-\tfrac{1}{6}x^2-\tfrac{13}{144}x^3-\tfrac{17}{288}x^4-\ldots\right\}, \quad (15.85)$$

the constant factor P_0 being clearly the mean tension at the centre of the disk. In terms of the constant c its value is

$$P_0=\tfrac{1}{4}(2p^2E^2)^{\frac{1}{4}}c=\left(\frac{p^2E^2}{128}\right)^{\frac{1}{4}}c. \quad (15.86)$$

In our equations we can use the constant P_0 instead of c since the former has the advantage of having a clear physical meaning. Thus

$$x=\frac{s_1}{c^3}=\frac{s}{c^3}\left(\frac{p^2}{128h^8E^2}\right)^{\frac{1}{4}}$$

$$=\frac{s}{P_0^3}\left(\frac{p^2E^2}{128}\right)^{\frac{3}{4}}\left(\frac{p^2}{128h^8E^2}\right)^{\frac{1}{4}}$$

$$=\frac{r^2}{128}\frac{p^2E}{h^2P_0^3}$$

$$=\frac{Hr^2}{P_0^3}\text{ say;} \quad (15.87)$$

and therefore

$$P'_1=P_0\left\{1-\frac{1}{2}\frac{Hr^2}{P_0^3}-\frac{1}{6}\frac{H^2r^4}{P_0^6}-\frac{13}{144}\frac{H^3r^6}{P_0^9}-\ldots\right\} \quad (15.88)$$

Let a denote the radius of the plate and let $P'_1=T$ at the rim. Then

$$T=P_0\left\{1-\frac{1}{2}\frac{Ha^2}{P_0^3}-\frac{1}{6}\frac{H^2a^4}{P_0^6}-\frac{13}{144}\frac{H^3a^6}{P_0^9}-\ldots\right\}. \quad (15.89)$$

If P_0 were given this equation would determine T directly, but if T is given it determines P_0 indirectly. By inverting the series we can express P_0 in terms of T. The equation for P_0 is

$$P_0 = T\left\{1 + \frac{1}{2}\frac{Ha^2}{T^3} - \frac{1}{3}\frac{H^2a^4}{T^6} + \frac{55}{144}\frac{H^3a^6}{T^9} - \right\} \quad . \quad . \quad . \quad (15.90)$$

The first approximation is

$$P_0 = T,$$

which is the usual assumption in dealing with stretched membranes. In getting this approximation it is assumed that Ha^2 is negligible in comparison with T^3 or with $P_0{}^3$. The assumption also means that P'_1 is constant over the whole of the disk.

The second approximation to the value of P_0 is

$$P_0 = T + \frac{1}{2}\frac{Ha^2}{T^2}$$

and the corresponding approximation to P'_1 is

$$\begin{aligned} P'_1 &= P_0(1 - \tfrac{1}{2}x) \\ &= P_0 - \frac{1}{2}\frac{Hr^2}{P_0{}^2} \\ &= \left(T + \frac{1}{2}\frac{Ha^2}{T^2}\right) - \frac{1}{2}\frac{Hr^2}{P_0{}^2} \\ &= T + \frac{1}{2}\frac{H}{T^2}(a^2 - r^2). \end{aligned}$$

The third approximations are

$$P_0 = T + \frac{1}{2}\frac{Ha^2}{T^2} - \frac{1}{3}\frac{H^2a^4}{T^5} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15.91)$$

and $$P'_1 = T + \frac{1}{2}\frac{H}{T^2}(a^2 - r^2) - \frac{1}{6}\frac{H^2}{T^5}(a^2 - r^2)(2a^2 - r^2). \quad . \quad (15.92)$$

Although equations (15.85) and (15.89) are true for any disk in which the flexural rigidity is negligible yet the inverted equations from (15.90) to (15.92) are true only on the assumption that P_0 is nearly equal to T. If P_0 and T are not nearly equal then it is necessary to retain equation (15.88), or the nearly equivalent equation

$$\begin{aligned} P'_1 &= \frac{P_0}{x}\left\{\tfrac{9}{2} - \tfrac{9}{2}(1 - \tfrac{7}{6}x)^{\frac{8}{7}} - 5x\right\} \\ &= \frac{P_0{}^4}{Hr^2}\left\{\frac{9}{2} - \frac{9}{2}\left(1 - \frac{7}{6}\frac{Hr^2}{P_0{}^3}\right)^{\frac{8}{7}} - 5\frac{Hr^2}{P_0{}^3}\right\} \quad . \quad . \quad . \quad (15.93) \end{aligned}$$

To get the deflexion w we must return to equation (15.73). Thus

$$\frac{1}{r}\frac{dw}{dr}=\theta=-\frac{p}{4hE}\frac{s}{\xi}=-\frac{p}{4h}\frac{1}{P'_1}$$

$$=-\frac{p}{4hP_0}\{1-\tfrac{1}{2}x-\tfrac{1}{6}x^2-\tfrac{13}{144}x^3-\ldots\}^{-1}$$

$$=-\frac{p}{4hP_0}\{1+\tfrac{1}{2}x+\tfrac{5}{12}x^2+\tfrac{55}{144}x^3+\tfrac{35}{96}x^4+\ldots\}$$

$$=-\frac{p}{4hP_0}\left\{1+\frac{1}{2}\frac{H}{P_0^3}r^2+\frac{5}{12}\frac{H^2}{P_0^6}r^4+\ldots\right\}. \quad . \quad . \quad (15.94)$$

Therefore, measuring w from the level of the middle of the plate in the direction of the pressure p, we get

$$w=-\frac{p}{8hP_0}\left\{r^2+\frac{1}{4}\frac{H}{P_0^3}r^4+\frac{5}{36}\frac{H^2}{P_0^6}r^6+\frac{55}{576}\frac{H^3}{P_0^9}r^8+\frac{7}{96}\frac{H^4}{P_0^{12}}r^{10}+\right\} \quad (15.95)$$

279. Deflexion of a disk due to a given symmetrical load. Approximate solution.

We have now found the deflexion of a disk under uniform pressure in the two extreme cases

(i) when the tension in the middle surface can be neglected;
(ii) when the flexural rigidity can be neglected.

There are, however, many intermediate cases where neither of these assumptions can be made, and it is our present object to get an approximate method of dealing with these intermediate cases. The labour required to get an accurate solution is so great that it is not worth while, particularly as the solution can only be expressed in the form of an infinite series in the end.

The method we shall use here is similar to that used in Chapter VI for a beam whose ends are held at a fixed distance apart. The present problem is similar to the beam problem. The method consists in assuming a reasonable expression for w and then deducing from the accurate equations of equilibrium an equation closely akin to an energy equation. This pseudo-energy equation, as we shall call it, determines the maximum deflexion, which is left undetermined in the assumed expression for w.

The equations of equilibrium for a load symmetrical about the centre are (15.34) and (15.35), which are rewritten here

$$p=E'I\nabla_1{}^4w-2hE\frac{1}{r}\frac{d}{dr}\left(\frac{d\varphi}{dr}\frac{dw}{dr}\right), \quad . \quad . \quad . \quad (15.96)$$

$$\nabla_1{}^4\varphi=-\frac{1}{r}\frac{dw}{dr}\frac{d^2w}{dr^2}$$

$$=-\frac{1}{2r}\frac{d}{dr}\left(\frac{dw}{dr}\right)^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15.97)$$

Now if w were proportional to p (and it is proportional to p when the tension in the middle surface is neglected) the work that would be done by p on an area dA of the plate, assuming that p gradually increases from zero up to its maximum value, would be $\frac{1}{2}pwdA$. The integral of this would, on these assumptions, give the total energy in the plate. Now, although this integral does not give the total energy in the plate when the tension is taken into account, nevertheless we do get an accurate equation by multiplying both sides of (15.96) by $\frac{1}{2}wdA$ and integrating both sides. But

$$\tfrac{1}{2}wdA = \tfrac{1}{2}w(2\pi rdr) = \pi wrdr.$$

Then multiplying both sides of (15.96) by $wrdr$, omitting the factor π, and then integrating over the whole disk, we get

$$\int_0^a pwrdr = E'I\int_0^a wr\nabla_1^4 wdr - 2hE\int_0^a w\frac{d}{dr}\left(\frac{d\varphi}{dr}\frac{dw}{dr}\right)dr. \quad . \quad (15.98)$$

But since

$$\nabla_1^2 = \frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\right)$$

it follows that

$$\int_0^a wr\nabla_1^4 wdr = \int_0^a w\frac{d}{dr}\left(r\frac{d\nabla_1^2 w}{dr}\right)dr$$

$$= \left[wr\frac{d}{dr}(\nabla_1^2 w)\right]_0^a - \int_0^a r\frac{d\nabla_1^2 w}{dr}\frac{dw}{dr}dr$$

Let w be measured from the rim of the plate, so that w is zero at the rim. Then the integrated term in the last equation is zero at the upper limit because w is zero there, and it is zero at the lower limit because r is zero at that limit. Therefore

$$\int_0^a wr\nabla_1^4 wdr = -\int r\frac{d\nabla_1^2 w}{dr}\frac{dw}{dr}dr$$

$$= -\left[r\nabla_1^2 w\frac{dw}{dr}\right]_0^a + \int_0^a \nabla_1^2 w\frac{d}{dr}\left(r\frac{dw}{dr}\right)dr$$

$$= -\left[r\nabla_1^2 w\frac{dw}{dr}\right]_0^a + \int_0^a (\nabla_1^2 w)^2 rdr. \quad . \quad . \quad (15.99)$$

The integrated term is again zero at the centre but not at the rim except for a clamped disk.

Again, by integration by parts,

$$\int_0^a w\frac{d}{dr}\left(\frac{d\varphi}{dr}\frac{dw}{dr}\right)dr = \left[w\frac{d\varphi}{dr}\frac{dw}{dr}\right]_0^a - \int_0^a \frac{d\varphi}{dr}\left(\frac{dw}{dr}\right)^2 dr.$$

The integrated term is zero because w is zero at the rim, and because $\frac{dw}{dr}$ is zero at the centre. Therefore

$$\int_0^a w \frac{d}{dr}\left(\frac{d\varphi}{dr}\frac{dw}{dr}\right) dr = -\int_0^a \frac{d\varphi}{dr}\left(\frac{dw}{dr}\right)^2 dr. \quad . \quad . \quad . \quad . \quad (15.100)$$

Now by multiplying through equation (15.97) by rdr and integrating from 0 to r we get

$$r\frac{d}{dr}(\nabla_1{}^2\varphi) = -\frac{1}{2}\left(\frac{dw}{dr}\right)^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (15.101)$$

Therefore equation (15.100) becomes

$$\int_0^a w \frac{d}{dr}\left(\frac{d\varphi}{dr}\frac{dw}{dr}\right) dr = 2\int^a \frac{d\varphi}{dr} r\frac{d}{dr}(\nabla_1{}^2\varphi) dr$$

$$= 2\left[r\frac{d\varphi}{dr}\nabla_1{}^2\varphi\right]_0^a - 2\int_0^a \nabla_1{}^2\varphi \frac{d}{dr}\left(r\frac{d\varphi}{dr}\right) dr.$$

Since the radial tension is

$$P_1 = \frac{E}{r}\frac{d\varphi}{dr}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15\ 102)$$

and since we are assuming that this tension is zero at the rim, it follows that the integrated term is again zero at both limits. Thus we get finally

$$\int_0^a w \frac{d}{dr}\left(\frac{d\varphi}{dr}\frac{dw}{dr}\right) dr = 2\int_0^a (\nabla_1{}^2\varphi)^2 r dr. \quad . \quad . \quad . \quad (15.103)$$

We can now write equation (15.98) in either of the forms

$$\int_0^a pwrdr = E'I\int_0^a (\nabla_1{}^2 w)^2 rdr + 2hE\int_0^a \frac{d\varphi}{dr}\left(\frac{dw}{dr}\right)^2 dr$$

$$-E'I\left[r\frac{dw}{dr}\nabla_1{}^2 w\right]_{r=a}, \quad . \quad . \quad . \quad . \quad (15.104a)$$

or

$$\int_0^a pwrdr = E'I\int^a (\nabla_1{}^2 w)^2 rdr + 4hE\int_0^a (\nabla_1{}^2\varphi)^2 rdr$$

$$-E'I\left[r\frac{dw}{dr}\nabla_1{}^2 w\right]_{r=a}, \quad . \quad . \quad . \quad . \quad (15.104b)$$

and $\nabla_1{}^2\varphi$ is given in terms of w by equation (15.101). The second of these equations can be used only when P'_1 is zero at the rim.

For a disk free at the rim, or supported at the rim without clamping, one of the boundary conditions is

$$\frac{d^2w}{dr^2} + \frac{\sigma}{r}\frac{dw}{dr} = 0 \text{ where } r = a. \quad . \quad . \quad . \quad (15.105)$$

But
$$\frac{d^2w}{dr^2}+\frac{1}{r}\frac{dw}{dr}=\nabla_1{}^2 w. \quad \ldots \ldots \quad (15.106)$$

Therefore, at the rim,
$$\frac{1-\sigma}{r}\frac{dw}{dr}=\nabla_1{}^2 w,$$

and consequently
$$r\frac{dw}{dr}\nabla_1{}^2 w=(1-\sigma)\left(\frac{dw}{dr}\right)^2. \quad \ldots \ldots \quad (15.107)$$

This expression can be used in the integrated term in (15.104 a) and (15.104 b).

If, however, the plate is clamped at the rim the boundary condition which replaces (15.105) is
$$\frac{dw}{dr}=0 \text{ where } r=a.$$

Consequently, at the rim,
$$r\frac{dw}{dr}\nabla_1{}^2 w=0. \quad \ldots \ldots \quad (15.108)$$

Thus, for a plate free or supported at the rim, equation (15.104 b) becomes
$$\int_0^a pwrdr = E'I\int_0^a(\nabla_1{}^2 w)^2 r\,dr + 4hE\int_0^a(\nabla_1{}^2\varphi)^2 rdr$$
$$-(1-\sigma)E'I\left[\left(\frac{dw}{dr}\right)^2\right]_{r=a}; \quad \ldots \ldots \quad (15.109)$$

and this is still true for a plate *clamped* at the rim because the last term is zero in that case.

If we put
$$s=\frac{r}{a}$$

equation (15.109) can be put in the rather more convenient form
$$a^4\int_0^1 pwsds = E'I\int_0^1\left\{\frac{1}{s}\frac{d}{ds}\left(s\frac{dw}{ds}\right)\right\}^2 sds-(1-\sigma)E'I\left(\frac{dw}{ds}\right)^2_{s=1}$$
$$+4hE\int_0^1\left\{\frac{1}{s}\frac{d}{ds}\left(s\frac{d\varphi}{ds}\right)\right\}^2 sds; \quad \ldots \ldots \quad (15.110)$$

and the equation for φ is
$$s\frac{d}{ds}\left\{\frac{1}{s}\frac{d}{ds}\left(s\frac{d\varphi}{ds}\right)\right\}=-\frac{1}{2}\left(\frac{dw}{ds}\right)^2. \quad \ldots \ldots \quad (15.111)$$

Equation (15.110) is the pseudo-energy equation.

280. Plate supported at the rim under uniform pressure.

We shall illustrate the approximate method by using a simple expression for w, namely,

$$w = b\left(1 - \frac{r^2}{a^2}\right) = b(1 - s^2), \quad . \quad . \quad . \quad (15.112)$$

which is the deflexion for a small portion of a spherical surface. This makes w zero at the rim, but makes the bending moment constant across every section of the plate. It is therefore very far from satisfying the boundary condition that the bending moment across the rim is zero, but it will do to illustrate the method.

We have

$$\frac{dw}{ds} = -2bs.$$

Therefore, by (15.111)

$$\frac{1}{s}\frac{d}{ds}\left(s\frac{d\varphi}{ds}\right) = -\int 2b^2 s ds$$
$$= -b^2 s^2 + \text{H}.$$

Integrating again after multiplying by s

$$s\frac{d\varphi}{ds} = -\tfrac{1}{4}b^2 s^4 + \tfrac{1}{2}\text{H}s^2 + \text{K} \quad . \quad . \quad . \quad . \quad (15.113)$$

Now we know that

$$\text{E}r\frac{d\varphi}{dr} = r^2 \text{P}'_1,$$

and consequently

$$s\frac{d\varphi}{ds} = r\frac{d\varphi}{dr} = \frac{r^2}{\text{E}}\text{P}'_1,$$

which vanishes at the centre because $r = 0$, and at the rim because $\text{P}'_1 = 0$. Therefore

$$\left.\begin{aligned} \text{K} &= 0 \\ -\tfrac{1}{4}b^2 + \tfrac{1}{2}\text{H} &= 0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (15.114)$$

These give H and K. Substituting their values in (15.113) we get

$$\frac{1}{s}\frac{d}{ds}\left(s\frac{d\varphi}{ds}\right) = \tfrac{1}{2}b^2(1 - 2s^2)$$

Therefore

$$4h\text{E}\int_0^1\left\{\frac{1}{s}\frac{d}{ds}\left(s\frac{d\varphi}{ds}\right)\right\}^2 sds = h\text{E}b^4\int_0^1 (1 - 4s^2 + 4s^4)sds$$
$$= \tfrac{1}{6}h\text{E}b^4.$$

Also

$$\text{E}'\text{I}\int_0^1\left\{\frac{1}{s}\frac{d}{ds}\left(s\frac{dw}{ds}\right)\right\}^2 sds = \text{E}'\text{I}\int_0^1 16b^2 sds$$
$$= 8\text{E}'\text{I}b^2$$

$$a^4\int_0^1 pwsds = a^4pb\int_0^1(1-s^2)sds$$
$$=\tfrac{1}{4}a^4pb.$$

On substituting these values in (15.110) we get

$$\tfrac{1}{4}a^4pb = 8E'Ib^2-4b^2(1-\sigma)E'I+\tfrac{1}{6}hEb^4$$
$$=4E'Ib^2(1+\sigma)+\tfrac{1}{6}hEb^4 \quad . \quad . \quad . \quad . \quad . \quad (15.115)$$

Let d be written for the thickness of the plate. Then

$$I=\tfrac{1}{12}d^3,\quad h=\tfrac{1}{2}d.$$

Therefore

$$\tfrac{1}{4}a^4pb=\tfrac{1}{3}Ed^3b^2(1+\sigma)+\tfrac{1}{12}dEb^4,$$

whence

$$\frac{a^4p}{Ed^4}=\frac{4}{3(1-\sigma)}\frac{b}{d}+\frac{1}{3}\frac{b^3}{d^3} \quad . \quad . \quad . \quad . \quad . \quad (15.116a)$$

which equation determines b, and therefore also w.

One of the conditions used in determining φ in the problem just solved was that P'_1 is zero at the rim. But it is possible in many cases that the rim is attached to a body which does not yield to the radial tension set up by the load. In that case the radial displacement u is zero at the rim. That is, since the circumferential strain is

$$\beta_0=\frac{u}{r},$$

the condition at the rim is that β_0 is zero, whence

$$P'_2-\sigma P'_1=0$$

or

$$\frac{d^2\varphi}{ds^2}-\frac{\sigma}{s}\frac{d\varphi}{ds}=0 \text{ where } s=1.$$

Thus instead of the second of equations (15.114) we get

$$\tfrac{1}{2}H-\tfrac{3}{4}b^2=\sigma(\tfrac{1}{2}H-\tfrac{1}{4}b^2)$$

whence

$$H=\frac{1}{2}\left(\frac{3-\sigma}{1-\sigma}\right)b^2$$

Since P'_1 is not zero at the rim we must use (15.104 a). The only difference is the integral involving φ. This integral is

$$2hE\int_0^a\frac{d\varphi}{dr}\left(\frac{dw}{dr}\right)^2dr=2hEa^2\int_0^1\frac{d\varphi}{ds}\left(\frac{dw}{ds}\right)^2ds$$
$$=\frac{7-\sigma}{6(1-\sigma)}hEa^2b^4$$

Instead of equation (15.116*a*) we now get

$$\frac{a^4p}{Ed^4}=\frac{4}{3(1-\sigma)}\frac{b}{d}+\frac{7-\sigma}{3(1-\sigma)}\frac{b^3}{d^3} \quad . \quad . \quad . \quad (15.116b)$$

This result could be applied to the case of a disk attached to the end of a cylinder whose walls are much thicker than the disk provided that the pressure is so great that the deflexion at the middle is greater than the thickness of the disk.

A second form for w. Let us now take a more general expression for w, namely,

$$w=b(1+ms^2+ns^4) \quad . \quad . \quad . \quad . \quad . \quad (15.117)$$

By the same method as was used for the last value of w we get

$$\frac{a^4p}{d^4E}(6+3m+2n)=\frac{4}{1-\sigma^2}\frac{b}{d}\left\{(m+2n)^2(1+\sigma)+\tfrac{8}{3}n^2\right\}$$
$$+\frac{b^3}{d^3}\left\{m^4+4m^3n+\tfrac{20}{3}m^2n^2+\tfrac{16}{3}mn^3+\tfrac{12}{7}n^4\right\} \quad (15.118)$$

Since w must be zero at the rim, where $s=1$, we find that

$$1+m+n=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15.119)$$

Eliminating m from the last two equations we get

$$\frac{a^4p}{d^4E}(3-n)=\frac{4}{1-\sigma^2}\frac{b}{d}\left\{1-n)^2(1+\sigma)+\tfrac{8}{3}n^2\right\}$$
$$+\left(\frac{b}{d}\right)^3\left\{1+\tfrac{2}{3}n^2+\tfrac{1}{21}n^4\right\} \quad . \quad . \quad . \quad (15.120)$$

By putting $n=0$ in this we repeat the result in (15.116a). By giving other values to n we can make the deflexion satisfy any other condition we choose. We may, for example, choose n so as to make $\frac{d^2w}{dr^2}$ zero at the rim. This makes the bending moment at the rim small, but not zero. The value of n which satisfies this condition, as well as the condition expressed in (15.119), is $n=\frac{1}{5}$. Then we get, taking $\sigma=\frac{1}{4}$,

$$\frac{a^4p}{d^4E}=\frac{136}{175}\frac{4}{3(1-\sigma)}\frac{b}{d}+1{\cdot}1001\times\frac{b^3}{3d^3} \quad . \quad . \quad (15.121)$$

A third form for w. Let us next take the actual expression for the deflexion when the tension is neglected. This expression is obtained from equation (14.79) by measuring the deflexion from the rim instead of from the centre.

Thus we take

$$w=b\left(1-\frac{6+2\sigma}{5+\sigma}s^2+\frac{1+\sigma}{5+\sigma}s^4\right) \quad . \quad . \quad . \quad . \quad (15.122)$$

which becomes, if $\sigma=0{\cdot}25$,

$$w = b\left(1 - \tfrac{26}{21} s^2 + \tfrac{5}{21} s^4\right) \quad . \quad . \quad . \quad . \quad . \quad (15.123)$$

Now equation (15.120) gives, with $\frac{5}{21}$ for n,

$$\frac{a^4 p}{d^4 \mathrm{E}} = \tfrac{46}{21} \frac{4}{3(1-\sigma)} \frac{b}{d} + 1{\cdot}127 \times \frac{b^3}{3d^3} \quad . \quad . \quad . \quad (15.124)$$

It should be remarked that the coefficients of b and b^3 in (15.124) are respectively two per cent less and two and a half per cent greater than the corresponding coefficients in (15.121). The difference is so small then that one of these equations is as good as the other. Both equations differ considerably from (15.116a); but this is to be expected because the spherical form from which (15.116a) was derived is very different from the forms assumed for the other two. Moreover the plate would approach the spherical form only if the tension were very great, and therefore consequently the term containing $\left(\frac{b}{d}\right)^3$ large compared with the term containing $\frac{b}{d}$. It should be recognised that the term containing b in the pseudo-energy equation represents the effect of the rigidity of the plate, and the term containing b^3 represents the effect of the tension in the middle surface. Now in all cases where the rigidity is more important than the tension in the middle surface equation (15.124) is very accurate, and its accuracy increases as the ratio of b to d decreases. Since the coefficient of b^3 does not alter very greatly as the assumed form of the surface changes from that given by equation (15.122) to that of a segment of a sphere it follows that equation (15.124) must still remain fairly good even when the ratio of b to d is large. Moreover, in finding b from such an equation as (15.124) when the b^3-term is much larger than the b-term, the resulting error in b is approximately proportional to the error in the cube root of its coefficient, and there is only a four per cent difference between the cube roots of the coefficients of b^3 in equations (15.116a) and (15.124). It follows then that quite good results will be given by (15.124) in every case where a plate is supported at the rim without clamping on the assumption that $\sigma = 0{\cdot}25$. Since the coefficients are only approximate we may take, for the supported disk,

$$\frac{a^4 p}{d^4 \mathrm{E}} = 1{\cdot}35 \frac{b}{d} + 0{\cdot}375 \frac{b^3}{d^3} \text{ if } \sigma = \tfrac{1}{4}; \quad . \quad . \quad . \quad (15.125)$$

$$\frac{a^4 p}{d^4 \mathrm{E}} = 1{\cdot}50 \frac{b}{d} + 0{\cdot}380 \frac{b^3}{d^3} \text{ if } \sigma = \tfrac{1}{3} \quad . \quad . \quad . \quad . \quad (15.126)$$

If it is required to find b for a given pressure the curve connecting the two quantities $\frac{a^4 p}{d^4 \mathrm{E}}$ and $\frac{b}{d}$ should be plotted, and then the value of the latter quantity can be read off from the curve. Such a curve is shown for (15.125) in fig. 148. The straight line touching this curve at the origin shows the relation between the same quantities when the tension of the middle surface is neglected. It will be observed that the pressure given by the curve when $b = 2d$ is roughly twice the pressure given by the straight line. It should be recalled that b is the maximum deflexion in the disk and d is the thickness.

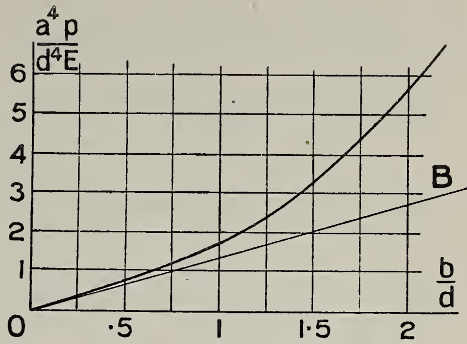


Fig. 148

An experimental confirmation of the preceding theory is supplied in a paper published in *"Engineering"* wherein Mr P. T. Steinthal* gives

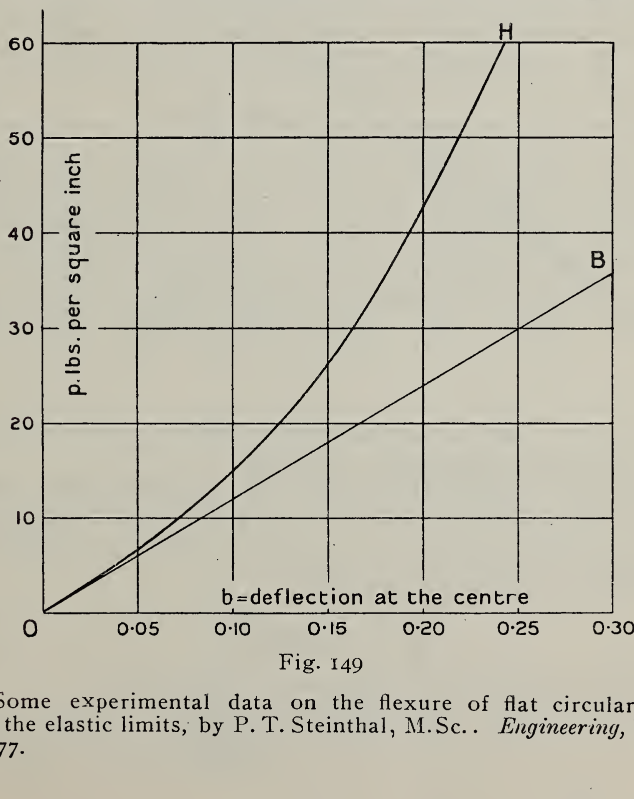


Fig. 149

* Some experimental data on the flexure of flat circular plates within the elastic limits, by P. T. Steinthal, M.Sc.. *Engineering,* Vol. 91, page 677.

the results of his experiments on the deflexions of thin disks under uniform pressures. Two of these curves are shown in figs 149 and 150. For fig. 149 the thickness of the disk is given as 0·142 of an inch, and the diameter as 12 inches. For fig. 150 the thickness is 0·268″ and the

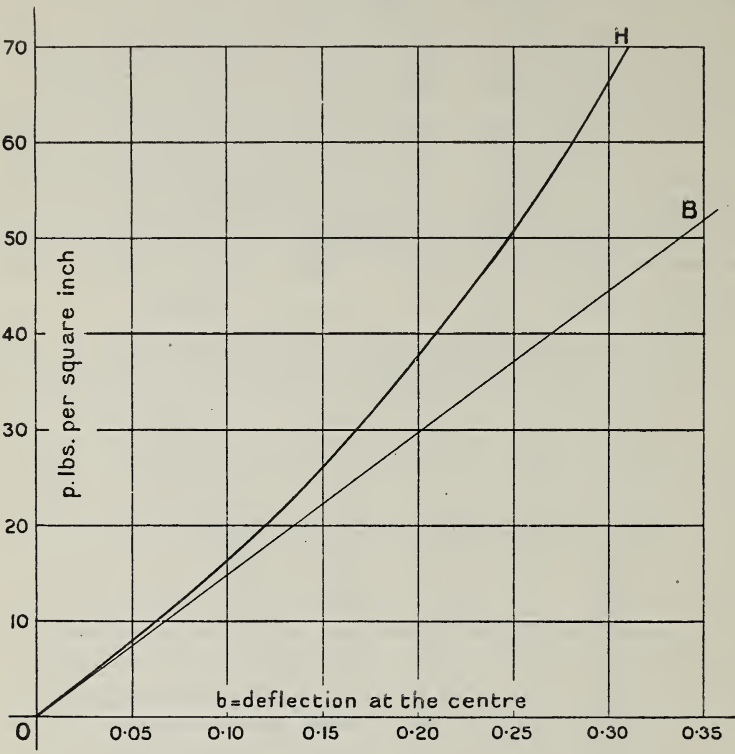


Fig. 150

diameter 18″. The general character of these two curves is precisely the same as that of the curve in fig. 148.

To make a numerical comparison between the theoretical results in (15.125), (15.126), and the results given by Mr Steinthal, let p_0 be written for the pressure on the assumption that this pressure is proportional to b. Thus for (15.125)

$$p_0 = 1\cdot 35 \frac{b}{d} \times \frac{d^4 E}{a^4} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15.127)$$

Therefore (15.125) and (15.126) can be written thus

$$p = p_0 \left\{ 1 + \frac{375}{1350} \frac{b^2}{d^2} \right\} \quad . \quad . \quad . \quad . \quad . \quad (15.128)$$

and

$$p = p_0\left\{1 + \frac{38}{150}\frac{b^2}{d^2}\right\}. \quad . \quad . \quad . \quad . \quad . \quad (15.129)$$

It will be seen that p_0 is the ordinate measured to the line OB in each figure, this line touching the p-curve at the origin. Thus p_0 would be the pressure if the effect of the stretching of the plate were always negligible; that is, the pressure on Poisson's theory.

Now at the point H in fig. 149

$$\frac{b}{d} = \frac{0{\cdot}242}{0{\cdot}142} = 1{\cdot}70$$

and

$$\frac{p}{p_0} = \frac{60}{28{\cdot}6} = 2{\cdot}10. \quad . \quad . \quad . \quad . \quad . \quad . \quad (15.130)$$

But equation (15.129) gives

$$\frac{p}{p_0} = 1{\cdot}73, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15.131)$$

when

$$\frac{b}{d} = 1{\cdot}70.$$

Also for the same value of $\frac{b}{d}$ equation (15.128) gives

$$\frac{p}{p_0} = 1{\cdot}80. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15.132)$$

The agreement between the result in (15.130) obtained from the experiments and the two results in (15.131) and (15.132) obtained by theory is good enough to give some support to the present theory.

We give another comparison, this time between the results shown in fig. 150 and those given by theory. From the figure we get

$$\frac{p}{p_0} = \frac{70}{44} = 1{\cdot}59, \quad . \quad . \quad . \quad . \quad . \quad . \quad (15.133)$$

when

$$\frac{b}{d} = \frac{0{\cdot}320}{0{\cdot}268} = 1{\cdot}19.$$

For the same value of $\frac{b}{d}$ equations (15.128) and (15.129) give respectively

$$\frac{p}{p_0} = 1{\cdot}39 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15.134)$$

and

$$\frac{p}{p_0} = 1{\cdot}36 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15.135)$$

Thus according to Steinthal's experiments the ratio of pressure to maximum deflexion increases at an even greater rate than according to the theory of this chapter.

The maximum stress.

The mean radial tension is

$$P'_1 = \frac{E}{r}\frac{d\varphi}{dr} = \frac{E}{a^2}\frac{1}{s}\frac{d\varphi}{ds} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15.136)$$

Therefore the total radial tension at the convex surface of the plate is, by such equations as (15.11),

$$P_1 = P'_1 - \frac{Eh}{1-\sigma^2}\left(\frac{d^2w}{dr^2} + \frac{\sigma}{r}\frac{dw}{dr}\right)$$

$$= \frac{E}{a^2}\left\{\frac{1}{s}\frac{d\varphi}{ds} - \frac{h}{1-\sigma^2}\left(\frac{d^2w}{ds^2} + \frac{\sigma}{s}\frac{dw}{ds}\right)\right\} \quad . \quad . \quad . \quad (15.137)$$

For a plate not clamped at the rim this has usually its maximum value at the centre of the plate. Now taking

$$w = b(1 + ms^2 + ns^4) \quad . \quad . \quad . \quad . \quad . \quad . \quad (15.138)$$

and using the conditions that

$$s\frac{d\varphi}{ds} = 0 \begin{cases} \text{where } s = 0 \\ \text{and } \quad s = 1 \end{cases}$$

we find that, at the centre of the plate,

$$\frac{1}{s}\frac{d\varphi}{ds} = (\tfrac{1}{4}m^2 + \tfrac{1}{3}mn + \tfrac{1}{6}n^2)b^2.$$

Since

$$1 + m + n = 0$$

this becomes

$$\frac{1}{s}\frac{d\varphi}{ds} = \tfrac{1}{12}\{n^2 + 2n + 3\}b^2.$$

Also, at the centre,

$$\frac{d^2w}{ds^2} + \frac{\sigma}{s}\frac{dw}{ds} = 2mb(1+\sigma) = -2(n+1)(1+\sigma)b.$$

Therefore

$$P_1 = \frac{E}{a^2}\left\{\tfrac{1}{12}b^2(n^2 + 2n + 3) + \frac{2bh}{1-\sigma}(n+1)\right\}, \quad . \quad (15.139)$$

whence

$$\frac{a^2 P_1}{d^2 E} = \frac{b}{d}\left\{\frac{1}{12}\frac{b}{d}(n^2 + 2n + 3) + \frac{n+1}{1-\sigma}\right\}. \quad . \quad . \quad . \quad . \quad (15.140)$$

The stress given by this equation will be generally the maximum stress in the disk. If the pressure p is constant equation (15.120) determines $\frac{b}{d}$, and then this last equation gives P_1.

Taking the same values of the constants as in (15.123), namely, $n = \frac{5}{21}$, $\sigma = \frac{1}{4}$, we find

$$\frac{a^2 P_1}{d^2 E} = \frac{104}{63}\frac{b}{d} + \frac{779}{2646}\frac{b^2}{d^2}$$

$$= 1{\cdot}65\frac{b}{d} + 0{\cdot}294\frac{b^2}{d^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (15{\cdot}141)$$

By giving values to $\frac{b}{d}$ the values of p and P_1 can be calculated from (15.125) and (15.141), and from these calculated values a curve showing the relation between $\frac{a^2 P_1}{d^2 E}$ and $\frac{a^4 p}{d^4 E}$ can be drawn. A table is given here

$\frac{b}{d}$	0	0·4	1	1·5	1·8	2
$\frac{a^4 p}{d^4 E}$	0	0·564	1·725	3·29	4·62	5 70
$\frac{a^2 P_1}{d^2 E}$	0	0·707	1·94	3·14	3·92	4·48

The curve is shown in fig. 151.

When the stretching of the middle surface is neglected the terms containing b^3 and b^2 are missing from equations (15.125) and (15.141). Consequently P_1 is in that case proportional to p, and the relation between them is shown by the straight line OA which touches at the origin the curve showing the relation on the present theory. It is to be observed that, for high pressures, the maximum stress corresponding to a given pressure is much smaller on the present theory than on Poisson's theory. It follows therefore that thin plates will bear much greater loads than Poisson's theory indicates.

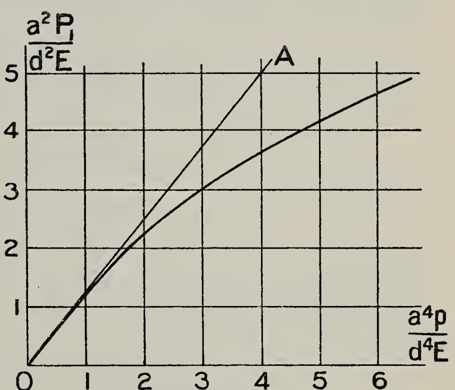


Fig. 151

281. Disk clamped at the rim under uniform pressure.

When the maximum deflexion is not more than about equal to the thickness of the plate the most suitable expression for w is the one given by Poisson's theory for a clamped disk under uniform pressure.

Measured from the plane of the rim this deflexion is, by equation (14 83),

$$w = b(1 - s^2)^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15.142)$$

This comes under the form (15.117), and, although the integrated term on the right hand side of equation (15.110) can be omitted for the clamped disk, nevertheless it makes no difference because $\frac{dw}{ds}$ is actually zero at the rim for the expression given in (15.142). It follows that (15.120) is correct for the clamped disk as well as for the sunported disk. In the present case $n = 1$, and therefore (15.120) gives

$$\frac{a^4 p}{d^4 E} = \frac{16}{3} \frac{1}{1-\sigma^2} \frac{b}{d} + \frac{6}{7} \frac{b^3}{d^3} \quad . \quad . \quad . \quad . \quad (15.143)$$

Also, by equation (15.140), the stress at the centre is given by the equation

$$\frac{a^2 P_1}{d^2 E} = \frac{2}{1-\sigma} \frac{b}{d} + \frac{1}{2} \frac{b^2}{d^2}. \quad . \quad . \quad . \quad . \quad . \quad (15.144)$$

But this is not the maximum stress for small deflexions. When the deflexion is so small that the tension in the middle surface is negligible the maximum stress occurs at the rim of the clamped disk, and its value is given by equation (14.84). On the present theory the maximum tension at the rim is, since the mean tension is zero there,

$$P_1 = \frac{Eh}{a^2(1-\sigma^2)} \left\{ \frac{d^2 w}{ds^2} + \frac{\sigma}{s} \frac{dw}{ds} \right\}_{s=1}$$

$$= \frac{Eh}{a^2(1-\sigma^2)} \times (8b),$$

so that

$$\frac{a^2 P_1}{d^2 E} = \frac{4}{1-\sigma^2} \frac{b}{d}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (15.145)$$

The stress given by this last equation will be greater than that given by (15.144) as long as

$$\frac{4}{1-\sigma^2} \frac{b}{d} > \frac{2}{1-\sigma} \frac{b}{d} + \frac{1}{2} \left(\frac{b}{d} \right)^2,$$

that is, as long as

$$\frac{b}{d} < \frac{4}{1+\sigma}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15.146)$$

If σ is $\frac{1}{3}$ the stress at the middle will exceed the stress at the rim only when the maximum deflexion b exceeds $3\,d$. For most plates this would be a very big deflexion, greater, in fact, than what is likely to occur. Usually then the maximum stress in a clamped disk is given by (15·145) and this equation has the same form as when the mean tension is not taken into account. It must be borne in mind, however, that the value of b corresponding to a given pressure p is less than when the mean tension is not taken into account.

Substituting the value of b from (15.145) in (15.143) we find

$$\frac{a^4 p}{d^4 \mathrm{E}} = \frac{4}{3}\frac{a^2 \mathrm{P}_1}{d^2 \mathrm{E}} + \frac{6}{7}\left(\frac{1-\sigma^2}{4}\frac{a^2 \mathrm{P}_1}{d^2 \mathrm{E}}\right)^3,$$

whence

$$\frac{3}{4}\frac{a^2}{d^2}p = \mathrm{P}_1\left\{1 + \frac{9(1-\sigma^2)^3}{896}\frac{a^4 \mathrm{P}_1{}^2}{d^4 \mathrm{E}^2}\right\} \quad . \quad . \quad . \quad . \quad (15.147)$$

which gives the relation between p and the stress at the rim.

When the mean tension is not taken into account the term containing $\mathrm{P}_1{}^3$ is missing from this last equation. It is clear then that, for a clamped disk, as well as for a supported disk, the maximum stress for a given pressure is less than on Poisson's theory.

282. Elliptic plate under uniform pressure and supported at the rim

For the supported elliptic plate, with principal axes $2a$ and $2b$, let us assume that

$$w = k\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right). \quad . \quad . \quad . \quad . \quad . \quad (15.148)$$

This deflexion is a reasonable one for pressures so large that the maximum deflexion is of the same order as the thickness, but is not so good for smaller pressures. The equation for φ is

$$\nabla_1{}^4\varphi = \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}$$

$$= -\frac{4k^2}{a^2 b^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15.149)$$

Let P_n denote the mean tensional stress in the middle surface at the edge of the plate in the direction normal to the rim. Then, if θ denote the inclination of the stress P_n to the x-axis,

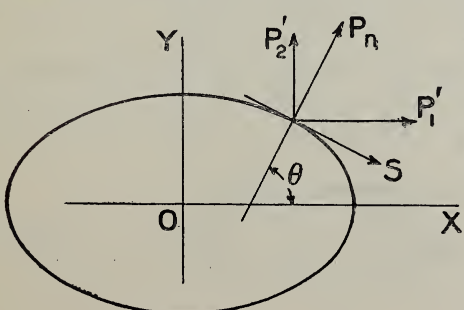


Fig. 152a

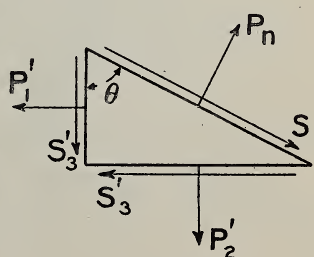


Fig. 152b

$$\cot\theta = -\frac{dy}{dx}$$

$$= \frac{x}{a^2} \cdot \frac{y}{b^2}.$$

Now by equation (1.22) the stress P_n is given by

$$P_n = P'_1 \cos^2\theta + P'_2 \sin^2\theta + 2 S'_3 \sin\theta\cos\theta.$$

The condition that $P^n = 0$ at the rim can therefore be put in the form

$$\frac{x^2}{a^4} P'_1 + \frac{y^2}{b^4} P'_2 + 2\frac{xy}{a^2b^2} S'_3 = 0. \quad . \quad . \quad . \quad (15.150)$$

Again the shear stress S at the edge in the direction shown in fig. 152 *a* is

$$S = (P'_1 - P'_2)\sin\theta\cos\theta + S'_3(\sin^2\theta - \cos^2\theta).$$

The condition that this should be zero at the rim is

$$\frac{xy}{a^2b^2}(P'_1 - P'_2) + \left(\frac{y^2}{b^4} - \frac{x^2}{a^4}\right) S'_3 = 0. \quad . \quad . \quad . \quad (15.151)$$

Now if the only forces acting on the rim of the plate are forces perpendicular to the plate then conditions (15.150) and (15.151) must be true at the boundary of the ellipse. Thus to find φ we have to solve (15.149) subject to these two conditions.

It is clear from (15.149) that φ must be of at least the fourth degree n x and y. Moreover, it is clear that the final expression for φ must be such that it will remain unaltered if a and b are interchanged, and if at the same time x and y are interchanged; then it follows that φ is a symmetrical function of $\frac{x}{a}$ and $\frac{y}{b}$. Let us therefore assume that

$$\varphi = A\left(\frac{x^4}{a^4} + \frac{y^4}{b^4}\right) + B\frac{x^2y^2}{a^2b^2} + C\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right). \quad . \quad . \quad . \quad (15.152)$$

With this value of φ

$$\left.\begin{aligned} P'_1 &= E\frac{\partial^2\varphi}{\partial y^2} = E\left\{12A\frac{y^2}{b^4} + 2B\frac{x^2}{a^2b^2} + \frac{2C}{b^2}\right\} \\ P'_2 &= E\frac{\partial^2\varphi}{\partial x^2} = E\left\{12A\frac{x^2}{a^4} + 2B\frac{y^2}{a^2b^2} + \frac{2C}{a^2}\right\} \\ S'_3 &= -E\frac{\partial^2\varphi}{\partial x\partial y} = -4EB\frac{xy}{a^2b^2}. \end{aligned}\right\} \quad . \quad . \quad (15.153)$$

Conditions (15.150) and (15.151) therefore become

$$\frac{2}{a^2b^2}\left\{B\left(\frac{x^4}{a^4} + \frac{y^4}{b^4}\right) + (12A - 4B)\frac{x^2y^2}{a^2b^2} + C\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)\right\} = 0 \quad . \quad (15.154)$$

and

$$\frac{xy}{a^2b^2}\left\{12A\left(\frac{y^2}{b^4}-\frac{x^2}{a^4}\right)+2B\frac{x^2-y^2}{a^2b^2}+2C\left(\frac{1}{b^2}-\frac{1}{a^2}\right)\right\}$$

$$-4B\left(\frac{y^2}{b^4}-\frac{x^2}{a^4}\right)\frac{xy}{a^2b^2}=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (15.155)$$

This last equation becomes, after the removal of a factor which is clearly not zero,

$$(6A-2B)\left(\frac{y^2}{b^4}-\frac{x^2}{a^4}\right)+B\frac{x^2-y^2}{a^2b^2}+C\left(\frac{1}{b^2}-\frac{1}{a^2}\right)=0. \quad . \quad (15.156)$$

Now equations (15.154) and (15.156) must be true all round the boundary of the ellipse whose equation is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15.157)$$

By means of (15.157) equation (15.154) can be put into the form

$$B\left(1-\frac{2x^2y^2}{a^2b^2}\right)+(12A-4B)\frac{x^2y^2}{a^2b^2}+C=0. \quad .$$

Since this must be identically true for all values of xy we get

$$12A-4B-2B=0,$$
$$B+C=0;$$

whence

$$B=2A,$$
$$C=-2A.$$

These values of the constants make the left hand side of (15.156) equal to

$$2A\left(\frac{1}{a^2}-\frac{1}{b^2}\right)\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right),$$

which is zero at the rim in consequence of (15.157).
Thus the solution

$$\varphi=A\left\{\frac{x^4}{a^4}+\frac{y^4}{b^4}+\frac{2x^2y^2}{a^2b^2}-2\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)\right\}$$

$$=A\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^2-A \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15.158)$$

satisfies both boundary conditions. We have to determine A so as to make φ satisfy also the differential equation (15.149). On substituting for φ in this equation we get

$$8A\left\{\frac{3}{a^4}+\frac{3}{b^4}+\frac{2}{a^2b^2}\right\}=-\frac{4k^2}{a^2b^2},$$

whence

$$A=-\frac{k^2a^2b^2}{6(a^4+b^4)+4a^2b^2}. \quad . \quad . \quad . \quad . \quad (15.159)$$

The expressions for the mean stresses are now

$$P_1 = E\frac{\partial^2\varphi}{\partial y^2} = \frac{4AE}{b^2}\left(\frac{x^2}{a^2}+\frac{3y^2}{b^2}-1\right), \quad . \quad . \quad . \quad (15.160)$$

$$P_2 = E\frac{\partial^2\varphi}{\partial x^2} = \frac{4AE}{a^2}\left(\frac{3x^2}{a^2}+\frac{y^2}{b^2}-1\right), \quad . \quad . \quad . \quad (15.161)$$

$$S_3 = -E\frac{\partial^2\varphi}{\partial x\partial y} = -8AE\frac{xy}{a^2b^2}. \quad . \quad . \quad . \quad . \quad (15.162)$$

The equation for the pressure is

$$p = E'I\nabla_1{}^4 w - 2hE\left\{\frac{\partial^2\varphi}{\partial y^2}\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2\varphi}{\partial x^2}\frac{\partial^2 w}{\partial y^2}-2\frac{\partial^2\varphi}{\partial x\partial y}\frac{\partial^2 w}{\partial x\partial y}\right\} \quad . \quad (15.163)$$

In order to determine the constant k we must multiply through this equation by $w\,dx\,dy$ and then integrate over the whole area of the plate. If, however, we substituted at once the value of w in $\nabla_1{}^4 w$ we should find that this expression is zero for the particular value of w that we have assumed. Consequently the integral of $w\nabla_1{}^4 w\,dx\,dy$ would be zero, and thus the term due to the rigidity would contribute nothing to the integrad of $pw\,dx\,dy$. But this rigidity term should clearly contribute an amount equal to twice the strain energy in the plate due to the bending of the plate on the assumption that the middle surface is unstretched, because it is clear that, on Poisson's theory, the integral of $p\,dx\,dy$ is twice the work done by the pressure p as the plate is gradually bent into its final state. This error arises through our assuming an expression for w which does not make the bending moment about the edge zero. But, of course, if we knew the correct expression we should not need an approximate method. The proper way to use the approximate method is to substitute for

$$\iint E'I w\nabla_1{}^4 w\,dx\,dy$$

twice the energy in the plate on Poisson's theory, which, by (14.154), is

$$\iint E'I\left[(\nabla_1{}^2 w)^2 - 2(1-\sigma)\left\{\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}-\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2\right\}\right]dx\,dy.$$

Then equation (15.163) gives

$$\iint pw\,dx\,dy = E'I\iint\left[(\nabla_1{}^2 w)^2 - 2(1-\sigma)\left\{\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}-\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2\right\}\right]dx\,dy$$

$$-2hE\iint\left\{\frac{\partial^2\varphi}{\partial y^2}\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2\varphi}{\partial x^2}\frac{\partial^2 w}{\partial y^2}-2\frac{\partial^2\varphi}{\partial x\partial y}\frac{\partial^2 w}{\partial x\partial y}\right\}w\,dx\,dy \quad (15.164)$$

With the values of w and φ that we have now got the last equation becomes

$$\iint pk\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)dx\,dy = 4k^2E'I\iint\left\{\frac{1}{a^4}+\frac{1}{b^4}+\frac{2\sigma}{a^2b^2}\right\}dx\,dy$$

$$+\frac{16EAhk^2}{a^2b^2}\iint\left(\frac{4x^2}{a^2}+\frac{4y^2}{b^2}-2\right)\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)dx\,dy. \quad (15.165)$$

But it was shown in (14.181) that, over the area of the ellipse,

$$\iint \frac{x^2}{a^2}\,dx\,dy = \iint \frac{y^2}{b^2}\,dx\,dy = \tfrac{1}{4}\pi ab.$$

Also

$$\iint \frac{x^4}{a^4}\,dx\,dy = \iint \frac{y^4}{b^4}\,dx\,dy = \tfrac{1}{8}\pi ab,$$

$$\iint \frac{x^2y^2}{a^2b^2}\,dx\,dy = \tfrac{1}{24}\pi ab.$$

Therefore (15.165) gives, for a uniform pressure p,

$$\tfrac{1}{2}pk\pi ab = 4k^2\mathrm{E'I}\pi ab\left\{\frac{1}{a^4}+\frac{1}{b^4}+\frac{2\sigma}{a^2b^2}\right\} - \frac{16}{3}\frac{\mathrm{EA}hk^2}{a^2b^2}\pi ab.$$

Consequently, since $\mathrm{I} = \tfrac{2}{3}h^3$,

$$\frac{a^2b^2p}{\mathrm{E}h^4} = \frac{16}{3}\frac{k}{h}\frac{a^4+b^4+2\sigma a^2b^2}{(1-\sigma^2)a^2b^2} - \frac{32}{3}\frac{\mathrm{A}k}{h^3},$$

or, if we write d for $2h$,

$$\frac{a^2b^2p}{\mathrm{E}d^4} = \frac{2}{3}\frac{k}{d}\frac{a^4+b^4+2\sigma a^2b^2}{(1-\sigma^2)a^2b^2} - \frac{16}{3}\frac{\mathrm{A}k}{d^3}$$

$$= \frac{2}{3}\frac{k}{d}\frac{a^4+b^4+2\sigma a^2b^2}{(1-\sigma^2)a^2b^2} + \frac{8}{3}\frac{k^3}{d^3}\frac{a^2b^2}{3(a^4+b^4)+2a^2b^2} \quad . \quad (15.166)$$

which is the equation for k. When $b = a$ this becomes, of course, the same equation as (15.116a), k here being the same as b in the earlier equation.

The two principal bending moments are

$$-\mathrm{M}_1 = \mathrm{E'I}k\left(\frac{2}{a^2}+\frac{2\sigma}{b^2}\right), \quad . \quad . \quad . \quad . \quad . \quad (15.167)$$

$$-\mathrm{M}_2 = \mathrm{E'I}k\left(\frac{2}{b^2}+\frac{2\sigma}{a^2}\right). \quad . \quad . \quad . \quad . \quad . \quad (15.168)$$

The tensional stresses at the surface of the plate parallel to the axes OX, OY, are

$$f_1 = -\frac{h\mathrm{M}_1}{\mathrm{I}} + \mathrm{P'}_1$$

$$= 2\mathrm{E'}hk\left(\frac{1}{a^2}+\frac{\sigma}{b^2}\right) + \frac{4\mathrm{AE}}{b^2}\left(\frac{x^2}{a^2}+\frac{3y^2}{b^2}-1\right) \quad . \quad . \quad (15.169)$$

and

$$f_2 = 2\mathrm{E'}hk\left(\frac{1}{b^2}+\frac{\sigma}{a^2}\right) + \frac{4\mathrm{AE}}{a^2}\left(\frac{3x^2}{a^2}+\frac{y^2}{b^2}-1\right). \quad . \quad . \quad (15.170)$$

Since A is negative the stresses P'_1 and P'_2 are positive in parts of the plate near the centre, and negative nearer the rim. The greatest values of f_1 and f_2 occur at the centre and these values are

$$f_1 = 2E'hk\left(\frac{1}{a^2}+\frac{\sigma}{b^2}\right)-\frac{4AE}{b^2}$$

$$= E'dk\left(\frac{1}{a^2}+\frac{\sigma}{b^2}\right)+\frac{2Ek^2a^2}{3(a^4+b^4)+2a^2b^2}, \quad . \quad . \quad (15.171)$$

$$f_2 = E'dk\left(\frac{1}{b^2}+\frac{\sigma}{a^2}\right)+\frac{2Ek^2b^2}{3(a^4+b^4)+2a^2b^2}. \quad . \quad . \quad (15.172)$$

Which of these is greater can only be determined when the ratio of b to a is given.

It may be useful to repeat here that the results just worked out for the elliptic plate should only be used for large deflexions, that is, when equations (14.177) and (14.188) make the maximum deflexion greater than the thickness of the plate.

283. Rectangular plate the edges of which have no displacement when the plate is loaded.

Let the equations to the edges of the middle surface be

$$x = \pm a,$$
$$y = \pm b,$$
$$z = 0 .$$

We are making the assumption that these edges are not displaced by the load. In earlier problems we have assumed that the mean tension normal to the edge was zero. The present alternative assumptions are that

$$u_0 = 0 \text{ where } x = \pm a, \quad . \quad . \quad . \quad . \quad . \quad (15.173)$$

$$v_0 = 0 \text{ where } y = \pm b \quad . \quad . \quad . \quad . \quad . \quad (15.174)$$

Let us assume that

$$w = k\cos\frac{\pi x}{2a}\cos\frac{\pi y}{2b}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15.175)$$

This makes w zero at the edges of the plate.

Then

$$\nabla_1{}^4\varphi = \left(\frac{\partial^2 w}{\partial x\partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}$$

$$= \frac{\pi^4 k^2}{16a^2b^2}\left\{\sin^2\frac{\pi x}{2a}\sin^2\frac{\pi y}{2b} - \cos^2\frac{\pi x}{2a}\cos^2\frac{\pi y}{2b}\right\}$$

$$= -\frac{\pi^4 k^2}{32a^2b^2}\left(\cos\frac{\pi x}{a}+\cos\frac{\pi y}{b}\right). \quad . \quad . \quad . \quad . \quad (15.176)$$

A particular integral of this equation, and moreover, one that suits our problem, is

$$\varphi = -\frac{k^2}{32a^2b^2}\left\{a^4 \cos\frac{\pi x}{a} + b^4 \cos\frac{\pi y}{b}\right\} + \mathrm{A}x^2 + \mathrm{B}y^2 \quad . \quad (15.177)$$

With this value of φ

$$\mathrm{P}'_1 = \mathrm{E}\frac{\partial^2\varphi}{\partial y^2} = \frac{\mathrm{E}k^2\pi^2}{32a^2}\cos\frac{\pi y}{b} + 2\mathrm{EB} \quad . \quad . \quad . \quad (15.178)$$

$$\mathrm{P}'_2 = \mathrm{E}\frac{\partial^2\varphi}{\partial x^2} = \frac{\mathrm{E}k^2\pi^2}{32b^2}\cos\frac{\pi x}{a} + 2\mathrm{EA} \quad . \quad . \quad . \quad (15.179)$$

$$\mathrm{S}'_3 = -\mathrm{E}\frac{\partial^2\varphi}{\partial x \partial y} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15.180)$$

The strains in the middle surface are, by (15.14), (15.15), (15.16),

$$\begin{aligned} a_0 &= \frac{1}{\mathrm{E}}(\mathrm{P}'_1 - \sigma\mathrm{P}'_2) \\ &= \frac{k^2\pi^2}{32}\left(\frac{1}{a^2}\cos\frac{\pi y}{b} - \frac{\sigma}{b^2}\cos\frac{\pi x}{a}\right) + 2\mathrm{B} - 2\sigma\mathrm{A} \quad . \quad . \quad (15.181) \end{aligned}$$

$$\beta_0 = \frac{k^2\pi^2}{32}\left(\frac{1}{b^2}\cos\frac{\pi x}{a} - \frac{\sigma}{a^2}\cos\frac{\pi y}{b}\right) + 2\mathrm{A} - 2\sigma\mathrm{B} \quad . \quad . \quad (15.182)$$

$$c_0 = \frac{\mathrm{S}'_3}{n} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15.183)$$

Therefore, by (15.5),

$$\begin{aligned} \frac{\partial u_0}{\partial x} = \frac{k^2\pi^2}{32}&\left(\frac{1}{a^2}\cos\frac{\pi y}{b} - \frac{\sigma}{b^2}\cos\frac{\pi x}{a}\right) + 2(\mathrm{B} - \sigma\mathrm{A}) \\ &- \frac{\pi^2k^2}{8a^2}\sin^2\frac{\pi x}{2a}\cos^2\frac{\pi y}{2b}. \end{aligned}$$

In order to have the stressed plate symmetrical about the y-axis we must make u_0 zero along that axis. That is, we must integrate the last equation and use the condition that $u_0 = 0$ where $x = 0$. This gives

$$\begin{aligned} u_0 = \frac{k^2\pi^2}{32}&\left(\frac{x}{a^2}\cos\frac{\pi y}{b} - \frac{\sigma a}{\pi b^2}\sin\frac{\pi x}{a}\right) + 2(\mathrm{B} - \sigma\mathrm{A})x \\ &- \frac{\pi^2k^2}{16a^2}\left(x - \frac{a}{\pi}\sin\frac{\pi x}{a}\right)\cos^2\frac{\pi y}{2b}. \quad . \quad . \quad . \quad . \quad (15.184) \end{aligned}$$

From the conditions that $u_0 = 0$ where $x = \pm a$ it follows now that

$$\begin{aligned} 0 &= \frac{k^2\pi^2}{32a}\cos\frac{\pi y}{b} + 2(\mathrm{B} - \sigma\mathrm{A})a - \frac{\pi^2k^2}{16a}\cos^2\frac{\pi y}{2b} \\ &= -\frac{k^2\pi^2}{32a} + 2(\mathrm{B} - \sigma\mathrm{A})a, \end{aligned}$$

whence

$$\mathrm{B} - \sigma\mathrm{A} = \frac{\pi^2k^2}{64a^2}.$$

Likewise the conditions that $v_0 = 0$ along the edges $y = \pm b$ leads to the equation

$$A - \sigma B = \frac{\pi^2 k^2}{64 b^2}.$$

From the last two equations we find that

$$A = \frac{\pi^2 k^2}{64(1-\sigma^2)}\left(\frac{1}{b^2} + \frac{\sigma}{a^2}\right) \quad . \quad . \quad . \quad . \quad . \quad (15.185)$$

$$B = \frac{\pi^2 k^2}{64(1-\sigma^2)}\left(\frac{1}{a^2} + \frac{\sigma}{b^2}\right) \quad . \quad . \quad . \quad . \quad . \quad (15.180)$$

The displacements u_0 and v_0 have been found from P'_1 and P'_2. We ought to verify that these values of u_0 and v_0 are consistent with equation (15.183). Thus, by (15.7),

$$c_0 = \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} + \frac{\partial w}{\partial x} \cdot \frac{\partial w}{\partial y}$$

But

$$\frac{\partial u_0}{\partial y} = -\frac{\pi^2 k^2}{32ab} \sin\frac{\pi x}{a} \sin\frac{\pi y}{b}$$

$$= \frac{\partial v_0}{\partial x}$$

and

$$\frac{\partial w}{\partial x}\frac{\partial w}{\partial y} = \frac{\pi^2 k^2}{4ab} \sin\frac{\pi x}{2a} \cos\frac{\pi y}{2b} \cos\frac{\pi x}{2a} \sin\frac{\pi y}{2b}$$

$$= \frac{\pi^2 k^2}{16ab} \sin\frac{\pi x}{a} \sin\frac{\pi y}{b}$$

Therefore

$$c_0 = 0.$$

Moreover

$$S'_3 = n\frac{\partial^2 \varphi}{\partial x \partial y} = 0.$$

Consequently equation (15.183) is satisfied.

The solution we have now got makes u_0 zero along one pair of edges, and v_0 zero along the other pair, but it does not make u_0 and v_0 zero along all the four edges. We have not therefore, solved the problem of a plate in which the particles of the edges of the middle surface are immovable. In our solution the particles slide along the edge but do not move in the direction perpendicular to the edge. It is very unlikely that these conditions would be realised in a practical problem, but nevertheless the stresses in our present solution at points not near the edge must be very nearly the same as if the edge particles were absolutely fixed.

The actual pressure corresponding to the present values of w and φ can be found by means of equation (15.27). This pressure is positive over the whole of the plate. We can, however, treat the pressure as constant and use equation (15.164) to give the value of k, and hence the value of the stresses.

When, as in the present case, the bending moment about the edge of the plate is zero, then the same result is obtained by multiplying (15.163) by $w\,dx\,dy$ and integrating as in equation (15.164). Thus, in the present case, the approximate method gives

$$\iint pw\,dx\,dy = E'I\iint w\nabla_1^4 w\,dx\,dy - 2hE\iint w\left\{\frac{\partial^2\varphi}{\partial y^2}\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2\varphi}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right\}dx\,dy \quad (15.187)$$

Now

$$\iint w\nabla_1^4 w\,dx\,dy = 4\int_0^a\int_0^b \frac{\pi^4 k^2}{16}\left(\frac{1}{a^2}+\frac{1}{b^2}\right)^2\cos^2\frac{\pi x}{2a}\cos^2\frac{\pi y}{2b}\,dx\,dy$$

$$= \frac{\pi^4 k^2}{4}\left(\frac{1}{a^2}+\frac{1}{b^2}\right)^2(\tfrac{1}{2}a)(\tfrac{1}{2}b)$$

$$= \frac{\pi^4 k^2 ab}{16}\left(\frac{1}{a^2}+\frac{1}{b^2}\right)^2 \quad (15.188)$$

$$-\iint w\left(\frac{\partial^2\varphi}{\partial y^2}\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2\varphi}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right)dx\,dy$$

$$= 4\int_0^a\int_0^b \frac{k^2\pi^2}{4}\cos^2\frac{\pi x}{2a}\cos^2\frac{\pi y}{2b}\left\{\frac{2B}{a^2}+\frac{2A}{b^2}\right\}dx\,dy$$

$$+4\int_0^a\int_0^b \frac{k^4\pi^4}{128}\cos^2\frac{\pi x}{2a}\cos^2\frac{\pi y}{2b}\left\{\frac{1}{a^4}\cos\frac{\pi y}{b}+\frac{1}{b^4}\cos\frac{\pi x}{a}\right\}dx\,dy$$

$$= \tfrac{1}{2}k^2\pi^2 ab\left(\frac{B}{a^2}+\frac{A}{b^2}\right)+\frac{k^4\pi^4 ab}{256}\left\{\frac{1}{a^4}+\frac{1}{b^4}\right\}$$

$$= \frac{k^4\pi^4 ab}{128(1-\sigma^2)}\left\{\frac{1}{a^4}+\frac{1}{b^4}+\frac{2\sigma}{a^2b^2}\right\}+\frac{k^4\pi^4 ab}{256}\left(\frac{1}{a^4}+\frac{1}{b^4}\right)$$

$$= \frac{k^4\pi^4 ab}{256(1-\sigma^2)}\left\{\frac{4\sigma}{a^2b^2}+(3-\sigma^2)\left(\frac{1}{a^4}+\frac{1}{b^4}\right)\right\}$$

Also, assuming p to be uniform over the plate,

$$\iint pw\,dx\,dy = 4pk\int_0^a\int_0^b\cos\frac{\pi x}{2a}\cos\frac{\pi y}{2b}\,dx\,dy$$

$$= \frac{16ab}{\pi^2}kp.$$

Therefore (15.187) becomes

$$\frac{16ab}{\pi^2}kp = \frac{\pi^4 E'Ik^2 ab}{16}\left(\frac{1}{a^2}+\frac{1}{b^2}\right)^2 + \frac{hk^4\pi^4 abE}{128(1-\sigma^2)}\left\{\frac{4\sigma}{a^2b^2}+(3-\sigma^2)\left(\frac{1}{a^4}+\frac{1}{b^4}\right)\right\} \quad (15.189)$$

whence

$$\frac{16}{\pi^6}p=\frac{E'Ik}{16}\left(\frac{1}{a^2}+\frac{1}{b^2}\right)^2+\frac{E'hk^3}{128}\left\{\frac{4\sigma}{a^2b^2}+(3-\sigma^2)\left(\frac{1}{a^4}+\frac{1}{b^4}\right)\right\} \quad (15.190)$$

which is the equation for k.

The principal bending·moments are

$$-M_1=-E'I\left(\frac{\partial^2 w}{\partial x^2}+\sigma\frac{\partial^2 w}{\partial y^2}\right)$$

$$=\frac{\pi^2 kE'I}{4}\left(\frac{1}{a^2}+\frac{\sigma}{b^2}\right)\cos\frac{\pi x}{2a}\cos\frac{\pi y}{2b},$$

$$-M_2=\frac{\pi^2 kE'I}{4}\left(\frac{1}{b^2}+\frac{\sigma}{a^2}\right)\cos\frac{\pi x}{2a}\cos\frac{\pi y}{2b}.$$

These couples are clearly greatest at the centre of the plate, and their values at this point are

$$-M_1=\frac{\pi^2 kE'I}{4}\left(\frac{1}{a^2}+\frac{\sigma}{b^2}\right),$$

$$-M_2=\frac{\pi^2 kE'I}{4}\left(\frac{1}{b^2}+\frac{\sigma}{a^2}\right).$$

Now if a is less than b then $-M_1$ is greater than $-M_2$. Again the mean tensions at the centre of the plate, where they have their greatest values, are

$$P'_1=\frac{\pi^2 Ek^2}{32(1-\sigma^2)}\left\{\frac{2-\sigma^2}{a^2}+\frac{\sigma}{b^2}\right\}$$

$$P'_2=\frac{\pi^2 Ek^2}{32(1-\sigma^2)}\left\{\frac{\sigma}{a^2}+\frac{2-\sigma^2}{b^2}\right\}$$

If a is less than b the greatest of these is P'_1. Thus the greatest stress in the plate is

$$f=P'_1-\frac{hM_1}{I}$$

$$=\frac{\pi^2 Ek^2}{32(1-\sigma^2)}\left\{\frac{(2-\sigma^2)k+8h}{a^2}+\frac{\sigma(k+8h)}{b^2}\right\} \quad . \quad . \quad . \quad (15.191)$$

This result can be used for a rectangular plate the edges of which are fastened to supports which do not yield except to let the plate turn about the edge so that there is no couple applied about the edge. This means that sufficient tensions are applied to the edges to prevent each edge from moving towards the opposite edge.

CHAPTER XVI.

STABILITY OF THIN PLATES.

284. Rectangular plate with thrusts parallel to the sides.

Suppose forces are applied to the edges of a rectangular plate, as shown in fig. 153, the force acting on each pair of edges being parallel to the other pair of edges. Let the thickness of the plate be $2h$, and let the edges be at $x=0$, $x=a$, $y=0$, $y=b$. Let the forces be $2h\text{P}$ per unit length on the edges of length b, and $2h\text{Q}$ per unit length on the edges of length a, both these forces being assumed to act towards the plate. Then it is clear that there are particular values of P and Q that will cause the plate to buckle just as a strut buckles under a thrust. It is required to find what values of P and Q will cause an infinitesimal buckling.

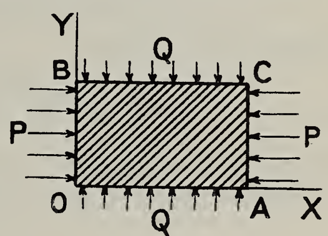


Fig. 153

Equation (15.23) applies to this problem. We need only put $p=0$, $\text{S}'_3=0$, $\text{P}'_1=-\text{P}$, $\text{P}'_2=-\text{Q}$. The resulting equation is

$$\frac{\text{EI}}{1-\sigma^2}\nabla_1{}^4 w = -2h\left(\text{P}\frac{\partial^2 w}{\partial x^2}+\text{Q}\frac{\partial^2 w}{\partial y^2}\right) \quad . \quad . \quad . \quad . \quad (16.1)$$

Suppose the edges are all supported but not clamped. Then the boundary conditions are

$$\left.\begin{aligned} &w=0 \text{ along all the edges;} \\ &\frac{\partial^2 w}{\partial x^2}+\sigma\frac{\partial^2 w}{\partial y^2}=0 \text{ along the edges } x=0,\ x=a; \\ &\frac{\partial^2 w}{\partial y^2}+\sigma\frac{\partial^2 w}{\partial x^2}=0 \text{ along the edges } y=0,\ y=b. \end{aligned}\right\} \quad . \quad (16.2)$$

All these conditions are satisfied by

$$w=\text{A}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16.3)$$

provided m and n are integers. This will also satisfy the differential equation if

$$\frac{\mathrm{EI}}{1-\sigma^2}\left(\frac{m^2\pi^2}{a^2}+\frac{n^2\pi^2}{b^2}\right)^2 = 2h\left(\frac{m^2\pi^2}{a^2}\mathrm{P}+\frac{n^2\pi^2}{b^2}\mathrm{Q}\right), \quad . \quad . \quad (16.4)$$

that is, if

$$\frac{m^2}{a^2}\mathrm{P}+\frac{n^2}{b^2}\mathrm{Q}=\frac{1}{3}\frac{\pi^2\mathrm{E}h^2}{1-\sigma^2}\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2 \quad . \quad . \quad . \quad . \quad (16.5)$$

Thus for the given shape of plate P and Q are not separately determined; only the combination of P and Q occurring on the left hand side of the last equation is determined. If, for example, Q were given, then the equation would determine P.

If $m=1$, $n=1$, then each of the median lines of the plate takes the form of half a sine wave, and the equation for P and Q is

$$\frac{\mathrm{P}}{a^2}+\frac{\mathrm{Q}}{b^2}=\frac{1}{3}\frac{\pi^2\mathrm{E}h^2}{1-\sigma^2}\left(\frac{1}{a^2}+\frac{1}{b^2}\right)^2 \quad . \quad . \quad . \quad . \quad . \quad (16.6)$$

Suppose $\mathrm{Q}=0$; then

$$\mathrm{P}=\frac{1}{3}\frac{\pi^2\mathrm{E}h^2a^2}{1-\sigma^2}\left(\frac{1}{a^2}+\frac{1}{b^2}\right)^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (16.7)$$

This is the least value of P that will buckle the plate when Q is zero.

Again suppose $\mathrm{Q}=\mathrm{P}$; then, still assuming $m=1$, $n=1$,

$$\mathrm{P}=\frac{1}{3}\frac{\pi^2\mathrm{E}h^2}{1-\sigma^2}\left(\frac{1}{a^2}+\frac{1}{b^2}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad (16.8)$$

whence

$$2h\mathrm{P}=\frac{2}{3}\frac{\pi^2\mathrm{E}h^3}{1-\sigma^2}\left(\frac{1}{a^2}+\frac{1}{b^2}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (16.9)$$

This is the least thrust per unit length, applied to all edges of the plate, that will cause the plate to buckle.

There is no reason why one of the stresses P or Q should not be negative, in which case the stress would be a tension; but if, for example, Q be a given tension, then, according to (16.6), P must be a still greater thrust to cause buckling than if Q were a thrust or zero.

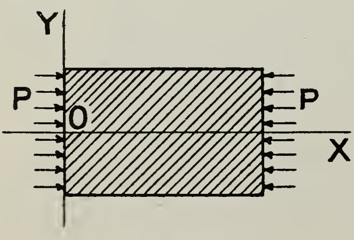


Fig. 154

285. Rectangular plate used as a strut.

Suppose that Q is zero and that the edges of length a are free and those of length b are supported. For this case let the axes be taken through the middle of one side of the plate as in figure 154. Then the boundary conditions are

$$\left.\begin{aligned} w=0 \\ \frac{\partial^2 w}{\partial x^2}+\sigma\frac{\partial^2 w}{\partial y^2}=0\end{aligned}\right\} \text{ where } x=0 \text{ and } x=a, \quad . \quad (16.10)$$

$$\left.\begin{aligned} \frac{\partial^2 w}{\partial y^2}+\sigma\frac{\partial^2 w}{\partial x^2}=0 \\ \frac{\partial}{\partial y}\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)=0\end{aligned}\right\} \text{ where } y=\pm\tfrac{1}{2}b \quad . \quad . \quad . \quad (16.11)$$

The set of conditions along the edges $x=0$ and $x=a$ are satisfied by

$$w=v\sin\frac{m\pi x}{a}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16.12)$$

where v is any function of y and m is an integer. Substituting this value of w in the differential equation (16.1) we get

$$\frac{\mathrm{EI}}{1-\sigma^2}\left\{\frac{d^4v}{dy^4}-2\frac{\pi^2m^2}{a^2}\frac{d^2v}{dy^2}+\frac{\pi^4m^4}{a^4}v\right\}=2h\frac{\pi^2m^2}{a^2}\mathrm{P}v,$$

that is,

$$\frac{d^4v}{dy^4}-2\frac{\pi^2m^2}{a^2}\frac{d^2v}{dy^2}+\frac{\pi^4m^4}{a^4}v=k^4v, \quad . \quad . \quad . \quad (16.13)$$

where
$$k^4=\frac{3\pi^2(1-\sigma^2)m^2\mathrm{P}}{a^2h^2\mathrm{E}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16.14)$$

To solve this put

$$v=\mathrm{A}e^{qy} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16.15)$$

This gives

$$\left(q^2-\frac{\pi^2m^2}{a^2}\right)^2=k^4,$$

from which we get four roots of the form $\pm q_1$, $\pm q_2$ satisfying the equations

$$\left.\begin{aligned} q_1{}^2-\frac{\pi^2m^2}{a^2}=k^2 \\ q_2{}^2-\frac{\pi^2m^2}{a^2}=-k^2\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (16.16)$$

Thus we find

$$v=\mathrm{A}\cosh q_1y+\mathrm{B}\sinh q_1y+\mathrm{C}\cosh q_2y+\mathrm{D}\sinh q_2y \quad (16.17)$$

It is easy to see that the boundary conditions (16.11) can be satisfied by taking v to be an even function of y, the form corresponding to symmetry about the axis of x. Thus we may take

$$v=\mathrm{A}\cosh q_1y+\mathrm{C}\cosh q_2y \quad . \quad . \quad . \quad . \quad (16.18)$$

The boundary conditions (16.11) give

$$\left(q_1{}^2-\sigma\frac{\pi^2m^2}{a^2}\right)\mathrm{A}\cosh\tfrac{1}{2}q_1b+\left(q_2{}^2-\sigma\frac{\pi^2m^2}{a^2}\right)\mathrm{C}\cosh\tfrac{1}{2}q_2b=0 \quad (16.19)$$

and

$$q_1\left(q_1{}^2-\frac{\pi^2m^2}{a^2}\right)\mathrm{A}\sinh\tfrac{1}{2}q_1b+q_2\left(q_2{}^2-\frac{\pi^2m^2}{a^2}\right)\mathrm{C}\sinh\tfrac{1}{2}q_2b=0 \quad (16.20)$$

By means of equation (16.16) these last two equations can be reduced to the forms

$$\left\{k^2+(1-\sigma)\frac{\pi^2m^2}{a^2}\right\}\mathrm{A}\cosh\tfrac{1}{2}q_1b-\left\{k^2-(1-\sigma)\frac{\pi^2m^2}{a^2}\right\}\mathrm{C}\cosh\tfrac{1}{2}q_2b=0 \quad (16.21)$$

$$q_1\mathrm{A}\sinh\tfrac{1}{2}q_1b-q_2\mathrm{C}\sinh\tfrac{1}{2}q_2b=0 \quad . \quad . \quad . \quad . \quad (16.22)$$

The elimination of the ratio A : C from these last two equations gives

$$q_2\left\{k^2+(1-\sigma)\frac{\pi^2m^2}{a^2}\right\}\tanh\tfrac{1}{2}q_2b=q_1\left\{k^2-(1-\sigma)\frac{\pi^2m^2}{a^2}\right\}\tanh\tfrac{1}{2}q_1b \quad (16.23)$$

This equation gives the buckling stress P corresponding to the form assumed for the shape of the plate. The smallest value of P is obtained by taking $m=1$.

To solve equation (16.23) it is probably best to express k^2 in terms of q_2; then the equation becomes

$$q_2\left\{(2-\sigma)\frac{\pi^2m^2}{a^2}-q_2{}^2\right\}\tanh\tfrac{1}{2}q_2b=q_1\left\{\sigma\frac{\pi^2m^2}{a^2}-q_2{}^2\right\}\tanh\tfrac{1}{2}q_1b \quad (16.24)$$

By means of (16.16) q_1 can be expressed in terms of q_2 thus

$$q_1{}^2=2\frac{\pi^2m^2}{a^2}-q_2{}^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16.25)$$

Now $\frac{1}{2}q_2b$ can be taken as variable and two curves can be plotted, with this as abscissa, the ordinates of which represent the values of the two sides of (16.24). The point of intersection of these curves determines q_2, and then (16.16) gives k^2.

It is easy to see that there is one positive real value of q_2 satisfying (16.24). For when q_2 is zero the left hand side is zero and the right hand side positive; again when $q_2{}^2a^2=\sigma\pi^2m^2$ the left hand side is positive and the right hand side zero. Consequently there is a value of q_2a lying between zero and $\sqrt{\sigma}\pi m$; there is a second value of q_2a lying between $\sqrt{(2-\sigma)}\pi m$ and $\sqrt{2}\pi m$, but this second root is merely the value of q_1a corresponding to the first root. Thus we can see, without going into detailed calculation, that

$$k^2=\frac{\pi^2m^2}{a^2}-q_2{}^2>\frac{\pi^2m^2}{a^2}-\sigma\frac{\pi^2m^2}{a^2},$$

and

$$k^2<\frac{\pi^2m^2}{a^2};$$

whence

$$\left.\begin{aligned}\mathrm{P}&>\frac{\pi^2m^2h^2\mathrm{E}}{3(1-\sigma^2)a^2}(1-\sigma)^2\\ \text{and}\qquad \mathrm{P}&<\frac{\pi^2m^2h^2\mathrm{E}}{3(1-\sigma^2)a^2}\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (16.26)$$

If σ is zero the equations for the plate agree with those for a beam, and therefore by putting $\sigma = 0$ in our result we ought to get the same value as Euler's theory of struts gives. When $\sigma = 0$ the two values between which P lies coincide, and when this common value is multiplied by the area of the section of the plate it becomes exactly Euler's critical thrust for a pinned-pinned beam of the same length and cross section, the smallest thrust arising by putting $m = 1$.

The equations suggest that there is a possible solution in which v is an odd function of y, involving only the sinh functions; but since, in this case, the median of the plate $y = 0$ would be straight, there can be no serious buckling of this type.

286. Buckling of a uniform circular plate under radial thrusts.

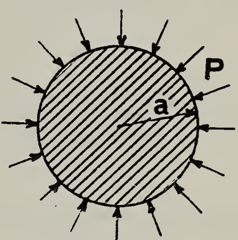


Fig. 155

Let the uniform radial thrust be P per unit area of the rim surface, and let a denote the radius of the plate. In equation (15.23) we must now put

$$P'_1 = -P, \quad P'_2 = -P, \quad S'_3 = 0.$$

The equation then becomes

$$\frac{EI}{1-\sigma^2}\nabla_1^4 w = -2hP\nabla_1^2 w,,$$

whence
$$\nabla_1^4 w = -k^2\nabla_1^2 w, \qquad (16.27)$$

where
$$k^2 = \frac{3(1-\sigma^2)P}{h^2E} \qquad (16.28)$$

Since w is a function of r only

$$\nabla_1^2 w = \frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)$$

Consequently two integrations of (16.27) give

$$\nabla_1^2 w = -k^2 w + B\log_e r + C \qquad (16.29)$$

Now if the disk has no central hole $\nabla_1^2 w$ must certainly be finite at the centre. This requires that B should be zero. Then

$$\nabla_1^2 w = -k^2(w-b), \qquad (16.30)$$

$k^2 b$ being written for C.

Now let

$$u = w - b;$$

then

$$\nabla_1^2 u = \nabla_1^2 w$$
$$= -k^2 u; \qquad (16.31)$$

that is,

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right) = -k^2u\,,$$

whence

$$\frac{1}{kr}\frac{d}{kdr}\left(kr\frac{du}{kdr}\right) = -u.$$

If we now put

$$s = kr \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16.32)$$

this last equation becomes

$$\frac{1}{s}\frac{d}{ds}\left(s\frac{du}{ds}\right) = -u \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16.33)$$

This can be solved by means of a series of the form

$$u = a_0 + a_1 s + a_2 s^2 + a_3 s^3 + \ldots$$
$$= \Sigma a_n s^n.$$

Substituting this in (16.33) we get

$$\Sigma n^2 a_n s^{n-2} = -\Sigma a_n s^n$$
$$= -\Sigma a_{n-2} s^{n-2}.$$

Equating coefficients of like powers of s on both sides we find

$$n^2 a_n = -a_{n-2}\,, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16.34)$$

whence $$a_n = -\frac{a_{n-2}}{n^2}.$$

Therefore, starting with a_0,

$$a_2 = -\frac{a_0}{2^2},$$

$$a_4 = -\frac{a_2}{4^2} = +\frac{a_0}{2^2.4^2},$$

etc.

If we put $n = 1$ in (16.34) we should find

$$a_{-1} = -a_1\,,$$

and thus the series would contain negative powers of r, which become infinite at the centre. To avoid this we must make

$$a_1 = 0\,,$$

and consequently the coefficients of all the odd powers of r must be zero. Therefore finally

$$u = a_0\left\{1 - \frac{s^2}{2^2} + \frac{s^4}{2^2.4^2} - \frac{s^6}{2^2.4^2.6^2} + \frac{s^8}{2^2.4^2.6^2\,8^2}\cdots\right\}, \quad . \quad (16.35)$$

the series in the brackets being convergent for all values of s. The series should be compared with the series for cos s, namely

$$\cos s = 1 - \frac{s^2}{1.2} + \frac{s^4}{1.2.3.4} - \frac{s^6}{1.2.3.4.5.6} + \ldots$$

The function of s represented by the infinite series in (16.35 is known as the Bessel function of zero order of the first kind, and is denoted by $J_0(s)$. Thus

$$J_0(s) = 1 - \frac{s^2}{2^2} + \frac{s^4}{2^2.4^2} - \frac{s^6}{2^2.4^2.6^2} + \ldots \quad (16.36)$$

This function has characteristics very similar to those of the cosine function. It is a periodic function, but, whereas $\cos s$ has a constant period and a constant amplitude, $J_0(s)$ has a variable period and a variable amplitude. The period, however, approaches the constant value 2π as s increases, but the amplitude is proportional to $s^{-\frac{1}{2}}$ when s is great.

On differentiating $J_0(s)$ we get

$$J'_0(s) = -\frac{s}{2} + \frac{s^3}{2^2.4} - \frac{s^5}{2^2.4^2.6} + \ldots \quad (16.37)$$

Now there is another Bessel function, called a Bessel function of the first order, which is defined by the equation

$$J_1(s) = \frac{s}{2} - \frac{s^3}{2^2.4} + \frac{s^5}{2^2.4^2.6} - \ldots \quad (16.38)$$

Thus we see that

$$J'_0(s) = -J_1(s) \quad (16.39)$$

These are the only Bessel functions we shall use in the present problem, but we shall later use functions of other orders.

Returning now to equation (16.28), we have found a solution of the form

$$w - b = a_0 J_0(kr) \quad (16.40)$$

Since (16.31) is a differential equation of the second order and (16.40) contains only one new constant of integration this latter does not give the complete solution of (16.30). It does, however, give the solution appropriate to a complete disk, that is, a disk with no central hole. If the disk had a central circular hole we should have needed to retain the term $B \log_e r$ in equation (16.29), and we should have needed the second solution of (16.30). We should thus have had two new constants in the expression for w by means of which we could have satisfied the boundary conditions at the edge of the hole in the disk.

287. Disk clamped at the rim.

If the disk is clamped at the rim the boundary conditions are

$$\left.\begin{aligned} w &= 0 \\ \frac{dw}{dr} &= 0 \end{aligned}\right\} \text{ where } r = a; \quad (16.41)$$

that is,

$$b + a_0 J_0(ka) = 0 \quad \ldots \quad (16.42)$$

$$a_0 k J'_0(ka) = 0 \quad \ldots \quad (16.43)$$

This last equation is equivalent to

$$J_1(ka) = 0, \quad \ldots \quad (16.44)$$

which has an infinite number of roots, since $J_1(s)$, like $J_0(s)$, is a periodic function of s. Tables of roots $J_n(x) = 0$ for several values of n are given in the appendix. The smallest root of (16.44) is

$$k_1 a = 3{\cdot}832 \quad \ldots \quad (16.45)$$

This determines P, and equation (16.42) determines only the constant b. Thus from (16.28) and (16.45) we find

$$P = \frac{3{\cdot}832^2}{3} \frac{h^2 E}{(1-\sigma^2)a^2}$$

$$= \frac{14{\cdot}68}{3} \frac{h^2 E}{(1-\sigma^2)a^2} \quad \ldots \quad (16.46)$$

The thrust per unit length of the rim is

$$2hP = 9{\cdot}79 \frac{h^3 E}{(1-\sigma^2)a^2}. \quad \ldots \quad (16.47)$$

The second root of (16.44) is

$$k_2 a = 7{\cdot}016, \quad \ldots \quad (16.48)$$

which gives

$$2hP = \frac{2 \times 7{\cdot}016^2}{3} \frac{h^3 E}{(1-\sigma^2)a^2}$$

$$= 32{\cdot}83 \frac{h^3 E}{(1-\sigma^2)a^2}. \quad \ldots \quad (16.49)$$

This last thrust arises when the shape of the disk is such that there is one circle, as well as the rim, where $w = 0$. The radius of this circle is given by putting

$$w = 0,$$

that is, by putting

$$b + a_0 J_0(k_2 r) = 0.$$

Substituting the value of b from (16.42) this becomes

$$a_0 \{ J_0(k_2 r) - J_0(k_2 a) \} = 0,$$

whence

$$J_0(k_2 r) = J_0(k_2 a) = J_0(7{\cdot}016). \quad \ldots \quad (16.50)$$

From tables of Bessel functions we find

$$J_0(7{\cdot}016) = 0{\cdot}29992$$

$$= 0{\cdot}300 \text{ nearly.}$$

Also

$$J_0(1{\cdot}869) = 0.300.$$

Therefore, for the smaller value of r,

$$k_2 r = 1{\cdot}869.$$

But

$$k_2 a = 7{\cdot}016.$$

Therefore

$$\frac{r}{a} = \frac{1{\cdot}869}{7{\cdot}016}$$

$$= 0{\cdot}265. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16.51)$$

Fig. 156a and 156b show roughly the shape of a diameter of the disk for the first and second cases.

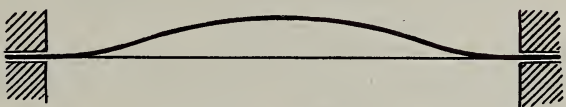


Fig. 156a

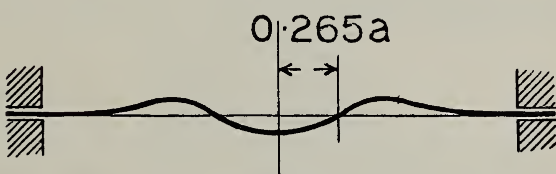


Fig. 156b

The second shape is just as unlikely in practice as the second form of a strut in Euler's theory. Only the first and smallest thrust has any useful application.

288. Plate not clamped at the rim.

In this case the bending moment is zero at the rim. Therefore the boundary conditions are

$$\left.\begin{aligned} w &= 0 \\ \frac{d^2w}{dr^2} + \frac{\sigma}{r}\frac{dw}{dr} &= 0 \end{aligned}\right\} \text{ where } r = a, \quad . \quad . \quad . \quad . \quad (16.52)$$

that is, since w is still correctly given by (16.40),

$$b + a_0 J_0(ka) = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (16.53)$$

$$k^2 J''_0(ka) + \sigma \frac{k}{a} J'_0(ka) = 0. \quad . \quad . \quad . \quad . \quad . \quad (16.54)$$

By (16.39) this last equation is equivalent to the following

$$ka J'_1(ka) + \sigma J_1(ka) = 0 \quad . \quad . \quad . \quad . \quad . \quad (16.55)$$

or, with s for ka,

$$s J'_1(s) + \sigma J_1(s) = 0. \quad . \quad . \quad . \quad . \quad . \quad (16.56)$$

Now from the definition of $J_1(s)$ in (16.38) we get

$$sJ_1(s)=\frac{s^2}{2}-\frac{s^4}{2^2.4}+\frac{s^6}{2^2.4^2.6}-\frac{s^8}{2^2.4^2.6^2.8}+\dots$$

whence, by differentiating both sides we get

$$sJ'_1(s)+J_1(s)=s-\frac{s^3}{2^2}+\frac{s^5}{2^2.4^2}-\frac{s^7}{2^2.4^2.6^2}+$$
$$=sJ_0(s).$$

Consequently

$$sJ'_1(s)=sJ_0(s)-J_1(s). \qquad (16.57)$$

By means of this last equation (16.56) can be written

$$sJ_0(s)=(1-\sigma)J_1(s) \qquad (16.58)$$

This is the equation which determines s, and therefore k. To get a numerical value we must assume a value for σ. If we assume that $\sigma=0.25$, then we have to solve

$$sJ_0(s)=0{\cdot}75\,J_1(s). \qquad (16.59)$$

From tables of Bessel functions we find

$$\left.\begin{aligned}2J_0(2)&=0{\cdot}4478\\ 0{\cdot}75\,J_1(2)&=0{\cdot}4325\end{aligned}\right\}$$
$$\left.\begin{aligned}2{\cdot}1\,J_0(2{\cdot}1)&=0{\cdot}3499\\ 0{\cdot}75\,J_1(2{\cdot}1)&=0{\cdot}4262\end{aligned}\right\}$$

Thus the root of (16.59) lies between 2 and 2·1, and by interpolation we find that it is approximately

$$ka=s=2{\cdot}017 \qquad (16.60)$$

and this is the smallest root, giving therefore the smallest value of k. Consequently the stress P which buckles the disk is

$$\begin{aligned}P&=\frac{k^2h^2E}{3(1-\sigma^2)}\\ &=\frac{2{\cdot}017^2}{3}\,\frac{h^2E}{(1-\sigma^2)a^2}\\ &=1{\cdot}441\,\frac{h^2E}{a^2},\end{aligned}$$

and the thrust per unit length of the rim is

$$2hP=2{\cdot}88\,\frac{h^3E}{a^2}. \qquad (16.61)$$

The corresponding thrust for a square plate with sides of length $2a$, equal to the diameter of the circle, is, by (16.9),

$$\begin{aligned}2hP&=\frac{2}{3}\,\frac{\pi^2h^3E}{(1-\sigma^2)}\cdot\frac{2}{(2a)^2}\\ &=3{\cdot}40\,\frac{h^3E}{a^2}. \qquad (16.62)\end{aligned}$$

If we assume that $\sigma = \frac{1}{3}$ the smallest root of (16.58) is

$$s = 2{\cdot}07\,,$$

whence

$$2h\text{P} = 3{\cdot}21\,\frac{h^3\text{E}}{a^2}, \qquad (16.63)$$

and the result for a square of side $2a$, with the same value of σ, is

$$2h\text{P} = 3{\cdot}70\,\frac{h^3\text{E}}{a^2}. \qquad (16.64)$$

289. Approximate method for disks.

Very good approximate values of k^2, and therefore of P, can be obtained from (16.27) by the same method as was used in the last chapter to find the deflexion due to a given pressure. The method consists, as before, in assuming a reasonable form for w and then deriving an energy equation which gives k^2. When P is constant over the disk the derived equation is, in this case, a true energy equation.

Multiplying (16.27) by $wrdr$ and integrating over the whole disk, we get

$$\int_0^a w\nabla_1{}^4 wrdr = -k^2\int_0^a w\nabla_1{}^2 wrdr \qquad (16.65)$$

But we have already shown, in proving equation (15.109), that, for a supported or clamped disk,

$$\int_0^a w\nabla_1{}^4 wrdr = \int_0^a (\nabla_1{}^2 w)^2 rdr - (1-\sigma)\left[\left(\frac{dw}{dr}\right)^2\right]_{r=a} \qquad (16.66)$$

Moreover, by integration by parts,

$$\begin{aligned}\int_0^a w\nabla_1{}^2 wrdr &= \int_0^a w\frac{d}{dr}\left(r\frac{dw}{dr}\right)dr\\ &= \left[wr\frac{dw}{dr}\right]_0^a - \int_0^a r\left(\frac{dw}{dr}\right)^2 dr\\ &= 0 - \int_0^a \left(\frac{dw}{dr}\right)^2 rdr, \qquad (16.67)\end{aligned}$$

the integrated term being zero because $w = 0$ at the rim and $r = 0$ at the centre.

Therefore (16.65) becomes

$$\int_0^a (\nabla_1{}^2 w)^2 rdr - (1-\sigma)\left[\left(\frac{dw}{dr}\right)^2\right]_{r=a} = k^2\int_0^a \left(\frac{dw}{dr}\right)^2 rdr \qquad (16.68)$$

An approximate value of w used in this last form of equation gives a very good value of k^2, and therefore of P.

Suppose

$$\begin{aligned}w &= c(a^2 - r^2)(a^2 - nr^2)\\ &= c\{a^4 - (n+1)a^2r^2 + nr^4\}. \qquad (16.69)\end{aligned}$$

Then

$$\frac{dw}{dr} = 2c\{-(n+1)a^2r + 2nr^3\}$$

$$= 2c(n-1)a^3 \text{ when } r = a\,,$$

$$\nabla_1{}^2 w = 4c\{-(n+1)a^2 + 4nr^2\}\,,$$

$$\int_0^a (\nabla_1{}^2 w)^2 r dr = 16c^2a^6\{\tfrac{1}{2}(n+1)^2 - 2n(n+1) + \tfrac{8}{3}n^2\}$$

$$= \tfrac{8}{3}c^2a^6(7n^2 - 6n + 3),$$

$$\int_0^a \left(\frac{dw}{dr}\right)^2 r dr = 4c^2a^8\{\tfrac{1}{4}(n+1)^2 - \tfrac{2}{3}n(n+1) + \tfrac{1}{2}n^2\}$$

$$= \tfrac{1}{3}c^2a^8(n^2 - 2n + 3).$$

Therefore (16.68) gives

$$\tfrac{8}{3}c^2a^6\{7n^2 - 6n + 3 - \tfrac{3}{2}(1-\sigma)(n-1)^2\} = \tfrac{1}{3}k^2c^2a^8(n^2 - 2n + 3),$$

whence

$$k^2a^2 = \frac{56n^2 - 48n + 24 - 12(1-\sigma)(n-1)^2}{n^2 - 2n + 3}. \quad . \quad . \quad (16.70)$$

The only suitable value of n for a disk with a clamped rim is $n = 1$, for this is the only value that makes $\frac{dw}{dr}$ zero at the rim. With this value the last equation gives

$$k^2a^2 = 16\,, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16.71)$$

whence $\qquad ka = 4\,,$

which is the approximate result corresponding to 3·832 given in (16·45) by the accurate method, the error being about four per cent.

For the disk not clamped at the rim we need only allow n to vary and find the minimum value of k^2a^2 given by (16.70). The justification for this is as follows.

In the equilibrium state the total potential energy is a minimum. Now the potential energy due to the strain of the plate and the work done by P is proportional to the excess of the left hand side of (16.68) over the right hand side. That is

$$V \propto c^2(\xi - \eta k^2a^2), \quad . \quad . \quad . \quad . \quad . \quad . \quad (16.72)$$

where

$$\xi = 56n^2 - 48n + 24 - 12(1-\sigma)(n-1)^2$$

$$\eta = n^2 - 2n + 3.$$

The conditions for a minimum value of V for variations in both c and n are

$$\frac{\partial V}{\partial(c^2)} = 0,$$

$$\frac{\partial V}{\partial n} = 0;$$

that is,

$$\xi-\eta k^2a^2=0, \qquad (16.73)$$

and

$$\frac{d\xi}{dn}=k^2a^2\frac{d\eta}{dn}=0. \qquad (16.74)$$

Eliminating k^2a^2 from these we get

$$\frac{1}{\xi}\frac{d\xi}{dn}-\frac{1}{\eta}\frac{d\eta}{dn}=0,$$

whence

$$\frac{d}{dn}\left(\frac{\xi}{\eta}\right)=0. \qquad (16.75)$$

This last equation gives n, and then (16.73) gives k^2a^2. Thus the method amounts to taking k^2a^2 as the minimum value of the fraction on the right hand side of (16.70).

If we put $\sigma=\frac{1}{3}$ in (16.70) we get

$$k^2a^2=16\,\frac{3m^2+4m+2}{m^2+2}, \qquad (16.76)$$

where $m=n-1$.

On putting

$$y=\frac{3m^2+4m+2}{m^2+2}$$

we find that

$$(3-y)m^2+4m+2(1-y)=0.$$

The extreme values of y that make m real are given by

$$4(3-y)\times 2(1-y)=4^2;$$

that is,

$$y^2-4y+1=0$$

whence

$$y=2\pm\sqrt{3}.$$

Thus the minimum value of k^2a^2 is

$$k^2a^2=16(2-\sqrt{3})$$
$$=4{\cdot}2872$$

from which

$$ka=2{\cdot}0705, \qquad (16.77)$$

which agrees excellently with 2·07 found from Bessel functions.

Let $\sigma=0{\cdot}25$. Then (16.70) becomes

$$k^2a^2=\frac{47n^2-30n+15}{n^2-2n+3}.$$

Taking the minimum value of the expression on the right we get

$$k^2a^2 = 4{\cdot}072,$$

whence
$$ka = 2{\cdot}018, \quad \ldots \ldots \quad (16.78)$$

which is also remarkably near the 2·017 we got in (16.60) by accurate methods.

290. The approximate method for plates of other shapes.

The equation (16.65) merely expresses the fact that the energy in the disk due to the bending is equal to the work done by the constant force $2h\mathrm{P}$ acting on each unit length of the rim, the radial displacement of the rim being assumed to be due to the buckling of the plate and not at all to the shortening of the radial lines; that is, the work done by the constant force $2h\mathrm{P}$ is calculated on the assumption that the curved radial lines in the middle surface have the same length as when they were straight, the displacement of the rim being therefore due to the bending of the radii alone. Thus the equation for the buckling load is correctly attained by assuming that the middle surface itself is inextensible. The reason for this is that the strain energy due to the compression of the middle surface up to the point when buckling begins is separately equal to the work done by the rim forces up to that time, and the two terms cancel out of the energy equation. We can now get similar equations for plates of other shapes if we make the assumption that the middle surface is incompressible.

291. Rectangular plate.

Let us return to the problem with which we began this chapter, for which the equation of equilibrium is (16.1). Instead of this equation we must now use the corresponding energy equation. To get this we may multiply both sides by $\frac{1}{2}w\,dx\,dy$ and integrate over the area of the plate. Thus

$$\frac{1}{2}\iint \mathrm{E'I}w\nabla_1{}^4 w\,dx\,dy = -h\iint\left(\mathrm{P}\frac{\partial^2 w}{\partial x^2} + \mathrm{Q}\frac{\partial^2 w}{\partial y^2}\right)w\,dx\,dy \quad . \ (16.79)$$

Now we have already found, in the case of the disk, that the left hand side of this equation is the total energy due to the bending of the disk. The proof for a plate of any shape requires a form of Green's theorem, but for the rectangular plate this theorem is easy to prove. Thus let

$$\nabla_1{}^2 w = \psi,$$

and let the sides be along the lines $x=0$, $x=a$, $y=0$, $y=b$. Then

$$\iint w\nabla_1{}^4 w\,dx\,dy = \iint w\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right)dx\,dy.$$

Now, by integration by parts,

$$\int_0^a w\frac{\partial^2\psi}{\partial x^2}\,dx = \left[w\frac{\partial\psi}{\partial x}\right]_0^a - \int_0^a \frac{\partial w}{\partial x}\frac{\partial\psi}{\partial x}\,dx.$$

The integrated term is zero because w is zero at both edges. Therefore, integrating by parts again,

$$\int_0^a w\frac{\partial^2\psi}{\partial x^2}dx = \left[-\psi\frac{\partial w}{\partial x}\right]_0^a + \int_0^a \psi\frac{\partial^2 w}{\partial x^2}dx;$$

consequently

$$\iint w\frac{\partial^2\psi}{\partial x^2}dxdy = -\int_0^b\left[\psi\frac{\partial w}{\partial x}\right]_0^a dy + \iint\psi\frac{\partial^2 w}{\partial x^2}dxdy,$$

and therefore

$$\iint w\nabla_1{}^4 w\,dxdy = -\int_0^b\left[\psi\frac{\partial w}{\partial x}\right]_0^a dy - \int_0^a\left[\psi\frac{\partial w}{\partial y}\right]_0^b dx$$

$$+\iint\psi^2 dxdy. \quad . \quad . \quad . \quad . \quad (16.80)$$

If the plate is clamped at the rim the two single integrals are zero, whereas if the rim is fixed without clamping, then, along the sides $x=0$, $x=a$, since the bending moment is zero,

$$0 = \frac{\partial^2 w}{\partial x^2} + \sigma\frac{\partial^2 w}{\partial y^2},$$

whence

$$\psi = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}$$

$$= (1-\sigma)\frac{\partial^2 w}{\partial y^2}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16.81)$$

and along the other pair of sides

$$\psi = (1-\sigma)\frac{\partial^2 w}{\partial x^2}.$$

Now since $w=0$ along all the sides it follows that

$$\frac{\partial^2 w}{\partial y^2} = 0,$$

along the two sides $x=0$, $x=a$. Consequently

$$\left[\psi\frac{\partial w}{\partial x}\right]_0^a = \left[(1-\sigma)\frac{\partial^2 w}{\partial y^2}\frac{\partial w}{\partial x}\right]_0^a$$

$$= 0$$

Likewise

$$\left[\psi\frac{\partial w}{\partial y}\right]_0^b = 0.$$

Then finally

$$\iint w\nabla_1{}^4 w\,dxdy = \iint\psi^2 dxdy \quad . \quad . \quad . \quad . \quad (16.82)$$

whether the plate is clamped or not.

Again, dealing with the other side of equation (16.79), we get

$$\int_0^a Pw\frac{\partial^2 w}{\partial x^2}dx = \left[Pw\frac{\partial w}{\partial x}\right]_0^a - \int_0^a P\left(\frac{\partial w}{\partial x}\right)^2 dx$$
$$= -P\int_0^a\left(\frac{\partial w}{\partial x}\right)^2 dx, \quad . \quad . \quad . \quad . \quad . \quad (16.83)$$

P being constant over the plate.

Therefore

$$\iint Pw\frac{\partial^2 w}{\partial x^2}dx\,dy = -P\iint\left(\frac{\partial w}{\partial x}\right)^2 dx\,dy. \quad . \quad . \quad . \quad (16.84)$$

Finally equation (16.79) becomes, for a rectangular plate,

$$\tfrac{1}{2}E'I\iint(\nabla_1{}^2 w)^2 dx\,dy = h\iint\left\{P\left(\frac{\partial w}{\partial x}\right)^2 + Q\left(\frac{\partial w}{\partial y}\right)^2\right\}dx\,dy. \quad . \quad (16.85)$$

The left hand side of this last equation is identical with the strain energy V given by (14.154) for a plate whose boundary is formed of straight lines which are all held so that w has the same value over the whole boundary. If we were dealing with a plate with a curved boundary the correct form of the left hand side of this last equation should, in every case, be the value of V given by (14.154). The simpler form for the case of the rectangular plate is due to the fact that the whole curvature is zero for a rectangular plate, or indeed for any plate with a rectilinear boundary. We can quickly put equation (16.85) to the test by using

$$w = A\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16.86)$$

the form that we have already found to be correct for the unclamped plate. Then

$$\nabla_1{}^2 w = -\pi^2 A\left(\frac{1}{a^2}+\frac{1}{b^2}\right)\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}; \quad . \quad (16.87)$$

$$\left(\frac{\partial w}{\partial x}\right)^2 = \frac{\pi^2}{a^2}A^2\cos^2\frac{\pi x}{a}\sin^2\frac{\pi y}{b};$$

$$\int_0^a\int_0^b(\nabla_1{}^2 w)^2 dx\,dy = \pi^4 A^2\left(\frac{1}{a^2}+\frac{1}{b^2}\right)^2\times\frac{ab}{4};$$

$$\int_0^a\int_0^b\left(\frac{\partial w}{\partial x}\right)^2 dx\,dy = \frac{\pi^2}{a^2}A^2\times\frac{ab}{4}.$$

Therefore (16.85) gives

$$\tfrac{1}{2}E'I\pi^4 A^2\left(\frac{1}{a^2}+\frac{1}{b^2}\right)^2\times\frac{ab}{4} = \frac{\pi^2 A^2 abh}{4}\left(\frac{P}{a^2}+\frac{Q}{b^2}\right), \quad . \quad (16.88)$$

whence

$$\frac{P}{a^2}+\frac{Q}{b^2}=\tfrac{1}{3}\pi^2E'h^2\left(\frac{1}{a^2}+\frac{1}{b^2}\right)^2, \quad . \quad . \quad . \quad . \quad (16.89)$$

which is the same equation as (16.7).

292. Rectangular plate clamped at all the edges.

Let the origin be taken at the middle of the plate, and let the sides be $x=\pm\frac{1}{2}a$, $y=\pm\frac{1}{2}b$. Then there are two simple forms for w that satisfy the boundary conditions, either of which may be used to give the approximate values of the smallest buckling thrusts. These forms are

$$w=c\cos^2\frac{\pi x}{a}\cos^2\frac{\pi y}{b}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16.90)$$

and

$$w=c(a^2-4x^2)^2(b^2-4y^2)^2. \quad . \quad . \quad . \quad . \quad . \quad (16.91)$$

From (16.90) we find that

$$\frac{\partial w}{\partial x}=-2c\frac{\pi}{a}\cos\frac{\pi x}{a}\sin\frac{\pi x}{a}\cos^2\frac{\pi y}{b},$$

$$\frac{\partial^2 w}{\partial x^2}=-2c\frac{\pi^2}{a^2}\cos\frac{2\pi x}{a}\cos^2\frac{\pi y}{b},$$

$$\nabla_1{}^2 w=-2c\pi^2\left\{\frac{1}{a^2}\cos\frac{2\pi x}{a}\cos^2\frac{\pi y}{b}+\frac{1}{b^2}\cos\frac{2\pi y}{b}\cos^2\frac{\pi x}{a}\right\} \quad (16.92)$$

Also, taking the integrals over quarter of the plate instead of the whole plate, which only amounts to omitting a factor 4, we get

$$\int_0^{\frac{a}{2}}\int_0^{\frac{b}{2}}(\nabla_1{}^2w)^2dx\,dy=\frac{\pi^4c^2ab}{16}\left(\frac{3}{a^4}+\frac{3}{b^4}+\frac{2}{a^2b^2}\right);$$

$$\int_0^{\frac{a}{2}}\int_0^{\frac{b}{2}}\left\{P\left(\frac{\partial w}{\partial x}\right)^2+Q\left(\frac{\partial w}{\partial y}\right)^2\right\}dx\,dy=\frac{3\pi^2c^2ab}{64}\left(\frac{P}{a^2}+\frac{Q}{b^2}\right).$$

Therefore (16.85) becomes

$$\tfrac{1}{3}E'h^2.\frac{\pi^4c^2ab}{16}\left(\frac{3}{a^4}+\frac{3}{b^4}+\frac{2}{a^2b^2}\right)=\frac{3\pi^2c^2ab}{64}\left(\frac{P}{a^2}+\frac{Q}{b^2}\right),$$

whence

$$\frac{P}{a^2}+\frac{Q}{b^2}=E'h^2\left(\frac{4\pi^2}{3a^4}+\frac{4\pi^2}{3b^4}+\frac{8\pi^2}{9a^2b^2}\right)$$

$$=E'h^2\left(\frac{13{\cdot}16}{a^4}+\frac{13{\cdot}16}{b^4}+\frac{8{\cdot}77}{a^2b^2}\right). \quad . \quad . \quad . \quad (16.93)$$

For a square plate

$$P+Q=\frac{32\pi^2}{9}\frac{E'h^2}{a^2}=35{\cdot}1\frac{E'h^2}{a^2}. \quad . \quad . \quad . \quad . \quad . \quad (16.94)$$

The corresponding result for an unclamped square plate, obtained by putting $b=a$ in (16.6), is

$$P+Q=\frac{4\pi^2}{3}\frac{E'h^2}{a^2}=\frac{3}{8}\left(\frac{32\pi^2}{9}\frac{E'h^2}{a^2}\right), \quad . \quad . \quad . \quad . \quad (16.95)$$

the factor $\frac{3}{8}$ corresponding to $\frac{1}{4}$ for struts.

Again, when the expression for w given in (16.91) is used, the equation corresponding to (16.93) is

$$\frac{P}{a^2}+\frac{Q}{b^2}=E'h^2\left(\frac{14}{a^4}+\frac{14}{b^4}+\frac{8}{a^2b^2}\right) \quad . \quad . \quad . \quad . \quad (16.96)$$

which gives, for a clamped square plate,

$$P+Q=36\frac{E'h^2}{a^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16.97)$$

The difference between the results expressed in (16.93) and (16.96) is so small that we may be sure they are both very near the truth. We may adopt, as a reliable equation for the clamped plate,

$$\frac{P}{a^2}+\frac{Q}{b^2}=E'h^2\left\{\frac{13}{a^4}+\frac{13}{b^4}+\frac{8}{a^2b^2}\right\}. \quad . \quad . \quad . \quad (16.98)$$

293. Finite deflexions.

The buckling thrusts already found in this chapter are the thrusts at which buckling begins. In order to produce deflexions that are not infinitesimal the thrusts have to be increased, and when the maximum deflexion of a plate is of the same order as the thickness the increase in the thrusts is usually of the same order as the buckling thrusts themselves. The reason for this is because, as we have already found in the last chapter, the energy in the plate due to the stretching of the middle surface is usually of the same order as the energy due to bending when the deflexion is the same order as the thickness. In the case of a thin rod also it was found that the thrust which will produce a finite deflexion is greater than the buckling thrust; yet the increase in the thrust does not become appreciable until the maximum deflexion becomes appreciable in comparison with the radius of curvature; the ratio of the deflexion to the thickness of the rod does not affect the thrust at all. Now since it is possible that the deflexion of a thin plate may be much greater than the thickness while the stresses are still not dangerous it follows that there is no buckling thrust for a plate in quite the same sense as there is for a thin rod. For a rod the buckling thrust is usually very near the thrust that causes collapse, whereas the buckling thrust for a plate may be but a small fraction of the thrust that causes collapse.

294. Finite deflexion of clamped disk.

It is worth while to get an approximate solution for one case of buckling with a finite deflexion. The clamped disk is probably the easiest case to deal with.

The problem of finding the deflexion of a plate under the action of boundary forces alone consists in solving the equations

$$\mathrm{E'I}\nabla_1{}^4 w = 2h\mathrm{E}\left\{\frac{\partial^2\varphi}{\partial y^2}\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2\varphi}{\partial x^2}\frac{\partial^2 w}{\partial y^2}-2\frac{\partial^2\varphi}{\partial x\partial y}\frac{\partial^2 w}{\partial x\partial y}\right\} \quad (16.99)$$

$$\nabla_1{}^4\varphi = \left(\frac{\partial^2 w}{\partial x\partial y}\right)^2-\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} \quad (16.100)$$

and adjusting the boundary conditions to suit the data of the problem. If the deflexion w is everywhere infinitesimal compared with h the equations simplify because we can neglect the terms on the right of equation (16.100), since these are of the second order in w. In that case the appropriate solution of this equation for a plate under thrusts at the edges is expressed by

$$\mathrm{E}\frac{\partial^2\varphi}{\partial y^2} = -\mathrm{P},$$

$$\mathrm{E}\frac{\partial^2\varphi}{\partial x^2} = -\mathrm{Q},$$

$$\frac{\partial^2\varphi}{\partial x\partial y} = 0,$$

P and Q being constants over the plate. In the earlier part of this chapter we have assumed these as obvious without any argument. The problem becomes, clearly, much more complicated when the mean stresses are not constant over the plate, and they cannot be constant when the right hand side of (16.100) is not negligible.

For a clamped disk of radius a under a uniform radial thrust at the rim let us assume that

$$w = c(a^2-r^2)^2 \quad (16.101)$$

which, as we know, satisfies the conditions that

$$w = 0,\ \frac{dw}{dr} = 0,$$

at the rim.

Now the equations corresponding to (16.99) and (16.100) for a disk are

$$\mathrm{E'I}\nabla_1{}^4 w = 2h\mathrm{E}\left\{\frac{1}{r}\frac{d\varphi}{dr}\frac{d^2 w}{dr^2}+\frac{1}{r}\frac{dw}{dr}\frac{d^2\varphi}{dr^2}\right\}. \quad (16.102)$$

$$\nabla_1{}^4\varphi = -\frac{1}{r}\frac{dw}{dr}\frac{d^2 w}{dr^2}. \quad (16.103)$$

The second of these can be written

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\nabla_1{}^2\varphi}{dr}\right) = -\frac{1}{r}\frac{dw}{dr}\frac{d^2 w}{dr^2},$$

which gives, after one integration,

$$r\frac{d(\nabla_1{}^2\varphi)}{dr} = -\frac{1}{2}\left(\frac{dw}{dr}\right)^2 . \quad . \quad . \quad . \quad . \quad (16.104)$$

To get the energy equation from (16.102) we multiply by $wr\,dr$ (omitting the factor π) and integrate over the whole area. Thus

$$E'I\int_0^a w\nabla_1{}^4 wr\,dr = 2hE\int_0^a w\frac{d}{dr}\left(\frac{d\varphi}{dr}\frac{dw}{dr}\right)dr,$$

which becomes

$$E'I\int_0^a (\nabla_1{}^2 w)^2 r\,dr - (1-\sigma)E'I\left[\left(\frac{dw}{dr}\right)^2\right]_{r=a}$$

$$= 2hE\left\{\left[w\frac{d\varphi}{dr}\frac{dw}{dr}\right]_0^a - \int_0^a \frac{d\varphi}{dr}\left(\frac{dw}{dr}\right)^2 dr\right\}$$

$$= -2hE\int_0^a \frac{d\varphi}{dr}\left(\frac{dw}{dr}\right)^2 dr, \quad . \quad . \quad . \quad . \quad . \quad (16.105)$$

the integrated term on the right vanishing because $w=0$ at the rim and $\frac{dw}{dr}=0$ at the centre.

Now with the value of w from (16.101) equation (16.104) becomes

$$r\frac{d(\nabla_1{}^2\varphi)}{dr} = -8c^2(a^4r^2 - 2a^2r^4 + r^6).$$

Therefore, integrating again,

$$\nabla_1{}^2\varphi = -8c^2(\tfrac{1}{2}a^4r^2 - \tfrac{1}{2}a^2r^4 + \tfrac{1}{6}r^6) + A,$$

that is,

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\varphi}{dr}\right) = -4c^2(a^4r^2 - a^2r^4 + \tfrac{1}{3}r^6) + A.$$

Integrating again

$$r\frac{d\varphi}{dr} = -c^2(a^4r^4 - \tfrac{2}{3}a^2r^6 + \tfrac{1}{6}r^8) + \tfrac{1}{2}Ar^2, \quad . \quad . \quad . \quad (16.106)$$

no new constant being necessary because both sides vanish where $r=0$.

Now, since $\frac{dw}{dr}=0$ where $r=a$ in this case, equation (16.105) becomes

$$E'I\int_0^a 64c^2(a^2-2r^2)^2 r\,dr$$

$$= -32hEc^2\int_0^a \left\{\tfrac{1}{2}Ar - c^2(a^4r^3 - \tfrac{2}{3}a^2r^5 + \tfrac{1}{6}r^7)\right\}r^2(a^2-r^2)^2 dr,$$

whence

$$\tfrac{32}{3}E'Ia^6 = -32hE\left\{\tfrac{1}{48}Aa^8 - \tfrac{1}{84}c^2a^{14}\right\} \quad . \quad . \quad (16.107)$$

Now the radial stress is

$$P'_1 = \frac{E}{r}\frac{d\varphi}{dr}; \quad . \quad . \quad . \quad . \quad . \quad . \qquad (16.108)$$

consequently, if P is the radial compressive stress at the rim, we find that

$$P = -(P'_1)_{r=a} = E(-\tfrac{1}{2}A + \tfrac{1}{2}c^2a^6) \quad . \quad . \quad . \qquad (16.109)$$

Therefore equation (16.107) gives

$$\tfrac{32}{3}E'I = 2hE\left(\frac{2}{3}\frac{Pa^2}{E} - \tfrac{1}{3}c^2a^8 + \tfrac{4}{21}c^2a^8\right),$$

whence

$$P = 8\frac{E'I}{ha^2} + \tfrac{3}{14}Ec^2a^6$$

$$= \frac{16}{3}\frac{E'h^2}{a^2} + \frac{3}{14}\frac{Ew_0{}^2}{a^2}, \quad . \quad . \quad . \quad . \qquad (16.110)$$

where w_0 indicates the maximum deflexion of the disk.

If the deflexion w_0 is infinitesimal the term involving $w_0{}^2$ is negligible in this last equation, and when this term is omitted the equation agrees with (16.71).

If $w_0 = 4h$, twice the thickness of the plate, then

$$P = \frac{16}{3}\frac{E'h^2}{a^2}\left\{1 + \tfrac{9}{14}(1-\sigma^2)\right\}, \quad . \quad . \quad . \quad . \qquad (16.111)$$

in which case the second term in the bracket is about 0·6 of the first. This means that the thrust at the rim required to produce a maximum deflexion equal to twice the thickness of the plate is about 1·6 of the buckling thrust. A plate whose radius is very much greater than its thickness might be well within the elastic limit if the maximum deflexion were twice the thickness. It follows therefore that the buckling thrust may quite easily be but a small fraction of the thrust that would cause a plate to collapse utterly.

295. The buckling of deep beams.

The following theory applies to beams whose cross-sections have one principal axis parallel to the applied loads. We shall always assume that the loads are vertical so that we can easily distinguish between the two principal axes. The expression *deep* applied to these beams is to be understood to mean that the moment of inertia of the section about the vertical principal axis is very much smaller than the moment of inertia about the horizontal principal axis.

The central axis of the beam in the unstrained state, that is, the line passing through the centres of gravity of the sections, is understood to be straight. Moreover, unless the contrary is distinctly stated, the beams are supposed to have uniform sections.

Let OX be taken through two points of the strained central axis, and let OY be a horizontal axis perpendicular to OX. The horizontal deflexion of a point on the central axis is denoted by y.

When loads are applied to a deep beam the vertical principal axis remains vertical if the loads are not too big. But there are critical systems of loading for which the untwisted state of the beam becomes unstable. In the buckled state the beams are twisted and bent sideways. There is thus no deflexion y until a critical state of loading is reached, whereas there is a vertical deflexion of the central axis for the smallest loads. Nevertheless, owing to the great difference between the two principal moments of inertia, the vertical deflexion will be small even when buckling begins.

Let ODD′B (fig. 157) be a plan of the central line of a buckled beam. The point D has coordinates x, y; and the point D′ has coordinates $x+dx$, $y+dy$. The beam is so twisted in the buckled state that the upper part of the beam is further from the vertical plane containing OX than the central line itself at all points except possibly at the ends of the beam; that is, if the twist of the beam at D is represented by a vector on the right-handed screw system, this vector points towards O when y is positive, and away from O when y is negative.

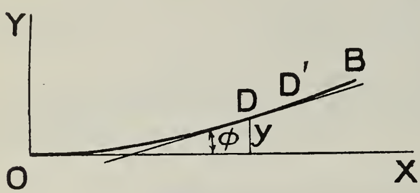


Fig. 157

Let I denote the moment of inertia of the section of the beam about is vertical principal axis, and let K denote the torsion coefficient of the beam. If τ denotes the twist per unit length at any point of the beam the relation between twist and torque Q is

$$Q = Kn\tau,$$

n being the modulus of rigidity. The constant K is calculated for a number of different sections in Chapter VII.

The action across the section at D of one portion of the beam on the other is a couple and a vertical force, the latter being the usual shearing force in beams. The couple can be resolved into three components, M about a vertical line, G in the vertical plane which touches the central line at D, and Q in a plane perpendicular to the other two. The vectors representing Q and G are horizontal vectors, and they are respectively along and perpendicular to the plan of the

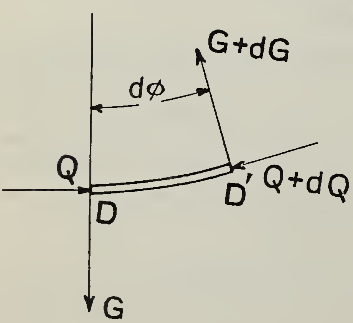


Fig. 158.

central line at D. Fig. 158 shows the plan of the element DD′ of the beam, and also the couples which have horizontal vectors. Taking moments about the vector representing $(Q + dQ)$ for the equilibrium of the element DD′ we get, to first order in $d\varphi$.

$$(Q + dQ) - Q\cos d\varphi + G\sin d\varphi = 0, \quad . \quad . \quad . \quad (16.112)$$

whence

$$dQ + G\,d\varphi = 0,$$

or

$$\frac{dQ}{dx} = -G\frac{d\varphi}{dx}.$$

But

$$\varphi = \tan\varphi = \frac{dy}{dx} \text{ nearly.}$$

Therefore

$$\frac{dQ}{dx} = -G\frac{d^2y}{dx^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (16.113)$$

The moment of the shearing force at D and the load on DD′ were neglected in forming equation (16.112) because these are quantities of smaller order than dQ or $G\,d\varphi$ when the load is applied to the central line of the beam.

The couples M and G can each be resolved into components having vectors along the two principal axes of the twisted section at D; Suppose AA′ is the principal axis which is vertical in the unstrained state. Then the component of G about AA′ is $G\sin\theta$, or approximately $G\theta$, where θ is the inclination of A′A to the vertical. Also the component of M about the same axis is $M\cos\theta$, which is approximately M. Thus the total couple about AA′ is approximately $G\theta + M$.

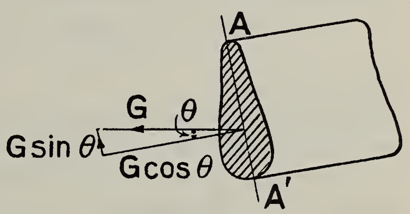


Fig. 159

The component curvature of the beam in the direction perpendicular to the principal axis AA′ is approximately

$$\frac{1}{\varrho} = \frac{d^2y}{dx^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16.114)$$

Since the couple producing this curvature is $(G\theta + M)$ we get, by equation (14.33),

$$E'I\frac{d^2y}{dx^2} = G\theta + M, \quad . \quad . \quad . \quad . \quad . \quad (16.115)$$

where

$$E' = \frac{E}{1 - \sigma^2}.$$

Now since the only forces acting on the beam are vertical forces, we find, by taking moments about a vertical axis for the equilibrium of an element of beam, that M is constant.

Again the component couple about the central axis of the beam at D is approximately Q. This, then, is the torque at D, and the corresponding twist per unit length is

$$\tau = \frac{d\theta}{dx} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16.116)$$

Therefore, from the relation between torque and twist,

$$Q = Kn\tau = Kn\frac{d\theta}{dx} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16.117)$$

We can make our results a little more general by assuming that there is a thrust P applied to the beam along the axis of x. In that case the couple M is not constant, but has the form given by the equation

$$M = N - Py,$$

exactly as for a strut. Then equation (16.115) can be written thus

$$E'I\frac{d^2y}{dx^2} = G\theta + N - Py, \quad . \quad . \quad . \quad . \quad (16.118)$$

N and P being constants.

By means of (16.117) equation (16.113) becomes

$$Kn\frac{d^2\theta}{dx^2} = -G\frac{d^2y}{dx^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16.119)$$

if K is constant.

When the loads on the beam are given G can be found. Then the two last equations determine y and θ as functions of x.

If P is zero y can be eliminated immediately from the last two equations. The eliminant is

$$E'IKn\frac{d^2\theta}{dx^2} = -G(G\theta + N) \quad . \quad . \quad . \quad (16.120)$$

The couple N is the couple applied about a vertical axis at each end of the beam. This couple does not come into action until the beam is buckled, and, moreover, it must remain zero when the beam is buckled unless the ends are clamped so that they cannot rotate about vertical axes. It is, of course, possible for a beam which has displacements in two planes to have totally different end conditions for displacements in the two planes. For example, if one end of the beam is free to rotate about a well-fitting hinge with a horizontal axis, the hinge itself being attached to a rigid body, then at that end the beam is pinned for displacements in the vertical plane, but clamped for displacements in the horizontal plane.

296. Beam of length l under a pair of balancing couples at the ends and a thrust P and no other forces.

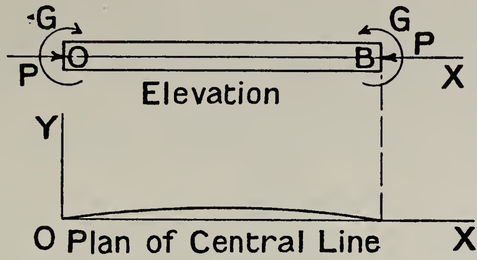


Fig. 160.

It is understood that the ends are held so that θ is zero at both ends. It is very easy to see that no torque is necessary to maintain this state of the ends. Thus the end conditions for θ are

$$\theta = 0 \text{ where } x = 0 \text{ and where } x = l \quad . \quad . \quad . \quad (16.121)$$

Also, the x-axis being taken through the ends of the middle line, the following end conditions hold:—

$$y = 0 \text{ where } x = 0 \text{ and where } x = l. \quad . \quad . \quad (16.122)$$

Now integrating (16.119) twice we get, since G is constant,

$$Kn\theta = -Gy + A_1 x + B_1 \quad . \quad . \quad , \quad . \quad (16.123)$$

In consequence of the conditions in (16.121) and (16.122) it follows that A_1 and B_1 are both zero.
Therefore

$$Kn\theta = -Gy \quad . \quad . \quad . \quad . \quad . \quad . \quad (16.124)$$

On substituting the value of θ given by the last equation in (16.118) we get

$$E'I\frac{d^2y}{dx^2} = -\frac{G^2}{Kn}y + N - Py,$$

whence

$$\frac{d^2y}{dx^2} = -\frac{1}{E'I}\left\{\left(\frac{G^2}{Kn} + P\right)y - N\right\}$$

$$= -m^2y + \frac{N}{E'I}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16.125)$$

where

$$m^2 = \frac{G^2}{E'IKn} + \frac{P}{E'I} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16.126)$$

Since we are assuming that there is no couple in a horizontal plane at either end of the beam the couple N is zero. Consequently

$$\frac{d^2y}{dx^2} = -m^2y \quad . \quad . \quad . \quad . \quad . \quad . \quad (16.127)$$

This equation has exactly the same form as the equation for the deflexion of a strut with pinned ends. Moreover the end conditions for y are exactly the same as for the strut. For a discussion of the solution of the equation we need only refer to Chapter VI where Euler's theory of struts is given. We deduce, by a repetition of the arguments used for the strut, that

$$y = A \sin mx, \quad . \quad . \quad . \quad . \quad . \quad . \quad (16.128)$$

where

$$m = \frac{s\pi}{l}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16.129)$$

s being an integer.

The smallest value of m gives, of course, the smallest buckling force. This smallest value occurs when $s = 1$. In that case

$$\frac{G^2}{Kn} + P = E'Im^2 = E'I\frac{\pi^2}{l^2} \quad . \quad . \quad . \quad . \quad . \quad (16.130)$$

This result includes the result for the strut, since we have only to put G zero to get the strut problem. If, however, P is zero, the last equation gives the couple which will buckle the beam in a plane perpendicular to the one in which the couple acts. This buckling couple is

$$G = \frac{\pi}{l}\sqrt{E'IKn} \quad . \quad . \quad . \quad . \quad . \quad . \quad (16.131)$$

Returning to the more general result in equation (16.130), we notice that P might conceivably be negative. Suppose then that $P = -T$, where T represents a tension. Then equation (16.130) becomes

$$\frac{G^2}{Kn} - T = E'I\frac{\pi^2}{l^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (16.132)$$

We thus see that a thrust and a couple each tend to buckle the beam when acting separately, and that their effects are added when they act together. A tension, however, partly or wholly neutralises the effect of the couple, so that it is possible to avoid buckling even when a big couple is applied provided a big enough tension is applied at the same time. It should be noticed that, since buckling depends on G^2, a change in the direction of G does not alter the effect of G

The couple and thrust in (16.130) are equivalent to a single force P applied at some point not on the axis of x. Thus suppose a pair of thrusts P are applied along the line parallel to the x-axis at a distance p above it; then the force at each end is equivalent to a force P along the x-axis and a couple pP. This action will buckle the beam if

$$\frac{p^2P^2}{Kn} + P = E'I\frac{\pi^2}{l^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (16.133)$$

For a given value of p this last equation has two real roots for P, one positive and one negative, and however small p is there is always a negative root. This shows that a beam can buckle under the action of a pair of opposing pulls applied along any line parallel to the central axis of the beam. Moreover, if p is very small the negative root of (16.133) has a large magnitude and it is approximately the same as would be got by dropping the constant term. Thus the approximate value of the negative root when p is very small is

$$P = -\frac{Kn}{p^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16.134)$$

It is remarkable that this depends only on the torsional rigitidy Kn and not at all on the flexural rigitidy E'I.

From (16.124) and (16.128) we find

$$Q = -mGA\cos mx.$$

When the beam is buckled A is not zero, and consequently Q is not zero at the ends of the beam. Other writers have concluded from this that buckling cannot take place unless couples act about OX at the ends of the beam. But the reader is referred to fig. 158 for the correct meaning of the torque Q; the torque is the component couple *about the tangent to the central line,* not about OX. Since OX is not the tangent to the central line at either end it follows that Q is not zero at either end. The torque at either end is, in fact, the component of the couple G about the tangent to the central line at that end.

Beam clamped at both ends.

If, in addition to the couple G and the thrust P, a couple is applied at each end in a horizontal plane so as to keep $\frac{dy}{dx}$ zero, the new conditions are that N in equation (16.125) is not zero, and

$$\frac{dy}{dx} = 0 \text{ where } x = 0 \text{ and where } x = l. \quad . \quad . \quad (16.135)$$

The conditions in (16.121) and (16.122) still hold, and therefore (16.124) is still true. We have therefore to solve (16.127) with the conditions that y and $\frac{dy}{dx}$ are zero at both ends. But again this is exactly the strut problem with both ends clamped but with $\frac{G^2}{Kn} + P$ instead of P. Therefore, by repeating the reasoning for the clamped strut, we find that the beam buckles when

$$\frac{G^2}{Kn} + P = 4E'I\frac{\pi^2}{l^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (16.136)$$

Again if P is zero this gives

$$G = \frac{2\pi}{l}\sqrt{E'IKn} \quad . \quad . \quad . \quad . \quad . \quad . \quad (16.137)$$

297. Beam of length l fixed at one end and free at the other, and carrying a load R at the free end.

Since the end thrust P is zero we can use equation (16.120) at once. Moreover the couple N is certainly zero because there is no couple in a horizontal plane at the free end. Thus equation (16.120) becomes

$$\mathrm{E'IK}n\frac{d^2\theta}{dx^2} = -\mathrm{G}^2\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16.138)$$

We shall use a short symbol for the oft recurring expression E′IKn. We shall put

$$c^2 = \mathrm{E'IK}n. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16.139)$$

Then when P is zero we have to solve the equation

$$\frac{d^2\theta}{dx^2} = -\frac{\mathrm{G}^2}{c^2}\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16.140)$$

Let the origin be taken at the free end. Then the bending moment at x in a vertical plane is

$$\mathrm{G} = \mathrm{R}x, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16.141)$$

and the sufficient end conditions are

$$\theta = 0 \text{ where } x = l, \quad . \quad . \quad . \quad . \quad . \quad (16.142)$$

$$\mathrm{Q} = 0, \text{ that is, } \frac{d\theta}{dx} = 0, \text{ where } x = 0 \quad . \quad . \quad (16.143)$$

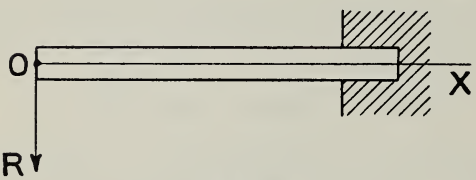


Fig. 161

The differential equation (16.140) now becomes

$$\frac{d^2\theta}{dx^2} = -\frac{\mathrm{R}^2}{c^2}x^2\theta \quad (16.144)$$

Putting

$$s = \frac{1}{2}\frac{\mathrm{R}}{c}x^2 \quad (16.145)$$

we find

$$\frac{d\theta}{dx} = \frac{d\theta}{ds}\cdot\frac{ds}{dx} = \frac{\mathrm{R}}{c}x\frac{d\theta}{ds},$$

$$\frac{d^2\theta}{dx^2} = \frac{\mathrm{R}}{c}\frac{d\theta}{ds} + \frac{\mathrm{R}}{c}x\frac{d}{dx}\left(\frac{d\theta}{ds}\right)$$

$$= \frac{\mathrm{R}}{c}\frac{d\theta}{ds} + \frac{\mathrm{R}^2}{c^2}x^2\frac{d^2\theta}{ds^2}$$

$$= \frac{\mathrm{R}}{c}\left\{\frac{d\theta}{ds} + 2s\frac{d^2\theta}{ds^2}\right\}.$$

Therefore (16.144) becomes

$$2s\frac{d^2\theta}{ds^2}+\frac{d\theta}{ds}=-2s\theta \quad . \quad . \quad . \quad . \quad (16.146)$$

Next putting

$$\theta=zs^{\frac{1}{4}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16.147)$$

we transform (16.146) into

$$\frac{d^2z}{ds^2}+\frac{1}{s}\frac{dz}{ds}+\left(1-\frac{1}{16s^2}\right)z=0 \quad . \quad . \quad . \quad (16.148)$$

Now the equation for Bessel functions* of order n is

$$\frac{d^2z}{ds^2}+\frac{1}{s}\frac{dz}{ds}+\left(1-\frac{n^2}{s^2}\right)z=0 \quad . \quad . \quad . \quad . \quad (16.149)$$

When n is not an integer the complete solution of this is

$$z=AJ_n(s)+BJ_{-n}(s), \quad . \quad . \quad . \quad . \quad . \quad (16.150)$$

the functions $J_n(s)$ and $J_{-n}(s)$ being Bessel functions of order n and $-n$ respectively. The function $J_n(s)$ is defined by means of an infinite series thus

$$J_n(s)=\frac{s^n}{2^n\Gamma(n+1)}\left\{1-\frac{s^2}{2^2(n+1)}+\frac{s^4}{2^4\lfloor 2\,(n+1)(n+2)}-\right\} (16.151)$$

The quantity $\Gamma(n+1)$ in the denominator is the gamma function. The values of log $\Gamma(x)$ are tabulated in Williamson's *Integral Calculus.*

In many problems the factor $2^n\Gamma(n+1)$ is unimportant; it is merged into the arbitrary constant.

Equation (16.148) is identical with (16.149) if $n=\frac{1}{4}$. Therefore the solution of (16.148) is

$$z=AJ_{\frac{1}{4}}(s)+BJ_{-\frac{1}{4}}(s).$$

Consequently

$$\theta=s^{\frac{1}{4}}\left\{AJ_{\frac{1}{4}}(s)+BJ_{-\frac{1}{4}}(s)\right\}$$

$$=x^{\frac{1}{2}}\left\{A_1J_{\frac{1}{4}}(s)+B_1J_{-\frac{1}{4}}(s)\right\} \quad . \quad . \quad . \quad (16.152)$$

It is worth while to get this result directly from equation (16.144) without making use of Bessel's equation. Indeed (16.144) is, in many respects, a simpler form of equation than Bessel's equation. Let us put

$$m^4=\frac{R^2}{c^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16.153)$$

To solve

$$\frac{d^2\theta}{dx^2}+m^4x^2\theta=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (16.154)$$

* See Byerley's *Fourier Series and Spherical Harmonics* (Ginn and Co); or Watson's *Theory of Bessel Functions* (Cambridge University Press). See also Appendix A.

put

$$\theta = \Sigma a_r x^r \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16.155)$$

Then

$$\frac{d^2\theta}{dx^2} = \Sigma r(r-1)a_r x^{r-2}$$

Therefore our differential equation gives

$$\Sigma\{r(r-1)a_r x^{r-2} + m^4 a_r x^{r+2}\} = 0, \quad . \quad . \quad . \quad (16.156)$$

which is equivalent to

$$\Sigma\{r(r-1)a_r x^{r-2} + m^4 a_{r-4} x^{r-2}\} = 0 \quad . \quad . \quad . \quad (16.157)$$

Since this is an identity the coefficient of x^{r-2} must be zero. Therefore

$$r(r-1)a_r + m^4 a_{r-4} = 0, \quad . \quad . \quad . \quad . \quad . \quad (16.158)$$

whence, if $r(r-1)$ is not zero,

$$a_r = -\frac{m^4}{r(r-1)} a_{r-4} \quad . \quad . \quad . \quad . \quad . \quad (16.159)$$

If, however, $r = 0$ or $r = 1$, then a_{r-4} is zero. It follows therefore from (16.158) that a_{-4} and a_{-3} are zero. Next by putting $r = -4$ or $r = -3$ it follows from the same equation that a_{-8} and a_{-7} are also zero. It is clear then that ascending series may begin at $r = 0$ or $r = 1$. The coefficients in the series beginning with $r = 0$ are a_0, a_4, a_8, etc. The relation between these coefficients is given by (16.159). Thus

$$a_4 = -\frac{m^4}{3 \cdot 4} a_0,$$

$$a_8 = -\frac{m^4}{7 \cdot 8} a_4 = +\frac{m^8}{3 \cdot 4 \cdot 7 \cdot 8} a_0.$$

Therefore one series is

$$a_0 \left\{ 1 - \frac{m^4 x^4}{3 \cdot 4} + \frac{m^8 x^8}{3 \cdot 4 \cdot 7 \cdot 8} - \frac{m^{12} x^{12}}{3 \cdot 4 \cdot 7 \cdot 8 \cdot 11 \cdot 12} + \right\}$$

Likewise the series beginning with $r = 1$ is

$$a_1 \left\{ x - \frac{m^4 x^5}{4 \cdot 5} + \frac{m^8 x^9}{4 \cdot 5 \cdot 8 \cdot 9} - \ldots \right\}$$

Therefore

$$\theta = a_0 \left\{ 1 - \frac{m^4 x^4}{3 \cdot 4} + \frac{m^8 x^8}{3 \cdot 4 \cdot 7 \cdot 8} - \ldots \right\}$$
$$+ a_1 \left\{ x - \frac{m^4 x^5}{4 \cdot 5} + \frac{m^8 x^9}{4 \cdot 5 \cdot 8 \cdot 9} - \ldots \right\} \quad . \quad . \quad . \quad . \quad (16.160)$$

This last equation is identical with (16.152)

Now the condition in (16.143) gives

$$a_1 = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16.161)$$

and the condition in (16.142) next gives

$$0 = 1 - \frac{m^4 l^4}{3.4} + \frac{m^8 l^8}{3.4.7.8} - \ldots \quad . \quad . \quad . \quad (16.162)$$

This last equation determines $m^4 l^4$, which in turn determines R. The equation for $m^4 l^4$ has an infinite number of roots corresponding to the infinite number of possible forms into which the beam can buckle. The smallest root is the only one that matters in the buckling problem.

When u is written for $m^4 l^4$ equation (16.162) becomes

$$0 = 1 - \frac{u}{3.4} + \frac{u^2}{3.4.7.8} - \frac{u^3}{3.4.7.8.11.12} + \ldots \quad . \quad . \quad (16.163)$$

Now if u is positive and not greater than 12 the series for u can be written

$$\left(1 - \frac{u}{12}\right) + \frac{u^2}{3.4.7.8}\left(1 - \frac{u}{11.12}\right) + \ldots$$

which is clearly positive. Therefore the smallest root of (16.163) must be greater than 12. Again when $u = 20$ the series is

$$1 - \frac{20}{12} + \frac{20^2}{3.4.7.8}\left\{1 - \frac{20}{11.12} + \frac{20^2}{11.12.15.16} - \right\}$$

$$= -\frac{2}{3} + \frac{25}{42}\left\{1 - \frac{20}{11.12} + \frac{20^2}{11.12.15.16} - \right\}$$

$$< -\frac{2}{3} + \frac{25}{42},$$

which is negative. Thus the series changes sign somewhere between $u = 12$ and $u = 20$. Now for this range of values of u the term containing the first power of u is the greatest term in the series. Then let us write equation (16.163) in the form

$$u = 12 + \frac{u^2}{7.8} - \frac{u^3}{7.8.11.12} + \frac{u^4}{7.8.11.12.15.16} - \quad . \quad . \quad . \quad (16.164)$$

We may try $u = 16$. Then the right hand side

$$= 12 + \frac{32}{7} - \frac{128}{7.3.11} + \frac{128}{7.3.11.15} -$$

$$= 12 + 4·5714 - 0·5541 + 0·0369 - 0·00155 + 0·00005$$

$$= 16·053\,.$$

This is nearly identical with u, and it is very likely to be a better value than the assumed value 16. Then let us next try $u = 16·1$. Then the right hand side of (16.164).

$$= 12 + 4·62874 - 0·56457 + 0·03787 - 0·00150 + 0·00004$$

$$= 16·1006.$$

It is clear then that 16·1 is very near the true value of the root. Newton's method, which is illustrated below, may be used to get a closer approximation to the root of such a series.

Let $u_1 = 16{\cdot}1$, the value of u that has been used in calculating the terms in the last approximation. Also let

$$f(u) = 12 - u + \frac{u^2}{7.8} - \frac{u^3}{7.8.11.12} + \dots ;$$

and let the exact root of equation (16.163) be $(u_1 + \alpha)$. Then,

$$f(u_1 + \alpha) = 0, \quad \dots\dots \quad (16.165)$$

whence, neglecting higher powers of α than the first, we get

$$f(u_1) + \alpha f'(u_1) = 0 \quad \dots\dots \quad (16.166)$$

Therefore

$$\alpha = -\frac{f(u_1)}{f'(u_1)} = -\frac{u_1 f(u_1)}{u_1 f'(u_1)}. \quad \dots\dots \quad (16.167)$$

The second form for α is much more convenient for calculation than the first.

Now $$f(u_1) = 16{\cdot}1006 - 16{\cdot}1000$$
$$= 0{\cdot}0006$$

Also

$$u_1 f'(u_1) = -u_1 + \frac{2u_1{}^2}{7.8} - \frac{3u_1{}^3}{7.8.11.12} + \dots$$

The terms in this series are quickly obtained from the terms in $f(u_1)$ since the only labour necessary is to multiply the terms in the latter series by 1, 2, 3, etc., respectively. In this way we find

$$u_1 f'(u_1) = -8{\cdot}39.$$

Therefore

$$\alpha = \frac{0{\cdot}0006 \times 16{\cdot}1}{-8{\cdot}39} = -0{\cdot}00116, \quad \dots \quad (16.168)$$

and consequently

$$u = u_1 + \alpha = 16{\cdot}09884 \quad \dots\dots \quad (16.169)$$

This determines the smallest value of G to a sufficiently good approximation. Thus

$$Rl^2 = cm^2 l^2 = c\sqrt{u}$$
$$= 4{\cdot}01235\sqrt{E'IKn} \quad \dots\dots \quad (16.170)$$

The root of equation (16.163) could have been found in another way. This equation is equivalent to

$$J_{-\frac{1}{4}}\left(\tfrac{1}{2}u^{\frac{1}{2}}\right) = 0$$

Stokes' formula for the first root of this equation gives

$$\tfrac{1}{2}u^{\frac{1}{2}} = 2\cdot006, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16.171)$$

which agrees with (16.170).

298. A beam of length l carries a load R at the middle and is supported at the ends, the only couples at the ends being such torsional couples as will keep the ends from twisting.

The ends are to be regarded as pinned both for horizontal and vertical displacements. Also the load R must be understood to be applied at the middle point of the section half way between the ends.

Let x be measured from one end of the beam towards the other end. Then, at a point between the origin and the load R, the couple G is the moment of the supporting force $\frac{1}{2}$R at the origin. Thus

$$G = -\tfrac{1}{2}Rx$$

and therefore equation (16.120) becomes, since N is zero in this case,

$$\frac{d^2\theta}{dx^2} = -\frac{R^2x^2}{4c^2}\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16.172)$$

If we now write

$$m^4 = \frac{R^2}{4c^2} = \frac{R^2}{4E'IKn} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16.173)$$

this last differential equation is the same as (16.154). The solution of the equation is therefore given by (16.160). Since there is a point of discontinuity at the middle of the beam it is best, if possible, to find two conditions in the half of the beam to which our solution applies by means of which the constants a_0 and a_1 can be determined. Two such conditions are

$$\theta = 0 \text{ where } x = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16.174)$$

$$\frac{d\theta}{dx} = 0 \text{ where } x = \tfrac{1}{2}l \quad . \quad . \quad . \quad . \quad . \quad . \quad (16.175)$$

The second of these conditions follows from the obvious fact that θ has either a maximum or a minimum value at the middle of the beam.

The condition (16.174) applied to (16.160) gives

$$a_0 = 0.$$

Then condition (16.175) gives

$$1 - \frac{u}{4} + \frac{u^2}{4.5.8} - \frac{u^3}{4.5.8.9.12} + \ldots = 0, \quad . \quad . \quad (16.176)$$

where

$$u = \frac{m^4 l^4}{10}.$$

The least root of (16.176) is

$$u = 4{\cdot}482 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16.177a)$$

We therefore find

$$R^2 l^4 = 4m^4 l^4 c^2 = 64 uc^2,$$

whence

$$R l^2 = 16{\cdot}94\sqrt{E'IKn} \quad . \quad . \quad . \quad . \quad . \quad (16{\cdot}177b)$$

299. The boundary conditions for a clamped beam.

When there is no end thrust on a beam the differential equations for θ and y are (16.120) and (16.119), which are re-written here:

$$E'IKn\frac{d^2\theta}{dx^2} = -\,G(G\theta + N), \quad . \quad . \quad . \quad . \quad (16.178)$$

$$Kn\,\frac{d^2\theta}{dx^2} = -\,G\frac{d^2y}{dx^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16.179)$$

Now the constant N in the first of these equations is an unknown constant of the nature of a constant of integration. When, therefore, θ is found as a function of x from (16 178), the expression contains three unknown constants, and we have usually only two boundary conditions involving θ alone. These are not sufficient to determine the ratios of the constants and the buckling load involved in G. We are thus obliged to use some of the boundary conditions foı y Now one more condition is all we need, and since $\frac{dy}{dx}$ is zero at both ends of the beam this condition can be expressed in the form

$$\int_0^l \frac{d^2y}{dx^2}\,dx = 0\,;$$

that is,

$$\int_0^l \frac{Kn}{G}\frac{d^2\theta}{dx^2}\,dx = 0. \quad . \quad . \quad . \quad (16.180)$$

This last condition introduces no new constant; it theref ·e supplies the extra condition that was needed to determine the bu kling load without finding y or $\frac{dy}{dx}$.

The limits for the integral in (16.180) are from one end to the other of the beam, and it is obviously not necessary that the origin should be at one end, although the limits given in the integral are suitable for that case.

If the beam is loaded symmetrically about the middle we are entitled to assume that $\frac{dy}{dx} = 0$ at the middle as well as at the ends. In that case we may take the range of integration in (16.180) between one end and the middle of the beam. Thus, whether the origin is at the middle or at one end, the extra condition for this case is

$$\int_0^{\frac{1}{2}l} \frac{Kn}{G} \frac{d^2\theta}{dx^2} dx = 0 \quad . \quad . \quad . \quad . \quad . \quad (16.181)$$

If Kn is constant this factor may, of course, be omitted

300. A beam of length l, supported at the ends, carries a concentrated load R at the middle point and the ends are clamped for horizontal displacements.

This case differs from the last in that the couple N in equation (16.120) is not zero, and $\frac{dy}{dx}$ is zero at each end. The couple G has exactly the some value as in the last problem.

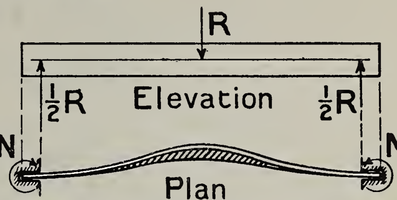


Fig. 162

With the origin at one end equation (16.120) becomes, for the present problem,

$$c^2 \frac{d^2\theta}{dx^2} = \tfrac{1}{2} Rx (N - \tfrac{1}{2} Rx\theta),$$

whence

$$\frac{d^2\theta}{dx^2} = -m^4x^2\theta + 6bm^3x, \quad . \quad . \quad . \quad (16.182)$$

where

$$m^4 = \frac{R^2}{4c^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16.183)$$

$$mb = \frac{RN}{12cm^2} = \frac{N}{6c} \quad . \quad . \quad . \quad . \quad . \quad . \quad (16.184)$$

The solution of (16.182) in a series is

$$\theta = a_0 \left\{ 1 - \frac{m^4x^4}{3 \cdot 4} + \frac{m^8x^8}{3 \cdot 4 \cdot 7 \cdot 8} - \dots \right\}$$
$$+ a_1 \left\{ x - \frac{m^4x^5}{4 \cdot 5} + \frac{m^8x^9}{4 \cdot 5 \cdot 8 \cdot 9} - \dots \right\}$$
$$+ bm^3 \left\{ x^3 - \frac{m^4x^7}{6 \cdot 7} + \frac{m^8x^{11}}{6 \cdot 7 \cdot 10 \cdot 11} - \dots \right\} \quad (16.185)$$

The conditions which must be satisfied at the ends of the half beam to which the solution applies are

$$\theta = 0 \text{ where } x = 0, \quad . \quad . \quad . \quad . \quad . \quad (16.186)$$

and

$$\frac{d\theta}{dx} = 0 \text{ where } x = \tfrac{1}{2}l. \quad . \quad . \quad . \quad . \quad . \quad (16.187)$$

We have, in addition, the condition given in (16.181) namely

$$\int_0^{\frac{1}{2}l} \frac{1}{G}\frac{d^2\theta}{dx^2} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16.188)$$

To satisfy the first of these conditions a_0 must be zero. Also, by means of (16.120), the last condition can be put into the form

$$0 = \int_0^{\frac{1}{2}l} (G\theta + N)\,dx$$

$$= \int_0^{\frac{1}{2}l} (N - \tfrac{1}{2}Rx\theta)\,dx$$

$$= \int_0^{\frac{1}{2}l} (6mbc - m^2cx\theta)\,dx$$

$$= c\int_0^{\frac{1}{2}l} (6b - mx\theta)\,m\,dx. \quad . \quad . \quad . \quad . \quad . \quad (16.189)$$

Now let u be written for $\frac{1}{16}m^4l^4$, and let

$$X = 1 - \frac{u}{4} + \frac{u^2}{4.5.8} - \frac{u^3}{4.5.8.9.12} + \quad . \quad . \quad . \quad . \quad . \quad (16.190)$$

$$Y = 3 - \frac{u}{6} + \frac{u^2}{6.7.10} - \frac{u^3}{6.7.10.11.14} + \quad . \quad . \quad . \quad . \quad (16.191)$$

$$U = \frac{u}{3} - \frac{u^2}{4.5.7} + \frac{u^3}{4.5.8.9.11} - \frac{u^4}{4.5.8.9.12.13.15} - . \quad (16.192)$$

$$V = 6 - \frac{u}{5} + \frac{u^2}{6.7.9} - \frac{u^3}{6.7.10.11.13} + \quad . \quad . \quad . \quad . \quad (16.193)$$

Then the condition (16.187) gives

$$a_1X + \tfrac{1}{4}bm^3l^2Y = 0;$$

and the condition (16.189) gives

$$-\frac{2a_1}{m^2l}U + \tfrac{1}{2}bmlV = 0.$$

The elimination of the ratio $a_1 : b$ from the last two equations leads to the equation

$$XV + YU = 0 \quad . \quad . \quad . \quad . \quad (16.194)$$

The smallest root of this equation can be found by calculating values of the functions X, Y, V, U, for different values of u. The arithmetic involved is rather laborious. It will be found that this smallest root is approximately

$$u = 10{\cdot}47 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16.195)$$

Therefore the smallest buckling load is R given by

$$Rl^2 = 2m^2l^2\sqrt{E'IKn} = 8\sqrt{10{\cdot}47}\sqrt{E'IKn}$$

$$= 25{\cdot}89\sqrt{E'IKn}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16.196)$$

301. A beam fixed at one end and free at the other carries a load W uniformly distributed along its length.

Let w be the load per unit length, so that $wl = \mathrm{W}$. The origin is taken at the free end and x measured towards the fixed end. Therefore

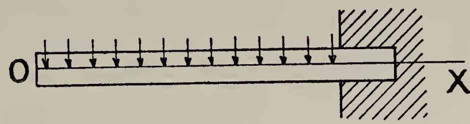


Fig. 163

$$\mathrm{G} = \tfrac{1}{2} w x^2.$$

Then the differential equation (16.120) becomes

$$\mathrm{E'IK}n \frac{d^2\theta}{dx^2} = -\tfrac{1}{4} w^2 x^4 \theta,$$

whence

$$\frac{d^2\theta}{dx^2} = -m^6 x^4 \theta, \quad \ldots \quad (16.197)$$

where

$$m^6 = \frac{w^2}{4\mathrm{E'IK}n}. \quad \ldots \quad (16.198)$$

The end conditions are that the torque is zero at $x = 0$ and θ is zero at the other end; that is,

$$\frac{d\theta}{dx} = 0 \text{ where } x = 0, \quad \ldots \quad (16.199)$$

$$\theta = 0 \text{ where } x = l \quad \ldots \quad (16.200)$$

The solution of equation (16.197) is

$$\theta = \mathrm{A}\left\{1 - \frac{m^6 x^6}{5.6} + \frac{m^{12} x^{12}}{5.6.11.12} - \ldots\right\}$$
$$+ \mathrm{B}x\left\{1 - \frac{m^6 x^6}{6.7} + \frac{m^{12} x^{12}}{6.7.12.13} - \ldots\right\}. \quad (16.201)$$

Condition (16.199) makes $\mathrm{B} = 0$. Then the other condition gives

$$0 = 1 - \frac{m^6 l^6}{5.6} + \frac{m^{12} l^{12}}{5.6.11.12} - \frac{m^{18} l^{18}}{5.6.11.12.17.18} + \ldots \quad (16.202)$$

The smallest root of this is

$$m^6 l^6 = 41{\cdot}30, \quad \ldots \quad (16.203)$$

whence

$$\frac{\mathrm{W}^2 l^4}{4\mathrm{E'IK}n} = 41{\cdot}30,$$

and therefore

$$\mathrm{W}l^2 = 12{\cdot}85\sqrt{\mathrm{E'IK}n}. \quad \ldots \quad (16.204)$$

302. A beam supported, without clamping, at both ends, carries a load W distributed uniformly along its length.

If we take the origin at one end the bending moment is

$$G = -\tfrac{1}{2} wx(l-x). \quad \ldots\ldots \quad (16.205)$$

Therefore the differential equation for θ is

$$E'IKn \frac{d^2\theta}{dx^2} = \tfrac{1}{4} w^2 x^2 (l-x)^2 \theta. \quad \ldots\ldots \quad (16.206)$$

Now it is more convenient to take the origin at the middle of the rod. Then putting

$$x_1 = x - \tfrac{1}{2} l, \quad \ldots\ldots \quad (16.207)$$

we get

$$E'IKn \frac{d^2\theta}{dx_1^{\,2}} = -\tfrac{1}{4} w^2 (x_1^{\,2} - \tfrac{1}{4} l^2)^2 \theta \quad \ldots \quad (16.208)$$

For convenience we shall use a dimensionless variable instead of x_1. Thus we put

$$x_1 = \tfrac{1}{2} ls, \quad \ldots\ldots \quad (16.209)$$

and then our differential equation becomes

$$\frac{d^2\theta}{ds^2} = -m^2 (s^2 - 1)^2 \theta, \quad \ldots\ldots \quad (16.210)$$

where

$$m^2 = \frac{W^2 l^4}{16^2 E'IKn}. \quad \ldots\ldots \quad (16.211)$$

Now there is a solution of the type

$$\theta = A\{1 + a_2 s^2 + a_4 s^4 + a_6 s^6 + \ldots\}, \quad \ldots \quad (16.212)$$

and this satisfies the condition

$$\frac{d\theta}{dx} = 0 \text{ where } x = 0 \quad \ldots\ldots \quad (16.213)$$

We have only to satisfy the additional conditions

$$\theta = 0 \text{ where } x = \pm \tfrac{1}{2} l,$$

that is,

$$\theta = 0 \text{ where } s = \pm 1. \quad \ldots\ldots \quad (16.214)$$

On substituting the assumed series for θ in equation (16.210) and equating the coefficients of the several powers of s we get

$$\begin{aligned} 2a_2 &= -m^2 \\ 4 \cdot 3 a_4 &= -m^2(a_2 - 2) = \tfrac{1}{2} m^2 (m^2 + 4) \\ 6 \cdot 5 a_6 &= -m^2 (a_4 - 2a_2 + 1) \\ &= -\frac{1}{\lfloor 4} m^2 (m^4 + 28 m^2 + 24). \end{aligned}$$

Thus all the coefficients a_2, a_4, a_6, etc., can be expressed in terms of m^2, which is a mere number. Then condition (16.214) gives

$$0 = 1 + a_2 + a_4 + a_6 + . \quad \ldots\ldots \quad (16.215)$$

If we write

$$f(m^2) = 1 + a_2 + a_4 + a_6 + \quad . \quad . \quad . \quad . \quad (16.216)$$

we can calculate $f(1)$, $f(2)$, $f(3)$, etc., and plot the curve $z = f(x)$ An approximate value of m^2 can be found from the curve. The smallest root is very near 3, and by successive approximations it is found to be

$$m^2 = 3 \cdot 131. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16.217)$$

Therefore

$$Wl^2 = 16\sqrt{3 \cdot 131}\sqrt{E'IKn}$$
$$= 28 \cdot 31\sqrt{E'IKn}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (16.218)$$

303. A beam free at one end and fixed at the other carries a uniformly distributed load W and a much smaller load R at the free end.

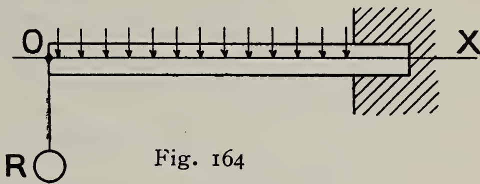


Fig. 164

Writing wl for W and taking the origin at the free end we get

$$G = \tfrac{1}{2}wx^2 + Rx$$
$$= \tfrac{1}{2}w\left\{\left(x + \frac{R}{w}\right)^2 - \frac{R^2}{w^2}\right\}. \quad . \quad . \quad . \quad . \quad (16.219)$$

Now we are making the assumption that R is small in comparison with wl. It follows then that $\frac{R^2}{w^2}$ is small in comparison with l^2, and therefore in comparison with $\left(x + \frac{R}{w}\right)^2$ except in the region near the end $x = 0$, and this is the region where G is itself small. Without serious error therefore we may take

$$G = \tfrac{1}{2}w\left(x + \frac{R}{w}\right)^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (16.220)$$

The differential equation for θ is consequently

$$E'IKn\frac{d^2\theta}{dx^2} = -\tfrac{1}{4}w^2\left(x + \frac{R}{w}\right)^4\theta.$$

In this put

$$z = x + \frac{R}{w}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16.221)$$

then

$$E'IKn\frac{d^2\theta}{dz^2} = -\tfrac{1}{4}w^2z^4\,\theta,$$

whence

$$\frac{d^2\theta}{dz^2} = -m^6z^4\,\theta, \quad . \quad . \quad . \quad . \quad . \quad . \quad (16.222)$$

where

$$m^6 = \frac{w^2}{4E'IKn}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16.223)$$

The end conditions are

$$\frac{d\theta}{dx} = 0 \text{ where } x = 0, \quad . \quad . \quad . \quad . \quad (16.224)$$

$$\theta = 0 \text{ where } x = l. \quad . \quad . \quad . \quad . \quad . \quad . \quad (16.225)$$

Now the solution of (16.222) is given in equation (16.201). In the present case

$$\theta = A\left\{1 - \frac{m^6 z^6}{5.6} + \frac{m^{12} z^{12}}{5.6.11.12} - \ldots\right\}$$
$$+ Bz\left\{1 - \frac{m^6 z^6}{6.7} + \frac{m^{12} z^{12}}{6.7.12.13} - \ldots\right\}. \quad . \quad . \quad (16.226)$$

The boundary condition at the free end is

$$\frac{d\theta}{dz} = 0 \text{ where } z = \frac{R}{w}.$$

Since we are neglecting all except the first power of $\frac{R}{w}$ in our equations this makes

$$B = 0.$$

The second condition (16.225) leads to

$$0 = \left\{1 - \frac{m^6 z_1{}^6}{5.6} + \frac{m^{12} z_1{}^{12}}{5.6.11.12} - \ldots\right\} \quad . \quad . \quad . \quad (16{\cdot}227)$$

where

$$z_1 = l + \frac{R}{w}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16.228)$$

Thus the equation for $m^6 z_1{}^6$ is the same as the equation for $m^6 l^6$ in (16.202). Therefore, by (16.203),

$$m^6\left(l + \frac{R}{w}\right)^6 = 41{\cdot}30, \quad . \quad . \quad . \quad . \quad . \quad . \quad (16{\cdot}229)$$

whence

$$m^3\left(l + \frac{R}{w}\right)^3 = \sqrt{41{\cdot}30} \quad . \quad . \quad . \quad . \quad . \quad (16.230)$$

or, to the first power of $\frac{R}{w}$,

$$m^3 l^3\left(1 + \frac{3R}{wl}\right) = \sqrt{41{\cdot}30}.$$

Therefore

$$Wl^2\left(1 + \frac{3R}{W}\right) = 2\sqrt{41{\cdot}30}\sqrt{E'IKn};$$

that is,

$$Wl^2 + 3Rl^2 = 12{\cdot}86\sqrt{E'IKn}, \quad . \quad . \quad . \quad (16.231)$$

This last equation is correct for values of R that are much smaller than W; we cannot, of course, rely on the equation for other ratios of R to W. Nevertheless, if we actually put W = 0 in this last equation we find

$$Rl^2 = 4{\cdot}29\sqrt{E'IKn},$$

which differs by only 7 % from the result in (16.170) which was calculated on the assumption that W was zero. It seems likely therefore that (16.231) cannot be far from the truth even when R is not small in comparison with W. The equation

$$\frac{Wl^2}{12{\cdot}86} + \frac{Rl^2}{4{\cdot}012} = \sqrt{E'IKn}, \quad . \quad . \quad . \quad . \quad (16.232)$$

which is correct when R = 0, and correct when W = 0, and is nearly correct when $\frac{R}{W}$ is small, may safely be used when R bears any ratio to W. In fact, by solving this problem again with the contrary assumption that W is small compared with R we find

$$Rl^2 + 0{\cdot}291\,Wl^2 = 4{\cdot}012\sqrt{E'IKn},$$

which is equivalent to

$$\frac{Wl^2}{13{\cdot}8} + \frac{Rl^2}{4{\cdot}012} = \sqrt{E'IKn}. \quad . \quad . \quad . \quad . \quad . \quad (16.233)$$

This differs from (16.232) by very little, and that only in the small term. We thus get another confirmation of (16.232) as an approximate formula.

304. A beam, supported without clamping at the ends, carries a load R at the middle in addition to a much smaller load W distributed uniformly along the length.

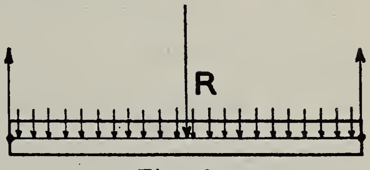


Fig 165.

This can be solved by first neglecting W—the solution for which case is given in (16.177)—and then adding a correction to θ for the extra term due to W in the bending moment G. This problem is solved in a paper in the *Philosophical Magazine* for February 1920.* The solution there found is

$$\frac{Rl^2}{16{\cdot}94} + \frac{Wl^2}{28{\cdot}6} = \sqrt{E'IKn}. \quad . \quad . \quad . \quad . \quad (16.234)$$

* *The Buckling of Deep Beams* by J. Prescott; Phil. Mag. Vol. XXXIX, February 1920.

Now the equation

$$\frac{Rl^2}{16{\cdot}94}+\frac{Wl^2}{28{\cdot}31}=\sqrt{E'IKn}\,. \quad . \quad . \quad . \quad . \quad (16.235)$$

is correct when R is zero and when W is zero, and it differs so little from (16.234) that it is very likely to be nearly correct for all values of the ratio R : W. It is therefore safe to use (16.235) as a good approximation in all cases.

305. Load not applied at the centre of the beam.

So far we have assumed that the loads acting on buckled beams were applied at the centre of gravity of the section in every case. It was on this assumption that equation (16.112) was obtained. Let us now suppose that the load wdx acting on the element dx is applied at a point fixed relatively to the beam, this point having coordinates x, p, q, before the beam is strained; p is measured in the direction of y, q is measured vertically upwards, that is, in the direction contrary to the load, and the x-axis passes through the centres of gravity of the sections of the unstrained beam. The infinitesimal element DD′ in fig. 157 is shown in fig. 166, the observer being supposed to look in the direction from D′ to D.

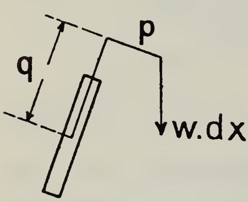


Fig. 166

This load wdx has a moment of amount $wdx\,(p+q\theta)$ about the same axis as $(Q+dQ)$ in fig. 158. This quantity should therefore be added to the left hand side of (16.112). This would give, instead of (16.113),

$$\frac{dQ}{dx}=-G\frac{d^2y}{dx^2}-w(p+q\theta) \quad (16.236)$$

Consequently (16.119) is changed to

$$Kn\frac{d^2\theta}{dx^2}=-G\frac{d^2y}{dx^2}-w(p+q\theta), \quad . \quad . \quad . \quad . \quad . \quad (16.237)$$

and when P is zero (16.120) is changed to

$$E'IKn\frac{d^2\theta}{dx^2}=-G(G\theta+N)-E'Iw(p+q\theta). \quad . \quad . \quad (16.238)$$

This last equation, together with the boundary conditions, determines the buckling load. In general p is a given function of x, and w must be a given function oi x, except that it may possibly involve a constant factor the magnitude of which we are seeking to determine.

Now the solution of (16.238) has the form

$$\theta=Af_1(x)+Bf_2(x)+Nf_3(x)+\varphi(x) \quad . \quad . \quad . \quad (16.239)$$

where $\varphi(x)$ is the particular integral corresponding to $-E'Iwp$, A and B are arbitrary constants, and N is also, in effect, an arbitrary constant as we pointed out in Art. 299. For a beam which is not clampee at the ends N is zero, and therefore the number of arbitrary constants reduces to two, the same as the number of boundary conditions. When N is not zero there are three boundary conditions. Whether there are two constants or three they can all be expressed in terms of the values f_1, f_2, f_3, φ, and their differential coefficients (of first or second order) at both ends of the beam. That is, the two boundary conditions (or three if N is not zero) determine definite values of the arbitrary constants, giving a definite value of θ. Thus the beam is in equilibrium in a strained but not unstable state, *unless it happens that the arbitrary constants have infinite values.* Infinite values of the constants give an infinite value of θ which is the analytical way in which instability shows itself. Now the vanishing of a single expression involving f_1, f_2, f_3, and their differential coefficients at the ends of the beam is enough to make the constants all infinite. The equation expressing the condition that the arbitrary constants are all infinite is exactly the same equation as we should get to express the condition that the constants are not zero when $\varphi(x)$ is zero for all values of x. That is, the condition for instability is just the same whether $\varphi(x)$ (and therefore also p) is zero or not. To show this in a simple case let us suppose that the beam is not clamped, in which case N is zero. Let the end-conditions be that $\theta = 0$ at $x = 0$ and $x = l$. These conditions give

$$\left.\begin{aligned} A f_1(0) + B f_2(0) + \varphi(0) = 0 \\ A f_1(l) + B f_2(l) + \varphi(l) = 0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (16.240)$$

From these we get

$$A = \frac{\varphi(l) f_2(0) - \varphi(0) f_2(l)}{f_1(0) f_2(l) - f_1(l) f_2(0)} \quad . \quad . \quad . \quad . \quad (16.241)$$

$$B = -\frac{\varphi(l) f_1'(0) - \varphi(0) f_1(l)}{f_1(0) f_2(l) - f_1(l) f_2(0)} \quad . \quad . \quad . \quad . \quad (16.242)$$

Both A and B are infinite if

$$f_1(0) f_2(l) - f_1(l) f_2(0) = 0 \quad . \quad . \quad . \quad . \quad . \quad (16.243)$$

If we had put $\varphi(0) = 0$ and $\varphi(l) = 0$ in equations (16.240) and then eliminated the ratio A : B from the two equations, thus assuming that A and B are not zero, we should have arrived at the same equation (16.243), and this is the equation which determines the buckling load. We thus find that the condition for instability is independent of the function $\varphi(x)$ in this case, and a similar argument can be applied to any other case. It therefore follows that the coordinate p does not affect stability.

Although the theoretical buckling load is independent of p it is possible that A and B may be so big when the load is nearly, but not quite, equal to the buckling load that the beam has greater strains than it should reasonably stand. That is, if p is not zero, there is a range of values of the load, extending from a little below to a little above the buckling load, for which the beam may be regarded as unstable. When p is zero there is just one critical smallest load, and θ and y will remain absolutely zero for all loads smaller than this.

306. A beam of length l, fixed at $x=l$ and free at $x=0$, carries a load W uniformly distributed along its length at a constant height q above the centres of the sections.

This is the problem that was solved in Art. 301 with the modification that the load is applied at height q above the x-axis.

If w denotes the load per unit length equation (16.238) for this problem becomes

$$\mathrm{E'IK}n\frac{d^2\theta}{dx^2}=-(\tfrac{1}{4}w^2x^4+\mathrm{E'I}wq)\theta,$$

whence

$$\frac{d^2\theta}{dx^2}=-m^6x^4\theta-b^2\theta, \quad . \quad . \quad . \quad . \quad . \quad (16.244)$$

where

$$\left.\begin{aligned} b^2&=\frac{wq}{\mathrm{K}n} \\ m^6&=\frac{w^2}{4\mathrm{E'IK}n}\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (16.245)$$

and

Now q is supposed to be small. If q were zero the value of θ satisfying the boundary conditions of this problem would be, by (16.201),

$$\theta_1=\mathrm{A}\left\{1-\frac{m^6x^6}{5.6}+\frac{m^{12}x^{12}}{5.6.11.12}-\ldots\right\}. \quad . \quad . \quad (16.246)$$

Since b^2 is small the actual value of θ for the present problem will differ from θ_1 by a small function of x We can find a first approximation to this small function of x by using θ_1 for θ in the small term in equation (16.244). Thus we have to solve the equation

$$\frac{d^2\theta}{dx^2}+m^6x^4\theta=-b^2\theta_1. \quad . \quad . \quad . \quad . \quad . \quad (16.247)$$

A particular integral of this equation is

$$\theta_2=-\mathrm{A}b^2x^2\mathrm{F}(m^6x^6), \quad . \quad . \quad . \quad . \quad . \quad (16.248)$$

where

$$\mathrm{F}(s)=\frac{1}{1.2}-\frac{s}{7.8}\left\{\frac{1}{1.2}+\frac{1}{5.6}\right\}$$
$$+\frac{s^2}{13.14}\left\{\frac{1}{1.2.7.8}+\frac{1}{5.6.7.8}+\frac{1}{5.6.11.12}\right\}$$

$$-\frac{s^3}{19.20}\left\{\begin{array}{l}\dfrac{1}{1.2.7.8.13.14}+\dfrac{1}{5.6.7.8.13.14}\\ +\dfrac{1}{5.6.11.12.13.14}+\dfrac{1}{5.6.11.12.17.18}\end{array}\right\}$$

etc. . . . . (16.249)

Now

$$\theta=\theta_1+\theta_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16.250)$$

is a solution of (16.245) which satisfies the condition that the torque is zero at the end $x=0$, which is one of the conditions of the present problem. The other condition is that

$$\theta=0 \text{ where } x=l \quad . \quad . \quad . \quad . \quad . \quad . \quad (16.251)$$

Let $f(m^6x^6)$ denote the series in brackets in (16.246). Then the condition that has still to be satisfied is

$$f(m^6l^6)-b^2l^2\mathrm{F}(m^6l^6)=0, \quad . \quad . \quad . \quad . \quad (16.252)$$

an equation from which m^6l^6 has to be found. Now since b^2 is small a first approximation to the root of this last equation is the root of the equation

$$f(m^6l^6)=0,$$

which root is, by (16.203),

$$m^6l^6=41{\cdot}30 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16.253)$$

To get a second approximation to the root of (16.252) put

$$m^6l^6=41{\cdot}30+v. \quad . \quad . \quad . \quad . \quad . \quad . \quad (16.254)$$

Then (16.252) becomes

$$f(41{\cdot}30+v)-b^2l^2\mathrm{F}(41{\cdot}30+v)=0,$$

from which we get, on neglecting v^2 and b^2v,

$$f(41{\cdot}30)+vf'(41{\cdot}30)-b^2l^2\mathrm{F}(41{\cdot}30)=0.$$

Since the first term in this last equation is zero we find

$$v=b^2l^2\frac{\mathrm{F}(41{\cdot}30)}{f'(41{\cdot}30)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (16{\cdot}255)$$

Now

$$f'(s)=-\frac{1}{5.6}+\frac{2s}{5.6.11.12}-\frac{3s^2}{5.6.11.12.17.18}+$$

$$=-\frac{0{\cdot}4890}{30} \text{ when } s=41{\cdot}30.$$

Also

$$\mathrm{F}(41{\cdot}30)=0{\cdot}1888.$$

Therefore

$$v=-11{\cdot}54b^2l^2=-11{\cdot}54\frac{qw_1l^2}{\mathrm{K}n}, \quad . \quad . \quad . \quad (16.256)$$

where w_1 is the value of w given by (16.253), that is, by the equation

$$w_1{}^2 l^6 = 4 \times 41{\cdot}30\,\mathrm{E'IK}n. \qquad (16.257)$$

Thus (16.254) now becomes

$$m^6 l^6 = 41{\cdot}30 - 23{\cdot}08\,\frac{q}{l\mathrm{K}n}\sqrt{41{\cdot}30\,\mathrm{E'IK}n}. \qquad (16.258)$$

Therefore the buckling load is W given by the equation

$$\begin{aligned} \mathrm{W}l^2 = wl^3 &= 2m^3 l^3 \sqrt{\mathrm{E'IK}n} \\ &= 2\sqrt{\mathrm{E'IK}n}\sqrt{41{\cdot}30 + v} \\ &= 2\sqrt{41{\cdot}30}\sqrt{\mathrm{E'IK}n}\left\{1 + \frac{v}{2 \times 41{\cdot}30}\right\} \\ &= 12{\cdot}85\sqrt{\mathrm{E'IK}n} - 23{\cdot}08\,\frac{q}{l}\,\mathrm{E'I}. \qquad (16.259) \end{aligned}$$

Thus the correction to $\mathrm{W}l^2$ due to applying the load at height q above the central line of the beam is about $-23\,\frac{q}{l}\,\mathrm{E'I}$. If the load were applied at distance q below the central line the correction to $\mathrm{W}l^2$ would be an added, instead of a subtracted, quantity. The assumption we have made that v is small in comparison with 41·3 really amounts to the assumption that the correction to $\mathrm{W}l^2$ is small in comparison with the first approximation. Since Kn is equal to about 4E'I for a rectangular section we see now that our assumption is justified if $\frac{q}{l}$ is a small fraction.

307. An approximate method of finding the buckling load of deep beams.

A method was given in Chapter VI for finding the thrust that will buckle a given strut. The method is, in effect, an energy method. Equation (6.265) really states that the energy expended in bending the strut is equal to the work done by the thrust P in forcing the ends closer together. We can find a similar equation for a buckled beam.

The equations of equilibrium in the buckled state are (16.113) and (16.118), which are rewritten here

$$\frac{d\mathrm{Q}}{dx} = -\mathrm{G}\,\frac{d^2y}{dx^2}, \qquad (16.260)$$

$$\mathrm{E'I}\,\frac{d^2y}{dx^2} = \mathrm{G}\theta + \mathrm{N} - \mathrm{P}y. \qquad (16.261)$$

If there is no end thrust P, y is easily eliminated from these equations. Thus

$$\frac{d\mathrm{Q}}{dx} = -\frac{\mathrm{G}}{\mathrm{E'I}}(\mathrm{G}\theta + \mathrm{N}), \qquad (16.262)$$

For a beam clamped at both ends in a horizontal plane, so that

$$\frac{dy}{dx} = 0 \text{ at both ends}$$

we get, by (16.260),

$$\int_{-b}^{a} \frac{1}{G} \frac{dQ}{dx} dx = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (16.263)$$

the ends of the rod being assumed to be at $x = a$ and $x = -b$.

For a beam not clamped at both ends N is usually zero.

From (16.262) we get, by multiplying by $\frac{E'I}{G^2} \frac{dQ}{dx}$,

$$\frac{E'I}{G^2} \left(\frac{dQ}{dx}\right)^2 = -\left(\theta + \frac{N}{G}\right) \frac{dQ}{dx}. \quad . \quad . \quad . \quad . \quad (16.264)$$

Now integrating both sides of this last equation over the whole length of the rod we get

$$\int_{-b}^{a} \frac{E'I}{G^2} \left(\frac{dQ}{dx}\right)^2 dx = -\int_{-b}^{a} \theta \frac{dQ}{dx} dx - N \int_{-b}^{a} \frac{1}{G} \frac{dQ}{dx} dx. \quad . \quad (16.265)$$

The last term in the preceding equation is zero for an unclamped beam because N is zero, and it is zero for a clamped beam by (16.263). Also

$$\int_{-b}^{a} \theta \frac{dQ}{dx} dx = \Big[\theta Q\Big]_{-b}^{a} - \int_{-b}^{a} Q \frac{d\theta}{dx} dx. \quad . \quad . \quad . \quad (16.266)$$

Now at an end where the beam is so held that it cannot twist from the upright position the angle θ is zero; and at a free end the torque Q is zero. Therefore the integrated term in the last equation vanishes. Thus (16.266) becomes

$$\int \frac{E'I}{G^2} \left(\frac{dQ}{dx}\right)^2 dx = \int Q \frac{d\theta}{dx} dx$$

$$= \int \frac{Q^2}{Kn} dx, \quad . \quad . \quad . \quad . \quad (16.267)$$

the range of integration being from one end to the other of the beam.

The last equation is, of course, an accurate equation if the correct value of Q is used in the integrals. But even when approximate values of Q are used the proportional error in the buckling load is usually very much smaller than the error in Q. Moreover, the equation is just as good for beams with variable sections as for beams with constant sections.

For beams with constant sections, for which E′I and Kn are constants, equation (16.267) can be put into either of the forms

$$E'IKn \int \frac{1}{G^2} \left(\frac{dQ}{dx}\right)^2 dx = \int Q^2 dx \quad . \quad . \quad . \quad . \quad (16.268)$$

or

$$E'IKn\int\frac{1}{G^2}\left(\frac{d^2\theta}{dx^2}\right)^2 dx=\int\left(\frac{d\theta}{dx}\right)^2 dx. \quad . \quad . \quad . \quad (16.269)$$

Just as the buckling thrust of a strut is the least value of P that can be got from equation (6.265) so here the true buckling load of a beam is the smallest load given by (16.267). It will be seen from the examples worked out below that this approximate method can be made to give results in which the errors of calculation are insignificant in comparison with the probable errors in the elastic constants.

308. Illustrative examples.

(*a*) *The same problem as in Art. 298.*

For this case

$$G=-\tfrac{1}{2}Rx$$

and therefore equation (16.268) becomes

$$\frac{4E'IKn}{R^2}\int\frac{1}{x^2}\left(\frac{dQ}{dx}\right)^2 dx=\int Q^2\,dx. \quad . \quad . \quad . \quad . \quad (16.270)$$

Let $2a$ be written for the length of the beam, and let the integrals in this last equation be taken from the end $x=0$ to the middle $x=a$. This will be quite safe provided we choose a value for Q which is zero at the middle, as it clearly is in the actual problem. Now the differential equation for θ or for Q shows that each is expressible in powers of x^4. Moreover Q is not zero at the end $x=0$, from which it follows that the expression for Q begins with a constant term. Then let us take as the approximate value of the torque

$$Q=1+p\frac{x^4}{a^4}+q\frac{x^8}{a^8} \quad . \quad . \quad . \quad . \quad . \quad (16.271)$$

The condition that Q is zero at the centre gives

$$0=1+p+q \quad . \quad . \quad . \quad . \quad . \quad (16.272)$$

This relation between p and q reduces the two constants in Q to one independent constant. Of course it would be more correct to write kQ instead of Q in equation (16.271), but as this factor k would appear as $\frac{1}{k^2}$ in each side of (16.270) it would make no difference to the result. It is therefore omitted.

Now (16.270) gives

$$\frac{R^2}{4E'IKn}=\frac{\int_0^a \frac{1}{x^2}\left(4p\frac{x^3}{a^4}+8q\frac{x^7}{a^8}\right)^2 dx}{\int_0^a\left(1+p\frac{x^4}{a^4}+q\frac{x^8}{a^8}\right)^2 dx}$$

$$= \frac{\dfrac{16}{a^4}\displaystyle\int_0^a \left\{ p^2 \frac{x^4}{a^4} + 4pq \frac{x^8}{a^8} + 4q^2 \frac{x^{12}}{a^{12}} \right\} dx}{\displaystyle\int_0^a \left\{ 1 + 2p\frac{x^4}{a^4} + (p^2 + 2q)\frac{x^8}{a^8} + 2pq\frac{x^{12}}{a^{12}} + q^2 \frac{x^{16}}{a^{16}} \right\} dx}$$

$$= \frac{16}{a^4} \frac{\frac{1}{5}p^2 + \frac{4}{9}pq + \frac{4}{13}q^2}{1 + \frac{2}{5}p + \frac{1}{9}(p^2 + 2q) + \frac{2}{13}pq + \frac{1}{17}q^2}$$

$$= \frac{16.17}{a^4} \frac{117p^2 + 260pq + 180q^2}{9945 + 3978p + 1105(p^2 + 2q) + 1530pq + 585q^2}.$$

By means of (16.272) this reduces to

$$\frac{R^2 a^4}{34 E'IKn} = \frac{37p^2 + 100p + 180}{5p^2 + 44p + 260}. \quad . \quad . \quad . \quad (16.273)$$

The minimum value of R^2a^4 for different values of p given by the last equation is

$$R^2 a^4 = 4 \times 4\cdot 482\, E'IKn, \quad . \quad . \quad . \quad . \quad (16.274)$$

which agrees with (16.177a) to the last figure.

(*b*) *The same problem as in Art. 300.*

Let the origin be taken at one end and let the integrals be taken over half the beam as in the last example. Assume

$$Q = 1 + p\frac{x^2}{a^2} + q\frac{x^4}{a^4} + r\frac{x^6}{a^6}. \quad . \quad . \quad . \quad . \quad (16.275)$$

Since y is zero at the end $x = 0$ and at the middle $x = a$ it follows from (16.260) that

$$\int_0^a \frac{1}{G}\frac{dQ}{dx}\, dx = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (16.276)$$

which gives, since $G = -\frac{1}{2}Rx$

$$2p + \tfrac{4}{3}q + \tfrac{6}{5}r = \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16.277)$$

Also, because the torque is zero at the middle of the beam,

$$1 + p + q + r = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (16.278)$$

From the last two equations we get

$$p = \tfrac{1}{6}(9 - q), \quad . \quad . \quad . \quad . \quad . \quad . \quad (16.279)$$

$$r = -\tfrac{1}{6}(15 + 5q), \quad . \quad . \quad . \quad . \quad . \quad (16.280)$$

which leaves one undetermined constant in the expression for Q.

Equation (16.268) gives, for this case,

$$\frac{16 E'IKn}{R^2 a^4}\left\{p^2 + \tfrac{4}{3}pq + \tfrac{4}{5}q^2 + \tfrac{6}{5}pr + \tfrac{12}{7}qr + r^2\right\}$$

$$= 1 + \tfrac{2}{3}p + \tfrac{1}{5}p^2 + \tfrac{2}{5}q + \tfrac{2}{7}pq + \tfrac{2}{7}r$$

$$+ \tfrac{1}{9}q^2 + \tfrac{2}{9}pr + \tfrac{2}{11}qr + \tfrac{1}{13}r^2 \quad . \quad . \quad . \quad . \quad (16.281)$$

By means of (16.279) and (16.280) this last equation can be reduced to

$$\frac{R^2a^4}{32\,E'IKn}=\frac{70+40q+24q^2}{48{\cdot}410+12{\cdot}601q+2{\cdot}2875q^2}. \quad . \quad . \quad (16.282)$$

The minimum value of the right hand side for variations in q is 1·310. Therefore this method gives

$$\frac{R^2a^4}{4\,E'IKn}=10{\cdot}48 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16{\cdot}283)$$

which exceeds the value shown in (16.195) by only one per thousand.

309. Approximate method when the load is not applied to the central line.

When the load is at height q above the central line of the beam equation (16.260) is changed to

$$\frac{dQ}{dx}=-G\frac{d^2y}{dx^2}-qw\theta.$$

Then (16.262) is changed to

$$\frac{dQ}{dx}=-\frac{G}{E'I}(G\theta+N)-qw\theta.$$

Consequently, instead of (16.267), we get

$$\int\frac{E'I}{G^2}\left(\frac{dQ}{dx}\right)^2dx=\int\frac{Q^2}{Kn}dx-\int\frac{E'Iqw}{G^2}\theta\frac{dQ}{dx}dx. \quad . \quad . \quad (16.284)$$

In dealing with this last equation the term involving q must be treated as small. A first approximation to w must be got by neglecting q. The first approximation to θ must then be used in the integral involving q in (16.284). As an example we take the problem worked out in Art. 306. Let us assume that

$$\theta=\frac{x^{12}}{l^{12}}+c\frac{x^6}{l^6}-(1+c). \quad . \quad . \quad . \quad . \quad . \quad (16.285)$$

This satisfies the boundary conditions

$$\left.\begin{aligned}\theta&=0 \text{ where } x=l\\ Q&=0 \text{ where } x=0\end{aligned}\right\}. \quad . \quad . \quad . \quad . \quad . \quad (16.286)$$

The equation for w is

$$\frac{4E'IKn}{w^2}\int_0^l\frac{1}{x^4}\left(\frac{d^2\theta}{dx^2}\right)^2dx=\int_0^l\left(\frac{d\theta}{dx}\right)^2dx-\frac{4E'Iq}{w}\int_0^l\frac{\theta}{x^4}\frac{d^2\theta}{dx^2}dx. \quad (16.287)$$

Neglecting q we find

$$\frac{w^2l^6}{4E'IKn}=\frac{5c^2+20c+\frac{484}{17}}{\frac{c^2}{11}+\frac{4c}{17}+\frac{4}{23}}. \quad . \quad . \quad . \quad . \quad . \quad (16.288)$$

The fraction on the right side of this last equation has a minimum value when

$$c = -4{\cdot}108 , \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16.289)$$

and this value of c gives

$$\frac{w^2 l^6}{4\text{E}'\text{IK}n} = 41{\cdot}39 , \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16.290)$$

which differs by about 0·2 per cent from the more correct result 41·30.

Now substituting in equation (16.284) the value of c we have just found, that equation gives, after division by $\frac{12}{l}$,

$$92{\cdot}1 \frac{4\text{E}'\text{IK}n}{w^2 l^6} = 2{\cdot}225 + 2 \times 50{\cdot}3 \frac{\text{E}'\text{I}q}{wl^4}$$

$$= 2{\cdot}225\left(1 + \frac{2 \times 50{\cdot}3}{2{\cdot}225} \frac{\text{E}'\text{I}q}{wl^4}\right),$$

whence

$$\frac{4\text{E}'\text{IK}n}{w^2 l^6} = \frac{1}{41{\cdot}39}\left(1 + 2 \times 22{\cdot}6 \frac{\text{E}'\text{I}q}{wl^4}\right).$$

On substituting the approximate value for w from (16.290) in the term involving q in the last equation we get

$$\frac{4\text{E}'\text{IK}n}{w^2 l^6} = \frac{1}{41{\cdot}39}\left(1 + \frac{22{\cdot}6}{\sqrt{41{\cdot}39}} \frac{q}{l} \sqrt{\frac{\text{E}'\text{I}}{\text{K}n}}\right).$$

Therefore, finally,

$$\frac{wl^3}{\sqrt{\text{E}'\text{IK}n}} = 2\sqrt{41{\cdot}39}\left(1 + \frac{22{\cdot}6}{\sqrt{41{\cdot}39}} \frac{q}{l} \sqrt{\frac{\text{E}'\text{I}}{\text{K}n}}\right)^{-\frac{1}{2}}$$

$$= 2\sqrt{41{\cdot}39}\left(1 - \frac{22{\cdot}6}{2\sqrt{41{\cdot}39}} \frac{q}{l} \sqrt{\frac{\text{E}'\text{I}}{\text{K}n}}\right) \text{ approximately}$$

$$= 2\sqrt{41{\cdot}39} - 22{\cdot}6 \frac{q}{l} \sqrt{\frac{\text{E}'\text{I}}{\text{K}n}}.$$

The coefficient 22·6 in the small term is near enough to 23·08 obtained from the exact expression for θ in Art. 306 to justify the use of the approximate method.

CHAPTER XVII.

CYLINDERS WITH THIN WALLS.

310. Equilibrium of a cylinder which is strained into another cylinder with a cross-section of different shape.

The general theory of the bending of a circular cylinder with thin walls is so complicated that it is not worth while to deduce the simpler cases from the more general theory. In spite of the fact that it will result in a certain amount of repetition we shall not therefore attack the general problem until we have solved some of the simpler problems. We shall start with the problem of a cylinder bent into another cylinder with a different cross section.

We are assuming that no forces act on the cylinder in the direction of the generating lines.

Let $2h$ denote the thickness. Just as in dealing with flat plates the surface at distance h from both faces is called the middle surface of the cylinder. Let two sections of the strained cylinder be taken perpendicular to the middle surface, each containing a generating line of that surface. The points A and B in fig. 167 are on these generating lines, and the plane of the figure is perpendicular to the generating lines. Let the length of the arc AB be infinitesimal and equal to ds. Let Q denote the mean compressive stress across the section at A. Let M denote the bending moment at A, this bending moment being a couple in a plane parallel to the plane of the figure. F is the shearing force per unit length across the section, its direction being perpendicular to the middle surface. A pressure p is supposed to act on the convex side of the cylinder. The lines OA, OB, are normals to the strained middle surface and contain an angle $d\varphi$. The forces and couple at B are $Q+dQ$, $F+dF$, and $M+dM$

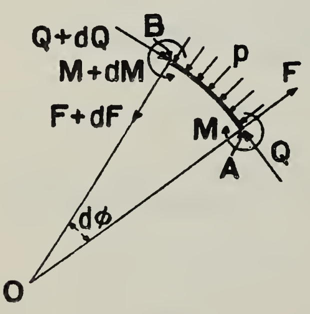


Fig. 167

By taking moments about B and neglecting small quantities of the second order we get

$$dM + F ds = 0 \quad \ldots\ldots\ldots \quad (17.1)$$

whence

$$\frac{dM}{ds} = -F \qquad (17.2)$$

Again by resolving along the normal BO we get

$$(F + dF) + pds - F\cos d\varphi - 2hQ\sin d\varphi = 0,$$

or, again neglecting quantities of the second and higher orders, and consequently writing 1 for $\cos d\varphi$ and $d\varphi$ for $\sin d\varphi$,

$$dF + pds - 2hQ\,d\varphi = 0, \qquad (17.3)$$

whence

$$\frac{dF}{ds} - 2hQ\frac{d\varphi}{ds} + p = 0 \qquad (17.4)$$

Next resolving in the direction of $(Q + dQ)$ at B we get

$$2h(Q + dQ) - 2hQ\cos d\varphi + F\sin d\varphi = 0$$

from which, with the same approximations as before, we get

$$2h\frac{dQ}{ds} + F\frac{d\varphi}{ds} = 0 \qquad (17.5)$$

From (17.2) we find, by differentiating with respect to s,

$$\frac{dF}{ds} = -\frac{d^2M}{ds^2} \qquad (17.6)$$

This last equation enables us to put (17.4) into the form

$$\frac{d^2M}{ds^2} + 2hQ\frac{d\varphi}{ds} = p \qquad (17.7)$$

Now in dealing with flat plates we proved that the bending moment M across any section is related to the radius of curvature ϱ of the perpendicular section by the equation

$$M = \frac{E'I}{\varrho}$$

If the radius of curvature of the originally flat plate changes from ϱ_0 to ϱ and the bending moment from M_0 to M, then clearly the change of bending moment is

$$M - M_0 = E'I\left(\frac{1}{\varrho} - \frac{1}{\varrho_0}\right) \qquad (17.8)$$

Now if the plate had originally a radius of curvature ϱ_0 the *changes* in the strains, and therefore also in the stresses, are clearly just the same as if the plate had been originally flat. Therefore the change of bending moment is still proportional to the change of curvature even though the original bending moment M_0 is zero. Consequently

$$M = E'I\left(\frac{1}{\varrho} - \frac{1}{\varrho_0}\right) \qquad (17.9)$$

is the relation between the bending moment and the curvature when the natural radius of curvature is ϱ_0.

The curvature of the section in fig. 167 is

$$c = \frac{d\varphi}{ds}.$$

Let c_0 denote the natural curvature of this element. Then,

$$M = E'I(c - c_0) \quad \dots \dots \quad (17.10)$$

Writing ξ for the change of curvature $(c - c_0)$, equation (17.7) can now be written in the form

$$E'I \frac{d^2\xi}{ds^2} + 2hQ(c_0 + \xi) = p \quad \dots \quad (17.11)$$

Also equation (17.5) can be written

$$2h \frac{hQ}{ds} = -F(c_0 + \xi) \quad \dots \dots \quad (17.12)$$

Moreover (17.2) enables us to express F in terms of ξ thus

$$F = -E'I \frac{d\xi}{ds} \quad \dots \dots \quad (17.13)$$

In the cases that are of most use in practice ξ is a small fraction of c_0 and can be neglected in comparison with c_0. Then (17.12) becomes approximately

$$2h \frac{dQ}{ds} = -Fc_0 \quad \dots \dots \quad (17.14)$$

311. Collapse of a long tube under external pressure.

Let us now apply the equations we have found to a cylinder which is circular in the unstrained state with a curvature

$$c_0 = \frac{1}{r} = \text{constant}.$$

Then, by means of (17.13) and (17.14), we get

$$Q = -\frac{1}{2h}\int Fc_0\, ds = \frac{E'I}{2h} c_0 \xi + Q_0, \quad \dots \quad (17.15)$$

Q_0 being a constant.
Now neglecting ξ^2 in (17.11) we find

$$E'I \frac{d^2\xi}{ds^2} + 2hQ_0c_0 + 2hQ_0\xi + E'Ic_0^2\xi = p,$$

or

$$E'I \frac{d^2\xi}{ds^2} + (2hQ_0 + E'Ic_0^2)\xi = p - 2hQ_0c_0 \quad \dots \quad (17.16)$$

If p is constant we might expect that the cylinder remains circular when it is strained, but with a slightly diminished radius. This is true as long as p does not exceed a definite pressure, which we can easily discover, but at that particular pressure the circular form becomes unstable, just as at strut becomes unstable for a particular thrust. To find this pressure we must solve (17.16). The solution is clearly

$$\xi = \frac{p - 2hQ_0c_0}{E'Ic_0{}^2 + 2hQ_0} + H\cos(ms + k), \quad . \quad . \quad . \quad (17.17)$$

where

$$m^2 = \frac{2hQ_0 + E'Ic_0{}^2}{E'I} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17.18)$$

Now ξ must clearly be such that its value is repeated when s increases by $2\pi r$, the circumference of the cylinder. Thus

$$m \times 2\pi r = 2n\pi, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17.19)$$

n being an integer; that is,

$$\frac{2hQ_0 + E'Ic_0{}^2}{E'I} r^2 = n^2,$$

whence
$$2hQ_0 = \frac{E'I}{r^2}(n^2 - 1) \quad . \quad . \quad . \quad (17.20)$$

The relation between Q_0 and p can be got from our equations by assuming that the strained cylinder is circular, since these quantities must have the same value just before and just after the cylinder becomes unstable. Thus in equation (17.4) we may assume that F does not vary with s and that $Q = Q_0$. Then we get approximately

$$\frac{Q_0}{r} = \frac{p}{2h} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17.21)$$

Therefore

$$p = \frac{E'I}{r^3}(n^2 - 1), \quad . \quad . \quad . \quad . \quad (17.22)$$

and, moreover, equation (17.17) becomes

$$\xi = H\cos(n\theta + k), \quad . \quad . \quad . \quad (17.23)$$

θ being the angle subtended by the arc s of the section of the middle surface of the cylinder at its axis.

Now n must be an integer and it must be greater than unity because, if n is equal to unity, equation (17.20) makes Q_0 zero, and therefore makes the corresponding value of p zero. But the ring could not collapse with a zero pressure. In fact, $n = 1$ can only apply to a split cylinder and not to a closed one. Then the least value of p occurs when $n = 2$, and this value is

$$p = \frac{3E'I}{r^3}$$

$$= \frac{2Eh^3}{(1 - \sigma^2)r^3} \quad . \quad . \quad . \quad . \quad . \quad . \quad (17.24)$$

The corresponding shape of the strained cross-section is elliptical.

The pressure given in (17.24) is the external pressure which would cause the collapse of a long cylindrical tube such as a boiler-flue. For short tubes the end-conditions must be taken into account. The investigation of the stability of such short tubes will have to be postponed till we have worked out the theory of the tube with changes in both its principal curvatures.

312. Stability of thin ring under external pressure.

The foregoing reasoning that has been used for a tube would apply equally well to a thin ring if E′ were replaced by E, and if I were regarded as the moment of inertia of the cross-section of the ring and p the external thrust per unit length applied to the ring. The thrust that would cause collapse of the ring is therefore

$$p = \frac{3EI}{r^3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17.25)$$

313. Hollow circular cylinder with thin walls having its strains symmetrical about its axis.

Let the x-axis be taken along the axis of the cylinder itself, x being measured from any convenient plane of particles which lie in a circular section of the middle surface of the tube. Let r be the radius of the middle surface of the cylinder before strain; and let the ring of particles which was at x before strain be at $(x+u)$ after strain and have a radius $(r+w)$.

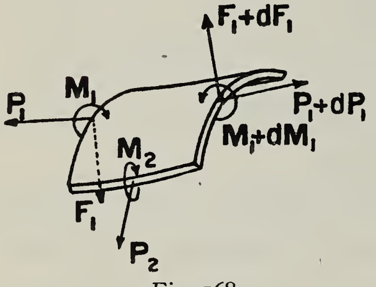


Fig. 168

Let P_1, P_2, be the tensional stresses in the middle surface of an element of the cylinder. The stress P_1 is not exactly parallel to the x-axis except at points of the tube where the tangent plane to the strained middle surface is parallel to that axis. Just as in a flat plate these stresses P_1 and P_2 are the mean stresses across the sections on which they act.

Let M_1 and M_2 denote the bending moments per unit length across the sections on which P_1 and P_2 act; that is, $M_2 dx$ and $M_1 dy$ are the moments of the tensional stresses about elements of length dx and dy in the middle surface. On account of the symmetry about the axis the torque across these sections is zero. Let F_1 be the mean shear force per unit length of the section across which P_1 acts, the direction of F_1 being radial as shown in fig. 168. The shear force F_2 across the perpendicular section is zero from symmetry.

Let the element we are dealing with subtend an angle $d\theta$ at the axis of the cylinder, so that $dy = rd\theta$.

Suppose that a pressure p per unit area acts on the convex surface of the cylinder; or if a pressure acts on both sides let p denote the excess of the external over the internal pressure.

Now let us find the equations of equilibrium of the element with dimensions dx and $rd\theta$. Before we write down equations it had better be made clear that P_1 has a radial component. The inward radial component force due to P_1 acting on an area $2hrd\theta$ is approximately

$$\left(P_1 \frac{dw}{dx}\right) \times 2hrd\theta .$$

The outward radial component of $(P_1 + dP_1)$ at the other end of the element is

$$2hrd\theta \left\{ P_1 \frac{dw}{dx} + \frac{d}{dx}\left(P_1 \frac{dw}{dx}\right) dx \right\}$$

Thus the excess outward radial force due to P_1 and $(P_1 + dP_1)$ is

$$2hrd\theta \frac{d}{dx}\left(P_1 \frac{dw}{dx}\right) dx .$$

Now resolving in the direction of $(F_1 + dF_1)$, and neglecting quantities of smaller order than those retained, we get

$$(dF_1) \times rd\theta + 2hrd\theta \frac{d}{dx}\left(P_1 \frac{dw}{dx}\right) dx$$
$$- (P_2 d\theta) \times 2hdx - prd\theta\, dx = 0 ,$$

whence

$$r \frac{dF_1}{dx} + 2hr \frac{d}{dx}\left(P_1 \frac{dw}{dx}\right) - 2hP_2 = pr \quad . \quad . \quad . \quad (17.26)$$

Again, resolving parallel to the axis of x, we find

$$2hrd\theta dP_1 - rd\theta \frac{d}{dx}\left(F_1 \frac{dw}{dx}\right) dx = 0 ,$$

from which

$$2h \frac{dP_1}{dx} - \frac{d}{dx}\left(F_1 \frac{dw}{dx}\right) = 0 \quad . \quad . \quad . \quad . \quad . \quad (17.27)$$

Next taking moments about the element $rd\theta$ where P_1 acts we get

$$rd\theta dM_1 + rd\theta F_1 dx = 0,$$

whence

$$\frac{dM_1}{dx} + F_1 = 0 . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17.28)$$

We have now arrived at the three equations of equilibrium, and we have next to express the stresses in terms of the displacements u and w.

The strains in the middle surface in the directions of P_1 and P_2 are

$$\alpha = \frac{du}{dx} \quad . \quad . \quad . \quad . \quad . \quad . \quad (17.29)$$

$$\beta = \frac{(r+w)d\theta - rd\theta}{rd\theta} = \frac{w}{r} \quad . \quad . \quad . \quad . \quad (17.30)$$

Therefore the expression for the stresses in terms of u and w are

$$P_1 = E'\left(\frac{du}{dx} + \sigma\frac{w}{r}\right), \quad . \quad . \quad . \quad . \quad . \quad (17.31)$$

$$P_2 = E'\left(\frac{w}{r} + \sigma\frac{du}{dx}\right), \quad . \quad . \quad . \quad . \quad . \quad (17.32)$$

where

$$E' = \frac{E}{1-\sigma^2}.$$

Also the changes in the principal curvatures of the middle surface are

$$\frac{d^2w}{dx^2}$$

and

$$-\frac{1}{r+w} + \frac{1}{r} = \frac{w}{r^2},$$

the changes of curvature being reckoned positive when they are in the directions of M_1 and M_2. Thus the expressions for M_1 and M_2 are

$$M_1 = E'I\left(\frac{d^2w}{dx^2} + \sigma\frac{w}{r^2}\right) \quad . \quad . \quad . \quad . \quad . \quad (17.33)$$

$$M_2 = E'I\left(\frac{w}{r^2} + \sigma\frac{d^2w}{dx^2}\right), \quad . \quad . \quad . \quad . \quad . \quad (17.34)$$

I being $\frac{2}{3}h^3$.

We have now got all the equations we need for the case of strain symmetrical about the axis. One of the equations, namely (17.27), can be integrated at once, and a simple result deduced from it. Thus we get, by integrating with respect to x,

$$2hP_1 - F_1\frac{dw}{dx} = \text{a constant} \quad . \quad . \quad . \quad . \quad (17.35)$$

Now equations (17.28) and (17.33) show that F_1 is of the same order as w, and consequently $F_1\frac{dw}{dx}$ is of the order w^2, and therefore negligible in our equations. Consequently (17.35) gives, to this degree of approximation,

$$P_1 = \text{constant}.$$

This enables us to write (17.26) thus

$$r\frac{dF_1}{dx} + 2hrP_1\frac{d^2w}{dx^2} - 2hP_2 = pr \quad . \quad . \quad . \quad . \quad (17.36)$$

Moreover, on eliminating $\frac{du}{dx}$ from (17.31) and (17.32) we get P_2 in terms of w thus

$$P_2 = \sigma P_1 + E\frac{w}{r} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (17.37)$$

By means of (17.28), (17.33), (17.37), equation (17.36) can now be written

$$-E'I\left(r\frac{d^4w}{dx^4} + \frac{\sigma}{r}\frac{d^2w}{dx^2}\right) + 2hrP_1\frac{d^2w}{dx^2} - 2h\left(\sigma P_1 + E\frac{w}{r}\right) = pr,$$

from which we get

$$\tfrac{1}{3}h^2\frac{d^4w}{dx^4} + \left(\frac{\sigma h^2}{3r^2} - \frac{P_1}{E'}\right)\frac{d^2w}{dx^2} + (1-\sigma^2)\frac{w}{r^2} + \frac{\sigma}{r}\frac{P_1}{E'} + \frac{p}{2hE'} = 0 \quad (17.38)$$

Suppose p is constant, and let us put

$$v = w + \frac{\sigma r P_1}{E} + \frac{r^2 p}{2hE} \quad . \; . \; . \; . \; . \; . \quad (17.39)$$

Then equation (17.38) becomes

$$\tfrac{1}{3}h^2\frac{d^4v}{dx^4} + \left(\frac{\sigma h^2}{3r^2} - \frac{P_1}{E'}\right)\frac{d^2v}{dx^2} + (1-\sigma^2)\frac{v}{r^2} = 0 \quad . \; . \quad (17.40)$$

314. Stability of a long tube under an axial thrust.

It is clear that there is a possibility, under favourable circumstances, of the crumpling of a tube under an axial thrust, the crumpling taking place in such a way that the axis remains straight and the strain is symmetrical about the axis. In this state of strain the longitudinal that these waves can be purely sine-waves. Thus (17.40) is satisfied by section of the middle surface has a wave form, and (17.40) shows that these waves can be pure sine-waves. Thus (17.40) is satisfied by

$$v = H\sin(mx + k) \quad . \; . \; . \; . \; . \; . \; . \quad (17.41)$$

provided that

$$\tfrac{1}{3}h^2m^4 - \left(\frac{\sigma h^2}{3r^2} - \frac{P_1}{E'}\right)m^2 + \frac{1-\sigma^2}{r^2} = 0,$$

that is, provided that

$$-\frac{P_1}{E'} = \tfrac{1}{3}h^2m^2 + \frac{1-\sigma^2}{m^2r^2} - \frac{\sigma h^2}{3r^2}.$$

Since P_1 is a tension we may write P for the thrust $-P_1$. Then

$$\frac{P}{E'} = \tfrac{1}{3}h^2m^2 + \frac{1-\sigma^2}{m^2r^2} - \frac{\sigma h^2}{3r^2} \quad . \; . \; . \; . \; . \quad (17.42)$$

There is a particular value of m for which P is a minimum, and this minimum P will obviously be the thrust at which this type of instability begins. The minimum value of P occurs when

$$\tfrac{1}{3}h^2m^2 = \frac{1-\sigma^2}{m^2r^2}$$

$$= \sqrt{\tfrac{1}{3}h^2m^2 \times \frac{1-\sigma^2}{m^2r^2}}$$

$$= \frac{h}{r}\sqrt{\tfrac{1}{3}(1-\sigma^2)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17.43)$$

This minimum value is then

$$P = E'\left\{\tfrac{1}{3}h^2m^2 + \frac{1-\sigma^2}{m^2r^2} - \frac{\sigma h^2}{3r^2}\right\}$$

$$= E'\left\{\frac{2h}{r}\sqrt{\tfrac{1}{3}(1-\sigma^2)} - \frac{\sigma h^2}{3r^2}\right\}$$

$$= \frac{2hE}{r}\sqrt{\frac{1}{3(1-\sigma^2)}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17.44)$$

approximately, since $\frac{h^2}{r^2}$ is negligible in comparison with $\frac{h}{r}$.

The total thrust on one end of the tube is

$$4\pi rhP = \frac{8\pi h^2E}{\sqrt{3(1-\sigma^2)}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (17.45)$$

This is the total thrust that would cause this kind of instability of a tube whose length is such that the end conditions have a negligible effect at the middle section of the tube. We have yet to discover whether the tube would collapse in some other way under a still smaller thrust, and this we can only discover when we have solved the problem of the collapse of a tube in the most general way, that is, with w as a function of both x and θ. We shall deal with the latter problem later in this chapter.

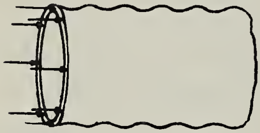


Fig. 169

The wave-length of the corrugations on the tube is

$$\lambda = \frac{2\pi}{m}$$

$$= \frac{2\pi\sqrt{hr}}{\sqrt{3(1-\sigma^2)}}$$

$$= 4{\cdot}8\sqrt{hr} \text{ approximately}, \quad . \quad . \quad . \quad (17.46)$$

which will usually be a small fraction of the radius. Fig. 169 gives an idea of the appearance of the tube when it collapses.

315. Stresses at the ends of a closed cylinder containing gas under pressure.

To find the stresses in a closed cylinder we must solve equation (17.40) and introduce the boundary conditions. The assumption has already been made in arriving at (17.40) that p is constant. To solve this equation put

$$v = Ae^{\frac{nx}{r}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17.47)$$

Then the equation gives

$$\tfrac{1}{3}h^2\frac{n^4}{r^4}+\left(\frac{\sigma h^2}{3r^2}-\frac{P_1}{E'}\right)\frac{n^2}{r^2}+\frac{1-\sigma^2}{r^2}=0,$$

whence

$$n^4+\left(\sigma-\frac{3r^2P_1}{h^2E'}\right)n^2+\frac{3(1-\sigma^2)r^2}{h^2}=0 \quad . \quad . \quad (17.48)$$

Now if the stress P_1 is due merely to the internal pressure the total tension across a circular section of the cylinder must be equal to the thrust due to p across the same area; that is,

$$2h \times 2\pi r P_1 = \pi r^2 p, \quad . \quad . \quad . \quad . \quad . \quad . \quad (17.49)$$

and therefore

$$\frac{3r^2P_1}{h^2E'}=\frac{3}{4}\frac{r^3p}{h^3E'}$$

Thus the equation for n is

$$n^4+\left(\sigma-\frac{3}{4}\frac{r^3p}{h^3E'}\right)n^2+\frac{3(1-\sigma^2)r^2}{h^2}=0 \quad . \quad . \quad (17.50)$$

Let us write

$$q=\sigma-\frac{3}{4}\frac{r^3p}{h^3E'}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17.51)$$

$$4m^4=\frac{3(1-\sigma^2)r^2}{h^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17.52)$$

so that the equation for n can be written

$$n^4+qn^2+4m^4=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (17.53)$$

From this

$$n^2=\frac{-q\pm i\sqrt{16m^4-q^2}}{2}. \quad . \quad . \quad . \quad . \quad . \quad (17.54)$$

In all practical cases $16m^4$ would be much greater than q^2. We shall therefore consider only the case where q^2 is negligible in comparison with $16m^4$. Then

$$n^2=\frac{-q\pm 4im^2}{2}$$

$$=\pm 2im \text{ nearly}$$

$$n=\pm(1\pm i)m \quad . \quad . \quad . \quad . \quad . \quad . \quad (17.55)$$

Therefore

$$v = Ae^{(1+i)\frac{mx}{r}} + Be^{(1-i)\frac{mx}{r}} + Ce^{-(1+i)\frac{mx}{r}} + De^{-(1-i)\frac{mx}{r}},$$

which can be written in the real form

$$v = e^{\frac{mx}{r}}\left(H\cos\frac{mx}{r} + K\sin\frac{mx}{r}\right) + e^{-\frac{mx}{r}}\left(H_1\cos\frac{mx}{r} + K_1\sin\frac{mx}{r}\right) \quad (17.56)$$

The constants H, K, H_1, K_1, are determined by the end conditions. At a clamped end, that is, an end where the tube is attached to a very rigid disk, the conditions are

$$w = 0, \quad \frac{dw}{dx} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17.57)$$

The effects of the conditions at one end at a considerable distance from that end must obviously be negligible. Let us suppose that $x = 0$ at one end and that the other end is at $x = l$, the assumption being that l is positive and very much greater than $\sqrt{hr}$. Now the terms having the factor $e^{\frac{mx}{r}}$ increase in magnitude as x increases, and since the effect of the conditions at $x = 0$ decreases as x increases it follows that the terms involving $e^{\frac{mx}{r}}$ must be almost wholly due to the conditions at the end $x = l$. But the effect of these conditions is negligible at $x = 0$, whence it follows that H and K must be so small that the terms in which they occur have no importance except near the end $x = l$. Therefore, near $x = 0$, we get

$$v = e^{-\frac{mx}{r}}\left(H_1\cos\frac{mx}{r} + K_1\sin\frac{mx}{r}\right), \quad . \quad . \quad . \quad . \quad (17.58)$$

the importance of which decreases as x increases.

Now suppose the end at $x = 0$ is rigidly fixed to a disk. Then the conditions (17.57) must be satisfied.

By means of equation (17.39) and these conditions we get

$$H_1 - \frac{\sigma r P_1}{E} - \frac{r^2 p}{2hE} = 0,$$

and

$$\frac{m}{r}(H_1 - K_1) = 0,$$

whence

$$H_1 = K_1 = \frac{\sigma r P_1}{E} + \frac{r^2 p}{2hE} = c \text{ say} \quad . \quad . \quad . \quad (17.59)$$

Therefore

$$w = ce^{-\frac{mx}{r}}\left(\cos\frac{mx}{r} + \sin\frac{mx}{r}\right) - c$$

$$= \sqrt{2}\,ce^{-\frac{mx}{r}}\sin\left(\frac{mx}{r} + \frac{\pi}{4}\right) - c \quad . \quad . \quad . \quad . \quad (17.60)$$

The greatest bending moment is M_1 at $x = 0$ and its value is

$$M_1 = E'I\left(\frac{d^2w}{dx^2} + \frac{\sigma w}{r^2}\right),$$

which becomes, when $x = 0$,

$$M_1 = -2E'Ic\frac{m^2}{r^2}$$

$$= -E'Ic\frac{\sqrt{3(1-\sigma^2)}}{hr}$$

$$= -\frac{EIc}{hr}\sqrt{\frac{3}{1-\sigma^2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17.61)$$

The maximum longitudinal stress at $x = 0$ is

$$f = P_1 - \frac{hM_1}{I}$$

$$= P_1 + \sqrt{\frac{3}{1-\sigma^2}}\left(\sigma P_1 + \frac{pr}{2h}\right), \quad . \quad . \quad . \quad (17.62)$$

which becomes, with the value of P_1 given by (17.49),

$$f = \frac{pr}{4h}\left\{1 + \sqrt{\frac{3}{1-\sigma^2}}(\sigma + 2)\right\}$$

$$= \frac{5pr}{4h} \text{ approximately} \quad . \quad . \quad . \quad . \quad . \quad (17.63)$$

The maximum value of the other principal stress differs very little from the value of P_2 in the region where the end-conditions have no effect. This value of P_2 can be obtained without using the equations of elasticity, and its magnitude is $\frac{pr}{2h}$. It follows therefore that the greatest stress in a cylinder with clamped ends which contains gas under pressure, occurs at the clamped ends, and its magnitude is given by (17.63).

316. Decay of the end conditions with distance from that end.

The index $-\frac{mx}{r}$ in the factor $e^{-\frac{mx}{r}}$, which occurs in (17.60), falls from zero to $-\sqrt[4]{\frac{3(1-\sigma^2)}{4}}$ as x increases from zero to $\sqrt{hr}$. Consequently the exponential quantity itself falls from unity to about $e^{-0\cdot9}$, that is to 0·41, in the same distance. As x increases by $2\sqrt{hr}$ the exponential factor drops to $(0\cdot41)^2$ — roughly one sixth of its value at $x = 0$. This shows how quickly the effects of the end conditions decay as the distance from the end increases.

A tube whose length is not less than $10\sqrt{hr}$ could quite reasonably be regarded as a long tube in this theory because the effect of the conditions at the ends falls to something not much more than 1 % half way between the ends.

317. Split tube.

The preceding analysis cannot be applied, without modification, to a split tube which is acted on by forces that bend it into a portion of a surface of revolution which is nearly another cylinder with a very different radius. In dealing with closed tubes we made the quite legitimate assumption that the increase in the radius was a small fraction of the radius itself. This increase must be small to keep the strains in the middle surface small. But when we come to deal with a split tube, or a plane sheet bent into a split tube, the change in the radius may be quite a big fraction of the initial or final radius while yet the strains are all small. We shall now find the equations suitable for the split tube.

A piece of a circular tube, the middle surface of which is bounded by two straight lines and two circular arcs, is bent into a surface of revolution by an external pressure p and by couples and forces applied to the edges that were straight in the unstrained state; to find the strain.

Let the axis of revolution of the strained sheet be taken as x-axis, and let the planes of the circular arcs at the ends be at $x=0$ and $x=l$. Let a denote the unstrained radius of the circular arcs, and ϱ the strained radius at x. Clearly the middle surface is very nearly a circular cylinder in the strained state. Then let us put $(r+w)$ for ϱ, w being a quantity which is small compared with either a or r. Then the changes in the principal curvatures of the middle surface are approximately

$$c_1 = \frac{d^2w}{dx^2} \quad \ldots \ldots \ldots \quad (17.64)$$

$$c_2 = \frac{1}{a} - \frac{1}{\varrho} = \frac{1}{a} - \frac{1}{r+w}$$

$$= \frac{1}{a} - \frac{1}{r} + \frac{w}{r^2} \quad \ldots \ldots \ldots \quad (17.65)$$

If we now assume that every circular arc in the strained middle surface subtends the same angle θ at the axis of revolution it follows that the length of the arc at x is

$$s = \rho\theta = (r+w)\theta$$

If the original length was l then the circumferential strain is

$$\beta = \frac{s-l}{l} = \frac{r\theta - l + w\theta}{l}$$

Since the magnitude of r has not yet been definitely fixed let us now make it satisfy the equation

$$r\theta = l;$$

then

$$\beta = \frac{w}{r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17.66)$$

Now equations (17.26), and (17.28) are still true, and P_1 is zero. The expressions for P_2 and M_1 are

$$P_2 = E\beta = E\frac{w}{r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17.67)$$

$$M_1 = E'I(c_1 + \sigma c_2) \quad . \quad . \quad . \quad . \quad . \quad . \quad (17.68)$$

The elimination of F_1, M_1, P_2, from (17.26), (17.28), (17.67), (17.68) gives

$$-E'Ir\left(\frac{d^4w}{dx^4} + \frac{\sigma}{r^2}\frac{d^2w}{dx^2}\right) - \frac{2hEw}{r} - rp = 0,$$

whence

$$\frac{d^4w}{dx^4} + \frac{\sigma}{r^2}\frac{d^2w}{dx^2} + \frac{3(1-\sigma^2)}{h^2r^2}w = -\frac{3(1-\sigma^2)p}{2Eh^3} \quad . \quad . \quad (17.69)$$

This equation differs from (17.38) only in having P_1 equal to zero, and the method of solution used in art. 315 gives, near the end $x=0$,

$$w = w_0 + w_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17.70)$$

where

$$w_0 = -\frac{pr^2}{2Eh} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17.71)$$

and

$$w_1 = e^{-\frac{mx}{r}}\left(H\cos\frac{mx}{r} + K\sin\frac{mx}{r}\right) \quad . \quad (17.72)$$

Since the end $x=0$ is free the bending moment M_1 and shearing force F_1 are zero there; that is,

$$\left.\begin{aligned} &\frac{d^2w}{dx^2} + \sigma\left\{\frac{w}{r^2} + \left(\frac{1}{a} - \frac{1}{r}\right)\right\} = 0 \\ \text{and} \quad &\frac{d^3w}{dx^3} + \frac{\sigma}{r^2}\frac{dw}{dx} = 0 \end{aligned}\right\} \text{ where } x=0. \quad . \quad (17.73)$$

These conditions give

$$-\frac{\sigma}{r^2}H + \frac{2m^2}{r^2}K = \sigma\left\{\frac{w_0}{r^2} + \frac{1}{a} - \frac{1}{r}\right\} \quad . \quad . \quad . \quad . \quad (17.74)$$

and

$$2\frac{m^3}{r^3}(H+K)+\frac{\sigma m}{r^3}(K-H)=0 \quad . \quad . \quad . \quad . \quad (17.75)$$

When σ is neglected in comparison with $2m^2$ this last equation gives

$$H=-K,$$

and then (17.74) becomes

$$\frac{2m^2+\sigma}{r^2}K=\sigma\left\{\frac{w_0}{r^2}+\frac{1}{a}-\frac{1}{r}\right\}.$$

Again neglecting σ in comparison with $2m^2$ we now get

$$-H=K=\frac{\sigma r^2}{2m^2}\left\{\frac{w_0}{r^2}+\frac{1}{a}-\frac{1}{r}\right\}. \quad . \quad . \quad . \quad . \quad . \quad (17.76)$$

Thus finally

$$w=w_0-\frac{\sigma r^2}{\sqrt{2}\,m^2}\left\{\frac{w_0}{r^2}+\frac{1}{a}-\frac{1}{r}\right\}e^{-\frac{mx}{r}}\cos\left(\frac{mx}{r}+\frac{\pi}{4}\right) \quad . \quad . \quad (17.77)$$

The maximum amplitude of the corrugations is

$$\frac{\sigma r^2}{\sqrt{2}\,m^2}\left\{\frac{w_0}{r^2}+\frac{1}{a}-\frac{1}{r}\right\}=\frac{\sqrt{2}\,\sigma hr}{\sqrt{3(1-\sigma^2)}}\left\{\frac{1}{a}-\frac{1}{r}-\frac{p}{2hE}\right\} \quad . \quad (17.78)$$

The results we have just obtained remain true if the final radius r is less than the initial radius a. If p is zero and a is infinite the results apply to a naturally flat plate bent into a portion of a nearly circular cylinder by couples M_2 and small forces P_2 acting on one pair of edges. In that case the maximum amplitude of the corrugations is

$$\frac{\sqrt{2}\,\sigma h}{\sqrt{3(1-\sigma^2)}},$$

a quantity independent of the final radius r; and the magnitude is about $0{\cdot}21h$ or $0{\cdot}29h$ according as σ is $\frac{1}{4}$ or $\frac{1}{3}$.

Since w is measured in the direction away from the axis, and since w is negative when $p=0$ and $a=\infty$, it is seen that one effect of bending a flat plate into a cylinder is to cause a sort of lip curl at the free edges, the curl being away from the axis of the cylinder. When the bent strip becomes narrow enough to be treated as a beam the two curling edges are so close together that they form one curve which is approximately a circle whose curvature we found in Art. 39 to bear the ratio σ to the curvature of the central line of the beam.

318. Thin complete cylinder with any state of strain.

So far we have assumed that the radial displacement w of the middle surface of a cylinder was a function of the distance x alone; or of the polar angle θ alone. We shall now generalise our equations by allowing w to be a function of both x and θ.

Let the axis of x be taken along the axis of the strained cylinder. If there is any uncertainty about which is the axis of the strained cylinder it is sufficient to take as x-axis any line relative to which the displacement w, mentioned below, is small for every particle of the middle surface.

Let cylindrical-polar coordinates be taken about the x-axis, and let the position of a particle in the middle surface when the cylinder is unstrained be x, r, θ; and let the position of the same particle when the cylinder is strained be $x+u_0+u$, $r+w_0+w$, $\theta+\eta$. It is convenient to split up the whole displacements into u_0+u, and w_0+w for reasons that will be seen later.

Let the compressive stresses in the middle surface in the directions parallel and perpendicular to the axis of the cylinder be (P_0+P) and (Q_0+Q) respectively. Also let the bending moments per unit length across the sections over which these stresses act be M_1 and M_2 respectively, these couples being positive when they tend to curve the surface away from the axis of the cylinder. Again let S denote the shear stress in the middle surface, and let H denote the torque. The quantities S and H were denoted by S_3 and Q in Chapter 14.

On the faces where M_1 and M_2 act there are shearing forces which we shall denote by F_1 and F_2 per unit length of the middle surface, just as we did in dealing with the flat plate in Chapter 14.

A small element of the cylinder may be treated in the same way as a small element of a flat plate; we have only to allow for the initial curvature of the cylinder when expressing the bending moments in terms of the curvature. For a naturally curved plate the bending moments are proportional to the changes of curvature, since the stresses giving rise to the bending moments are proportional to these changes of curvature.

Suppose a uniform pressure p acts on the convex surface of the cylinder, and a force $4\pi rhP_0$ acts on the ends of the cylinder in the direction parallel to the axis. Since the thickness is $2h$ this latter force gives rise to the stress P_0. Moreover, the pressure p gives rise to a constant circumferential stress which we have denoted by Q_0. Thus P_0 and Q_0 are stresses that exist while the cylinder remains a cylinder but with slightly altered radius. The partial displacements u_0 and w_0 are assumed to be due entirely to P_0 and Q_0. Therefore, since the circumferential strain due to w_0 is $\frac{w_0}{r}$,

$$-E\frac{\partial u_0}{\partial x}=(P_0-\sigma Q_0)=\text{constant} \quad . \quad . \quad . \quad . \quad (17.79)$$

$$-E\frac{w_0}{r}=(Q_0-\sigma P_0)=\text{constant} \quad . \quad . \quad . \quad . \quad (17.80)$$

Thus u_0 is a linear function of x, and w_0 is a constant.

Let us write u_1 and w_1 for (u_0+u) and (w_0+w). Then the longitudinal strains in the middle surface in the directions of x and θ are

$$\alpha=\frac{\partial u_1}{\partial x}=\frac{\partial u_0}{\partial x}+\frac{\partial u}{\partial x}, \quad \ldots \ldots \ldots \quad (17.81)$$

$$\beta=\lim_{d\theta\to 0}\frac{(r+w_1)d(\theta+\eta)-rd\theta}{rd\theta}$$

$$=\frac{w_1}{r}+\frac{\partial\eta}{\partial\theta}=\frac{w_0}{r}+\frac{w}{r}+\frac{\partial\eta}{\partial\theta}. \quad \ldots \ldots \quad (17.82)$$

Also the shear strain in the middle surface is, by analogy with (13.60)

$$c=r\frac{\partial\eta}{\partial x}+\frac{1}{r}\frac{\partial u_1}{\partial\theta}$$

$$=r\frac{\partial\eta}{\partial x}+\frac{1}{r}\frac{\partial u}{\partial\theta} \quad \ldots \ldots \ldots \quad (17.83)$$

since u_0 is not a function of θ.

The formula for the curvature of a curve in terms of polar co-ordinates (r, θ) is

$$\frac{1}{\varrho}=\frac{1}{r}\frac{1+\frac{2}{r^2}\left(\frac{dr}{d\theta}\right)^2-\frac{1}{r}\frac{d^2r}{d\theta^2}}{\left\{1+\frac{1}{r^2}\left(\frac{dr}{d\theta}\right)^2\right\}^{\frac{3}{2}}}. \quad \ldots \ldots \quad (17.84)$$

Now if the curve is nearly a circle, so that r differs very little from a constant, we may neglect squares of the differential coefficients of r. In that case the formula becomes

$$\frac{1}{\varrho}=\frac{1}{r}\left(1-\frac{1}{r}\frac{d^2r}{d\theta^2}\right).$$

Writing $(a+v)$ for r in this, we find, carrying the approximation as far as the first order in v and its differential coefficients,

$$\frac{1}{\varrho}=\frac{1}{a+v}\left(1-\frac{1}{a+v}\frac{d^2v}{d\theta^2}\right)$$

$$=\frac{1}{a}\left(1-\frac{v}{a}\right)\left(1-\frac{1}{a}\frac{d^2v}{d\theta^2}\right)$$

$$=\frac{1}{a}\left(1-\frac{v}{a}-\frac{1}{a}\frac{d^2v}{d\theta^2}\right). \quad \ldots \ldots \ldots \quad (17.85)$$

We can now write down the changes of curvature of the sections
We can now write down the changes of curvature of the sections
of the tube on which M_1 and M_2 act. These changes, reckoned positive when M_1 and M_2 are positive, are

$$c_1=\frac{\partial^2 w_1}{\partial x^2}=\frac{\partial^2 w}{\partial x^2} \quad \ldots \ldots \ldots \quad (17.86)$$

$$c_2 = -\frac{1}{r}\left(1 - \frac{w_1}{r} - \frac{1}{r}\frac{\partial^2 w_1}{\partial\theta^2}\right) + \frac{1}{r} \text{ by (17.85)},$$

$$= \frac{1}{r^2}\left(w_1 + \frac{\partial^2 w_1}{\partial\theta^2}\right) = \frac{1}{r^2}\left(w_0 + w + \frac{\partial^2 w}{\partial\theta^2}\right). \quad . \quad . \quad . \quad (17.87)$$

The twist per unit length of an element of the middle surface whose sides are approximately dx and $rd\theta$ is the angle through which the element $rd\theta$ at one end is twisted relative to the corresponding element at the other, divided by dx. Now, at the point x, r, θ, the inclination

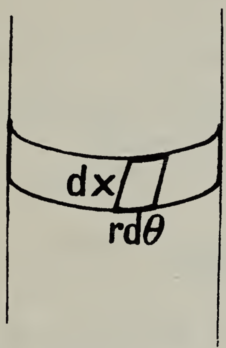


Fig. 170a

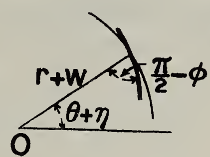


Fig. 170b

of the tangent at the middle of the element $rd\theta$ to the radius through that point is $(\frac{1}{2}\pi - \varphi)$, and φ is given by the equation

$$\frac{\partial w}{r\partial\theta} = \tan\varphi = \varphi \text{ nearly.} \quad . \quad . \quad . \quad . \quad . \quad . \quad (17.88)$$

But the radius through the element at x is turned through an angle η in the contrary direction to φ. Consequently the whole rotation of the element at x is $(\varphi - \eta)$, and the rotation of the corresponding element at $x + dx$ is $\varphi - \eta + d\varphi - d\eta$. That is, the twist per unit length about dx is

$$\tau = \frac{\partial\varphi}{\partial x} - \frac{\partial\eta}{\partial x}$$

$$= \frac{\partial}{\partial x}\left(\frac{1}{r}\frac{\partial w}{\partial\theta}\right) - \frac{\partial\eta}{\partial x}$$

$$= \frac{1}{r}\frac{\partial^2 w}{\partial x\partial\theta} - \frac{\partial\eta}{\partial x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17.89)$$

to the first order in w and its differential coefficients.

The relations between the stresses and strains are

$$\left.\begin{aligned} -\mathrm{E}\alpha &= \mathrm{P}_0+\mathrm{P}-\sigma(\mathrm{Q}_0+\mathrm{Q}) \\ -\mathrm{E}\beta &= \mathrm{Q}_0+\mathrm{Q}-\sigma(\mathrm{P}_0+\mathrm{P}) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (17.90)$$

$$nc = \mathrm{S}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17.91)$$

By means of (17.79), (17.80), (17.81), (17.82), these give

$$-\mathrm{E}\frac{\partial u}{\partial x} = \mathrm{P}-\sigma\mathrm{Q} \quad . \quad . \quad . \quad . \quad . \quad . \quad (17.92)$$

$$-\mathrm{E}\left(\frac{w}{r}+\frac{\partial \eta}{\partial \theta}\right) = \mathrm{Q}-\sigma\mathrm{P} \quad . \quad . \quad . \quad . \quad . \quad (17.93)$$

$$n\left(r\frac{\partial \eta}{\partial x}+\frac{1}{r}\frac{\partial u}{\partial \theta}\right) = \mathrm{S}. \quad . \quad . \quad . \quad . \quad . \quad (17.94)$$

The equations connecting the couples with the changes of curvature and the twist are, just as for flat plates (Chap. 14),

$$\mathrm{M}_1 = \mathrm{E'I}(c_1+\sigma c_2) \quad . \quad . \quad . \quad . \quad . \quad . \quad (17.95)$$

$$\mathrm{M}_2 = \mathrm{E'I}(c_2+\sigma c_1) \quad . \quad . \quad . \quad . \quad . \quad . \quad (17.96)$$

$$\mathrm{H} = 2n\mathrm{I}\tau = (1-\sigma)\mathrm{E'I}\tau. \quad . \quad . \quad . \quad . \quad . \quad (17.97)$$

319. Equations of equilibrium.

We have now to find the equations of equilibrium of an element of the cylinder of dimensions $dx \times ds$, the dimension ds being originally the same as $rd\theta$.

It is to be understood that the stresses $(\mathrm{P}_0+\mathrm{P})$ and $(\mathrm{Q}_0+\mathrm{Q})$ act in the directions of the strained elements whose original lengths were dx and $rd\theta$. If we took these stresses parallel to the unstrained directions of the elements they would not be stresses in the middle surface, but would be slightly inclined to this surface. The difference does not matter except when one of the stresses becomes very big, as P_0 does when the tube is buckled by an end-thrust. It has to be remembered that the theory of a thin plate, as also of a thin rod, was originally worked out only for a small element. The axes of reference for the plate were fixed relatively to the normal to the middle surface of the element. By taking the stresses to be always in the middle surface we are, in effect, shifting our axes of reference to suit the element we are dealing with.

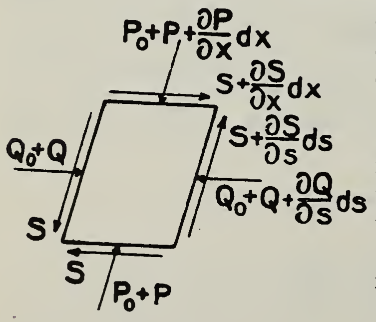


Fig. 171

Fig. 171 gives a view from the convex side of the element. Resolving in the direction of the x-axis we get

$$2hds\frac{\partial P}{\partial x}dx - 2hdx\frac{\partial S}{\partial s}ds - 2h(Q_0 + Q)dx\frac{\partial^2 u}{\partial s^2}ds = 0, \quad (17.98)$$

the quantity $\frac{\partial^2 u}{\partial s^2}ds$ being the difference of the inclinations of $(Q_0 + Q)$ and $(Q_0 + Q + \frac{\partial Q}{\partial s}ds)$ to the axis of x.

The correct form of the last term actually given in equation (17.98) is

$$2hdx\frac{\partial}{\partial s}\left\{(Q_0 + Q)\frac{\partial u}{\partial s}\right\}ds.$$

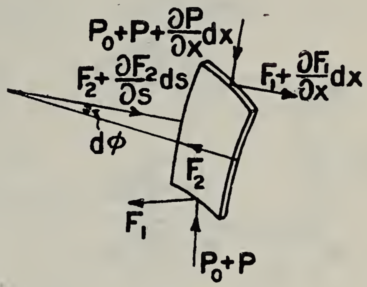


Fig. 172

But this differs from the last term actually given in that equation by

$$2hdx\frac{\partial Q}{\partial s}\frac{\partial u}{\partial s}ds,$$

which is a quantity of lower order than the other terms in the equation because Q itself is of the same order as u and w. This is not the case with Q_0, since this can have a finite value while u and w are zero. By a similar argument we ought to drop the term Q in the sum $(Q_0 + Q)$. Then neglecting Q and dividing through (17.98) by $2hdxds$ we get

$$\frac{\partial P}{\partial x} - \frac{\partial S}{\partial s} - Q_0\frac{\partial^2 u}{\partial s^2} = 0. \quad (17.99)$$

Next resolving the forces in the direction of the normal through the middle of the element and neglecting P in comparison with P_0, we get

$$2hdsP_0\frac{\partial \psi}{\partial x}dx - dx\frac{\partial F_2}{\partial s}ds - ds\frac{\partial F_1}{\partial x}dx$$
$$- 2hdx(Q_0 + Q)d\varphi + p\,ds\,dx = 0, \quad (17.100)$$

p being the external pressure per unit area on the element, and ψ the angle which $(P_0 + P)$ makes with the x-axis. But

$$\psi = \frac{\partial w}{\partial x}.$$

Therefore, on dividing equation (17.100) by $dxds$, we get

$$2hP_0\frac{\partial^2 w}{\partial x^2} - \frac{\partial F_2}{\partial s} - \frac{\partial F_1}{\partial x} - 2h(Q_0 + Q)\frac{d\varphi}{ds} + p = 0. \quad (17.101)$$

The reason why Q is not neglected in this last equation is because it is multiplied by the quantity $\frac{d\varphi}{ds}$, which is not of the same order as the displacements.

We have still to resolve along the unstrained direction of *ds*. This gives, again neglecting P in comparison with P_0,

$$2hdx\frac{\partial Q}{\partial s}ds + 2hP_0 ds\frac{\partial^2(r\eta)}{\partial x^2}dx - 2hds\frac{\partial S}{\partial x}dx - F_2\,dx\,d\varphi = 0;$$

whence $$2h\frac{\partial Q}{\partial s} + 2hrP_0\frac{\partial^2\eta}{\partial x^2} - 2h\frac{\partial S}{\partial x} - F_2\frac{d\varphi}{ds} = 0 \quad . \quad . \quad (17.102)$$

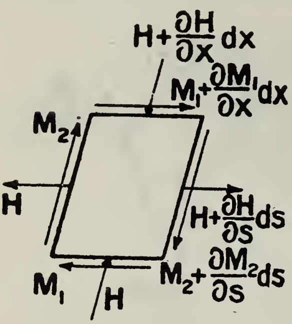


Fig. 173

The couples acting on the element are represented by vectors in Fig. 173, which is again a view from the convex side of the element.

Taking moments about one of the edges *dx* we get

$$dx\frac{\partial M_2}{\partial s}ds + ds\frac{\partial H}{\partial x}dx + F_2\,dx\,ds = 0;$$

from which

$$\frac{\partial M_2}{\partial s} + \frac{\partial H}{\partial x} + F_2 = 0. \quad . \quad (17.103)$$

Similarly, by taking moments about one of the elements *ds*, we arrive at the equation

$$\frac{\partial M_1}{\partial x} + \frac{\partial H}{\partial s} + F_1 = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (17.104)$$

The expression $\frac{d\varphi}{ds}$ occurs twice in the equations of equilibrium, and where it occurs it clearly means the curvature of the strained circular element; that is, by (17.87),

$$\frac{d\varphi}{ds} = \frac{1}{r}\left\{1 - \frac{w_1}{r} - \frac{1}{r}\frac{\partial^2 w_1}{\partial\theta^2}\right\}$$
$$= \frac{1}{r} \quad \frac{1}{r^2}\left(w_0 + w + \frac{\partial^2 w}{\partial\theta^2}\right)$$
$$= \frac{1}{r+w_0} - \frac{1}{r^2}\left(w + \frac{\partial^2 w}{\partial\theta^2}\right) \quad . \quad . \quad . \quad . \quad (17.105)$$

approximately, the terms in brackets on the last line being small when the form of the tube differs but slightly from that of a circular cylinder. These terms are, in fact, of the same order as Q and F_2. Consequently

$$(Q_0 + Q)\frac{d\varphi}{ds} = \frac{Q_0 + Q}{r + w_0} - \frac{Q_0}{r^2}\left(w + \frac{\partial^2 w}{\partial\theta^2}\right), \quad . \quad . \quad (17.106)$$

quantities of the second order in *w* being neglected. Also, to the same degree of approximation,

$$F_2 \frac{d\varphi}{ds} = \frac{F_2}{r + w_0} = \frac{F_2}{r} \text{ nearly} \quad . \quad . \quad . \quad . \quad (17.107)$$

Wherever else ds occurs in the equations of equilibrium it can be replaced by $rd\theta$ without affecting the accuracy.

Since Q_0 is the circumferential stress while the tube is a circular cylinder of radius $(r + w_0)$, that is, while w, F_1, F_2, are all zero, and since equation (17.101) must remain true for this particular condition of the tube, it follows from that equation and from (17.107) that

$$\frac{2hQ_0}{r + w_0} = p \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17.108)$$

When the difference between r and $(r + w_0)$ is neglected equation (17.101) therefore reduces to

$$2hP_0 \frac{\partial^2 w}{\partial x^2} - \frac{1}{r}\frac{\partial F_2}{\partial \theta} - \frac{\partial F_1}{\partial x} - \frac{2hQ}{r} + \frac{2hQ_0}{r^2}\left(w + \frac{\partial^2 w}{\partial \theta^2}\right) = 0 \quad (17.109)$$

The other equations of equilibrium, namely, (17.99), (17.102), (17.103), (17.104), can be written thus

$$\frac{\partial P}{\partial x} - \frac{1}{r}\frac{\partial S}{\partial \theta} - \frac{Q_0}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0, \quad . \quad . \quad . \quad . \quad (17.110)$$

$$\frac{2h}{r}\frac{\partial Q}{\partial \theta} + 2hrP_0\frac{\partial^2 \eta}{\partial x^2} - 2h\frac{\partial S}{\partial x} - \frac{F_2}{r} = 0, \quad . \quad . \quad . \quad (17.111)$$

$$\frac{1}{r}\frac{\partial M_2}{\partial \theta} + \frac{\partial H}{\partial x} + F_2 = 0, \quad . \quad . \quad . \quad . \quad . \quad (17.112)$$

$$\frac{\partial M_1}{\partial x} + \frac{1}{r}\frac{\partial H}{\partial \theta} + F_1 = 0. \quad . \quad . \quad . \quad . \quad . \quad (17.113)$$

The last five equations, together with the equations expressing the stresses and couples in terms of the displacements, are sufficient to solve any problem in which the displacements are small quantities. The important equations are the last five and equations (17.92) to (17.97).

Since the shear forces F_1 and F_2 occur only in the equations of equilibrium it is worth while to eliminate them. We can do this by taking their values from (17.112) and (17.113) and substituting in the other equations of equilibrium. This process transforms (17.109) and (17.111) into

$$2hP_0\frac{\partial^2 w}{\partial x^2} + \frac{1}{r^2}\frac{\partial^2 M_2}{\partial \theta^2} + \frac{2}{r}\frac{\partial^2 H}{\partial \theta \partial x} + \frac{\partial^2 M_1}{\partial x^2}$$
$$- \frac{2hQ}{r} + \frac{2hQ_0}{r^2}\left(w + \frac{\partial^2 w}{\partial \theta^2}\right) = 0. \quad . \quad . \quad . \quad . \quad (17.114)$$

and

$$\frac{2h}{r}\frac{\partial Q}{\partial \theta} + 2hrP_0\frac{\partial^2 \eta}{\partial x^2} - 2h\frac{\partial S}{\partial x} + \frac{1}{r^2}\frac{\partial M_2}{\partial \theta} + \frac{1}{r}\frac{\partial H}{\partial x} = 0 \quad . \quad (17.115)$$

The three equations of equilibrium not containing F_1 and F_2 are now (17.110), (17.114), (17.115).

320. Stability of a circular tube.

When the tube begins to buckle, due either to a pressure p on the outside of the tube or to an axial thrust applied at the ends, the displacements are small oscillatory functions of x and θ which can be expressed by the equations

$$\left.\begin{aligned} u &= A\cos m\theta\cos\frac{kx}{r} \\ r\eta &= B\sin m\theta\sin\frac{kx}{r} \\ w &= C\cos m\theta\sin\frac{kx}{r} \end{aligned}\right\} \quad . \quad . \quad (17.116)$$

With these expressions for the displacements equations (17.92) to (17.97) give

$$P = \frac{E'}{r}(kA-\sigma C-\sigma mB)\cos m\theta\sin\frac{kx}{r} \quad . \quad . \quad (17.117)$$

$$Q = \frac{E'}{r}(\sigma kA-C-mB)\cos m\theta\sin\frac{kx}{r}. \quad . \quad . \quad (17.118)$$

$$S = \frac{n}{r}(kB-mA)\sin m\theta\cos\frac{kx}{r}$$

$$= \tfrac{1}{2}(1-\sigma)\frac{E'}{r}(kB-mA)\sin m\theta\cos\frac{kx}{r}. \quad . \quad . \quad (17.119)$$

$$M_1 = \frac{E'I}{r^2}\left\{-k^2+\sigma(1-m^2)\right\}C\cos m\theta\sin\frac{kx}{r}+\frac{E'I}{r^2}\sigma w_0 \quad . \quad (17.120)$$

$$M_2 = \frac{E'I}{r^2}\left\{(1-m^2)-\sigma k^2\right\}C\cos m\theta\sin\frac{kx}{r}+\frac{E'I}{r^2}w_0 \quad . \quad . \quad (17.121)$$

$$H = (1-\sigma)\frac{E'I}{r^2}\left\{-mkC-kB\right\}\sin m\theta\cos\frac{kx}{r}. \quad . \quad . \quad (17.122)$$

On substituting these values in (17.110), (17.114), (17.115), and dividing each equation by a suitable factor we arrive at the following equations:—

$$\left\{m^2\frac{Q_0}{E'}+k^2+\tfrac{1}{2}(1-\sigma)m^2\right\}A-\tfrac{1}{2}(1+\sigma)mkB-\sigma kC = 0 \quad . \quad (17.123)$$

$$-\sigma kA+\left\{2(1-\sigma)mk^2f+m\right\}B-\left\{k^2\frac{P_0}{E'}+(m^2-1)\frac{Q_0}{E'}-1\right\}C$$
$$+f\left\{(m^2+k^2)^2-m^2-\sigma k^2\right\}C = 0 \quad . \quad . \quad . \quad (17.124)$$

$$-\tfrac{1}{2}(1+\sigma)mkA+\left\{-k^2\frac{P_0}{E'}+m^2+\tfrac{1}{2}(1-\sigma)(1+2f)k^2\right\}B$$
$$+mC+f\{m(m^2-1)+mk^2\}C=0, \quad . \quad . \quad . \quad (17.125)$$

where f is written for $\frac{h^2}{3r^2}$.

From the last three equations the two ratios A : B : C can be eliminated, the resulting equation giving a relation between P_0, Q_0, m, k, f, and the elastic constants. If Q_0 is zero then the equation gives P_0; and if P_0 is zero it gives Q_0, and therefore the pressure p. The constant k is something that depends on the end-conditions of the tube, whereas m must be an integer since w must have the same value when $\theta=0$ and when $\theta=2\pi$. The eliminant of the last three equations contains, as particular cases, the two results given in (17.20) and (17.42).

The equation obtained by eliminating A, B, C, from (17.123), (17.124), (17.125), is expressed by equating to zero the determinant

$$\begin{vmatrix} 1-k^2\dfrac{P_0}{E'}-(m^2-1)\dfrac{Q_0}{E'} & m+2(1-\sigma)mk^2f & \sigma k \\ +f\{(m^2+k^2)^2-m^2-\sigma k^2\} & & \\ m+mf(m^2+k^2-1) & m^2+\tfrac{1}{2}(1-\sigma)(1+2f)k^2-k^2\dfrac{P_0}{E'} & \tfrac{1}{2}(1+\sigma)mk \\ \sigma k & \tfrac{1}{2}(1+\sigma)mk & m^2\dfrac{Q_0}{E'}+k^2 \\ & & +\tfrac{1}{2}(1-\sigma)m^2 \end{vmatrix}$$

Now $\frac{P_0}{E'}$ and $\frac{Q_0}{E'}$ are certainly small fractions in any practical case. Moreover f is also a small fraction. Then, in expanding the determinant, we may safely neglect squares and products of these three small quantities. Expanding the determinant under these conditions we get, as the eliminant,

$$-\tfrac{1}{2}(1-\sigma)k^2\frac{P_0}{E'}\{(m^2+k^2)^2+m^2+2k^2+2\sigma k^2\}$$
$$-\tfrac{1}{2}(1-\sigma)\frac{Q_0}{E'}\{(m^2-1)(m^2+k^2)^2-m^2k^2\}$$
$$+\tfrac{1}{2}(1-\sigma)f\left\{\begin{matrix}(m^2+k^2)^4+m^4+3m^2k^2+2(1-\sigma)k^4\\ -2m^6-7m^4k^2-(7+\sigma-2\sigma^2)m^2k^4-\sigma k^6\end{matrix}\right\}$$
$$+\tfrac{1}{2}(1-\sigma)(1-\sigma^2)k^4=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17.126)$$

In order to find the pressure that will cause the tube to collapse when no axial force is applied we must put $P_0=0$ in the last equation.

Then, if the terms containing the small fraction f can be neglected, we get as a first approximation,

$$Q_0 = \frac{(1-\sigma^2)k^4 E'}{(m^2-1)(m^2+k^2)^2 - m^2k^2}$$
$$= \frac{k^4 E}{(m^2-1)(m^2+k^2)^2 - m^2k^2}. \quad . \quad . \quad . \quad (17.127)$$

Now the smallest value of m that is possible for a tube collapsing under an external pressure is $m=2$, which corresponds to an elliptical form of the section of the tube. Consequently, since Q_0 must be very much smaller than E, it follows from the approximate expression for Q_0 in the last equation that k must be a small fraction. Then regarding k as much smaller than m, and therefore taking only the terms independent of k in the coefficient of f, we get a more correct value of Q_0 from the following equation

$$\frac{Q_0}{E'} = \frac{(1-\sigma^2)k^4}{(m^2-1)(m^2+k^2)^2 - m^2k^2} + \frac{m^4(m^2-1)^2 f}{(m^2-1)(m^2+k^2)^2 - m^2k^2} \quad (17.128)$$

which becomes, when k is neglected in the denominators,

$$\frac{Q_0}{E'} = \frac{(1-\sigma^2)k^4}{m^4(m^2-1)} + (m^2-1)f \quad . \quad . \quad . \quad . \quad (17.129)$$

If the tube has a length l, and if the end conditions are such that $w=0$ at both ends, then

$$k = \frac{\pi r}{l}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17.130)$$

These end conditions would be realised if a thin circular plug of radius $(r+w_0)$ were inserted at the ends of the tube; for it is clear that w and M_1 are both zero at points where kx is equal to zero or πr, and these are precisely the end conditions when the thin plugs are in position. In such a case therefore the pressure is given by

$$\frac{p}{2hE'} = \frac{1}{r+w_0}\frac{Q_0}{E'} = \frac{1}{r}\frac{Q_0}{E'} \text{ nearly}$$
$$= \frac{1-\sigma^2}{m^4(m^2-1)}\frac{\pi^4 r^3}{l^4} + \frac{1}{3}(m^2-1)\frac{h^2}{r^3}. \quad . \quad . \quad (17.131)$$

Now the tube collapses when p has the least value consistent with this last equation, m being some integer greater than unity. But if m were not restricted to integral values we could find the minimum value of p consistent with (17.131) for variations in m^2. The condition that the expression for p should be a minimum is

$$\frac{1}{2hE'}\frac{dp}{d(m^2)} = 0;$$

that is,

$$\frac{2}{m^6(m^2-1)}+\frac{1}{m^4(m^2-1)^2}=\frac{1}{3(1-\sigma^2)}\frac{h^2l^4}{\pi^4r^6} \quad . \quad . \quad (17.132)$$

Now as m increases from 1 to ∞ the left hand side of this last equation steadily decreases from ∞ to 0, and consequently there is one, and only one, value of m satisfying the equation for any given values of h, l, and r. If this value of m is an integer the corresponding value of p is the collapsing pressure. But if the value of m satisfying (17.132) lies between the two integers m_1 and (m_1+1) then the true collapsing pressure is the smaller of the two values of p obtained by putting m_1 and (m_1+1) for m in (17.131).

Equation (17.132) shows that, for given values of h and r, m decreases as l increases, and for very big values of l the value of m given by that equation is less than 2. For such big values of l the collapsing pressure is given by $m=2$. Moreover, when l is infinite, equation (17.131) gives

$$p=\frac{2}{3}(2^2-1)\frac{E'h^3}{r^3}=\frac{2E'h^3}{r^3}, \quad . \quad . \quad . \quad . \quad (17.133)$$

just as in (17.24).

Thus we see that the section of a tube may collapse under an external pressure into a two-lobed or elliptical form, or a three-lobed form, or into a form having a greater number of lobes, according to the value of the fraction on the right hand side of (17.132). This explanation of the behaviour of tubes under pressure was first given by Mr. R. V. Southwell* in 1913. In the Royal Society paper Mr. Southwell also gave the theory of collapse of tubes under axial thrust. There is a slight difference between Mr. Southwell's equations of equilibrium and those given in this chapter, but the difference is in terms that have no importance.

The type of solution we have assumed in (17.116) cannot be made to satisfy the conditions that w and $\frac{\partial w}{\partial x}$ are both zero at the same point, and these are the conditions at a clamped end. This type of solution does, however, make $\frac{\partial w}{\partial x}$ zero where $kx=\pm\frac{\pi}{2}r$. If these two planes be taken as the ends of the tube, and if the tube has a length l, we get

$$kl=\frac{\pi}{2}r-\left(-\frac{\pi}{2}r\right)=\pi r.$$

*) "*On the General Theory of Elastic Stability,*", by R. V. Southwell B.A., Phil. Trans. Royal Society, Series A, Vol. 213, pp 187—244; and "*On the Collapse of Tubes under External Pressure*", by R. V. Southwell B.A., Phil. Mag. May 1913.

This gives the same value of k as we got for the plugged ends. It will be noticed, however, that neither w nor η is zero at the ends in this case, so that it would be very difficult to realise these end-conditions in practice.

In order to satisfy any other conditions except those at the plugged ends it would be necessary to get the general solution of our differential equations and then make this satisfy the given conditions. But the analysis is so cumbersome that it is doubtful if it is worth while in any case. The general value of w would have the form

$$w = \Sigma C e^{n_1 x} \cos m\theta \sin n_2 x, \quad . \quad . \quad . \quad . \quad (17.134)$$

the number of terms in the sum indicated by Σ being eight.

321. Collapse of tube under axial thrust.

If the pressure p is zero, and therefore also Q_0 zero then equation (17.126) gives the axial stress P_0 at which the tube begins to buckle. Now m may be zero or an integer. In any case we get

$$\frac{P_0}{E'} = \frac{(1-\sigma^2)k^2}{L} + \frac{N}{k^2 L} f. \quad . \quad . \quad . \quad . \quad (17.135)$$

L and N being the respective coefficients of $-\frac{1}{2}(1-\sigma)k^2 \frac{P_0}{E'}$ and $\frac{1}{2}(1-\sigma)f$ in (17.126). Now if m is not zero this last equation shows that k^2 must be small, since we know that P_0 is very much smaller than E′. In that case all powers of k^2 can be neglected in L and N. But if m is zero L contains a factor k^4, and therefore k^2 occurs in the denominator instead of in the numerator of the first term on the right of (17.135). The known fact that P_0 is much smaller than E′ now requires that k^2 should be big, and in this case we can neglect all except the highest powers of k. Then, retaining only the highest powers of k in the coefficient of f and in the term independent of f when m is zero, we get, as the approximate equations for P_0,

$$\frac{P_0}{E'} = \frac{(1-\sigma^2)k^2}{m^2(m^2+1)} + \frac{m^2(m^2-1)^2 f}{(m^2+1)k^2}, \quad . \quad . \quad . \quad (17.136)$$

when $m > 0$;

$$\frac{P_0}{E'} = \frac{1-\sigma^2}{k^2} + k^2 f \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17.137)$$

when $m = 0$.

The case where $m = 1$ is peculiar because the term containing f in (17.136) vanishes. The equation for this case is

$$\frac{P_0}{E'} = \frac{1}{2}(1-\sigma^2)k^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17.138)$$

The radial displacement for $m = 1$ is

$$w = C \cos\theta \sin\frac{kx}{r}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17.139)$$

This indicates that the whole circular section at x is displaced in the direction $\theta = 0$ through a distance $C \sin \frac{kx}{r}$, the section undergoing no change of shape or size. But this is precisely the displacement in Euler's theory of struts. It follows from (17.138) that the whole thrust on one end of the tube in this case is

$$4\pi r h P_0 = 2\pi r h k^2 E \quad . \quad . \quad . \quad . \quad . \quad , \quad (17.140)$$

Now if $w = 0$ at the ends of the tube where $x = 0$ and $x = l$, it follows that

$$\frac{kl}{r} = \pi. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17.141)$$

Consequently the whole thrust is

$$4\pi r h P_0 = \frac{2\pi^3 r^3 h E}{l^2}$$

$$= \frac{EI\pi^2}{l^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (17.142)$$

where I is the moment of inertia of the section of the tube about a diameter. This agrees exactly with Euler's theory of struts.

When m is not equal to unity k depends somehow on the end conditions of the tube and on the length of the tube. Usually these end conditions will be such that there must necessarily be a whole number of halfwave lengths on the tube. This means that k must usually not be less than the value determined by (17.141), and, if the present theory were strictly true in practice, kl would have to be an exact multiple of πr. But it is easily possible in practice, owing to slight irregularities not taken into account in this theory, for the tube to buckle so that kl is not an exact multiple of πr. The greater the number of waves on the curve the more easily could this happen. Then all that we can safely say is that kl cannot be less than πr and it may be very much greater. The more the ends of a given tube are constrained the bigger k is likely to be. Now regarding k as a variable, the least value of P_0 given by (17.136) is determined by

$$\frac{P_0}{E'} = 2\sqrt{\frac{(1-\sigma^2)k^2}{m^2(m^2+1)} \cdot \frac{m^2(m^2-1)^2 f}{(m^2+1)k^2}}$$

$$= \frac{2h}{r}\frac{m^2-1}{m^2+1}\sqrt{\tfrac{1}{3}(1-\sigma^2)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (17.143)$$

The corresponding value of k is the solution of the equation

$$\frac{(1-\sigma^2)k^2}{m^2(m^2+1)} = \frac{m^2(m^2-1)^2 f}{(m^2+1)k^2},$$

from which we get

$$k^4 = \frac{m^4(m^2-1)^2 f}{1-\sigma^2} = \frac{m^4(m^2-1)^2 h^2}{3(1-\sigma^2)r^2}. \quad . \quad . \quad . \quad (17.144)$$

Similarly the least value of P_0 given by (17.137) is determined by

$$\frac{P_0}{E'}=\frac{2h}{r}\sqrt{\tfrac{1}{3}(1-\sigma^2)}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (17.145)$$

and the corresponding value of k is given by

$$k^4=\frac{1-\sigma^2}{f}=\frac{3(1-\sigma^2)r^2}{h^2}. \quad . \quad . \quad . \quad . \quad (17.146)$$

The value of P_0 given by (17.145) is greater than that given by (17.143) for all values of m, but as m approaches infinity the two values approach equality. Moreover, the smaller m is, the smaller is the value of P_0 in (17.143). The least permissible value of m in (17.143) is 2, and this gives

$$\frac{P_0}{E'}=\frac{6}{5}\frac{h}{r}\sqrt{\tfrac{1}{3}(1-\sigma^2)}. \quad . \quad . \quad . \quad . \quad . \quad (17.147)$$

If this happens to be less than the value of $\frac{P_0}{E'}$ satisfying (17.138) then the tube will collapse at this stress provided the end conditions are suitable. But if the end conditions are not suitable, or if kl is less than πr, k being obtained from (17.144) by putting $m=2$, then the tube will not buckle at this stress and m will have some value greater than 2. If it has the value 3, in which case the section is a three lobed curve, the least possible value of P_0 is given by

$$\frac{P_0}{E'}=\frac{8}{5}\frac{h}{r}\sqrt{\tfrac{1}{3}(1-\sigma^2)}. \quad . \quad . \quad . \quad . \quad . \quad (17.148)$$

Thus we see that the shorter the tube is, the greater the value of m is likely to be, unless it becomes equal to zero, which gives the same value of P_0 as $m=\infty$ gives. The case $m=0$ is, of course, the case where the strain is symmetrical about the axis, which case has been dealt with separately earlier in this chapter. For a very short tube therefore the collapsing thrust is given by (17.145).

It should be recalled that we decided that k must always be small when m is not zero. Then as the length of the tube decreases k must get greater, but it can never be anything but a rather small fraction when m is not zero. Consequently m cannot get very big if it is connected with k by equation (17.144); that is, in practical cases m is never likely to be more than three or four. Unless the length is greater than twice the circumference it is impossible for k^2 to be treated as a small fraction. For such short tubes then k is not small, but big, and m is zero, and the collapsing thrust is still small. Thus if different lengths of the same tubing were tested the very long lengths would fail as Euler struts. Then, as the tubes were shortened, there would come a length which would fail by crumpling with $m=2$. Again a still shorter length might fail by crumpling with $m=3$. Still shorter lengths might possibly go on to $m=4$ and $m=5$, but there would come a stage at which the tube would fail at $m=0$. That is, after m had been increasing as the tube were gradually shortened there would come a critical length at which m would jump to zero from some number, such as three or four, and for all shorter tubes m would be zero.

322. Collapse of tube under combined pressure and axial thrust.

If m is not zero k must be small and therefore the important terms in equation (17.126) are given in the following equation

$$k^2(m^4+m^2)\frac{P_0}{E'}+m^4(m^2-1)\frac{Q_0}{E'}=(1-\sigma^2)k^4+m^4(m^2-1)^2 f,$$

whence

$$\frac{Q_0}{E'}=\frac{(1-\sigma^2)k^4}{m^4(m^2-1)}+(m^2-1)f-\frac{k^2(m^2+1)}{m^2(m^2-1)}\frac{P_0}{E'}. \quad (17.149)$$

Thus the existence of the thrust P_0 causes the tube to collapse for a smaller value of Q_0 than if P_0 were zero. If P_0 is negative, thus representing a tension, the pressure that will cause collapse will be greater than if P_0 were zero.

When k has been determined from the end conditions of the tube the collapsing value of Q_0 is the smallest value given by (17.149) when m is any integer above unity. When P_0 is fairly big the term containing P_0 in (17.149) is so important that m will be a small integer. But the only way to discover the smallest value of Q_0 is to substitute in turn $m=2, 3, 4, \ldots$ etc. in this equation and pick out the smallest value of Q_0. For given values of k, f, and P_0, it is easy to see that Q_0 has only one minimum value, and consequently it is easy to pick out the smallest value of Q_0.

323. Strain energy in a curved plate.

It has been already mentioned in Art. 211 that when we try to express the energy in a curved plate or rod in powers of the thickness and the curvatures of the middle surface we get into difficulties with the terms involving the cube of the thickness. Nevertheless we can get an expression which is a good enough approximation in nearly all cases.

Let ρ_1, ρ_2, denote the strained radii of curvature of two perpendicular sections at any point O of the middle surface, both sections containing the normal to the surface at that point. Let ρ'_1, ρ'_2, denote the unstrained radii of curvature of the same two sections, and let M_1, M_2, denote the bending moments across the sections. Also let τ denote the twist of the surface per unit length in the directions perpendicular to the two sections, and let τ_0 be the unstrained twist, and Q the torque in the strained state. Then the following are the approximate equations connecting the curvatures and twist with the couples.

$$M_1 = E'I(c_1 + \sigma c_2), \quad \ldots \quad (17.150)$$

$$M_2 = E'I(c_2 + \sigma c_1), \quad \ldots \quad (17.151)$$

$$Q = 2nI(\tau - \tau_0)$$
$$= (1-\sigma)E'I(\tau - \tau_0), \quad \ldots \quad (17.152)$$

where

$$c_1 = \frac{1}{\varrho_1} - \frac{1}{\varrho'_1}, \quad \ldots \quad (17.153)$$

$$c_2 = \frac{1}{\varrho_2} - \frac{1}{\varrho'_2}. \quad \ldots \quad (17.154)$$

Again let P_1, P_2, denote the tensional stress in the middle surface across the sections where M_1, and M_2 act. Also let S denote the shear stress on the same sections in the plane of the middle surface. Denoting the strains corresponding to P_1, P_2, S, by α, β, c, we have, by (15.14), (15.15), (15.16),

$$P_1 = E'(\alpha + \sigma\beta), \quad \ldots \quad (17.155)$$

$$P_2 = E'(\beta + \sigma\alpha), \quad \ldots \quad (17.156)$$

$$S = nc = \tfrac{1}{2}(1-\sigma)E'c \quad \ldots \quad (17.157)$$

Equations (17,150), (17,151), (17,152), are consistent with the corresponding equations for a naturally flat plate. Take, for example (14.32). We may write this in the form

$$M_1 = E'I\left\{\frac{1}{\varrho_1} + \frac{\sigma}{\varrho_2}\right\}. \quad \ldots \quad (17.158)$$

If M'_1 denotes the bending moment in the naturally flat plate corresponding to principal radii of curvature ϱ'_1 and ϱ'_2, we get

$$M'_1 = E'I\left(\frac{1}{\varrho'_1} + \frac{\sigma}{\varrho'_2}\right). \quad \ldots \quad (17.159)$$

Therefore by subtracting (17.159) from (17.158) we find

$$M_1 - M'_1 = E'I(c_1 + \sigma c_2).$$

This agrees exactly with the expressions for the curved plate because M'_1 is zero in that case.

By the method used in Art. 263 we can show that the energy in unit area of the bent surface due to the changes of curvature and twist is

$$\tfrac{1}{2}\left\{M_1 c_1 + M_2 c_2 + 2Q(\tau - \tau_0)\right\} \quad \ldots \quad (17.160)$$

Also the energy due to strains in the middle surface is, by (13,75),

$$\tfrac{1}{2} \times 2h\left\{P_1\alpha + P_2\beta + Sc\right\}. \quad \ldots \quad (17.161)$$

Thus the total energy per unit area is

$$V = h\{P_1\alpha + P_2\beta + Sc\} + \tfrac{1}{2}\{M_1c_1 + M_2c_2 + 2Q(\tau-\tau_0)\}$$
$$= E'h\{\alpha^2+\beta^2+2\sigma\alpha\beta+\tfrac{1}{2}(1-\sigma)c^2\}$$
$$+\tfrac{1}{2}E'I\{c_1{}^2+c_2{}^2+2\sigma c_1c_2+2(1-\sigma)(\tau-\tau_0)^2\}. \quad (17.162)$$

Since I contains a factor h^3 it might appear that the terms in the expression for V that contain I would become less and less important as h decreased in magnitude. Actually, however, the smaller h is, the more easily does the surface bend under given forces, and therefore the greater are the magnitudes of c_1 and c_2. It turns out then that the terms containing I really grow in importance as h decreases instead of waning as we might have expected in our first hasty judgment. In the problem of the buckling tube under an axial thrust the terms containing h and h^3 have equal importance.

324. The buckling tube problem by the minimum energy method.

The method consists in assuming reasonable expressions for the displacements — these expressions containing adaptable constants — and determining these constants by making the total potential energy of the internal and applied forces a minimum or maximum. In finding the strain energy we may, by the reasoning in Art. 146, neglect u_0 and w_0, because these are the displacements in an equilibrium position. Consequently, in the notation of Art. 318, we may calculate the strains in equation (17.162) by taking the displacements to be u, η, w, and not $u+u_0$, η, $w+w_0$.

We shall here deal only with the collapse of a tube under an axial thrust P_0. We have then to find the work done by P_0 due to the displacements u, η, w. To do this we have to find by how much the tube shortens in consequence of the strains; that is, we have to find the contraction of a line of particles which lie on a generating line of the cylinder in its unstrained state.

Let dx denote the distance between two particles m and m_1 in the middle surface of the cylinder just before buckling occurs. If we take $\frac{\partial u}{\partial x}$ to be the strain in the middle surface we may regard du as a displacement measured in this middle surface. Consequently the new length of the element which was dx in the unstrained state is $(dx+du)$. Now if dx_1 is the projection of $(dx+du)$ on the axis the relative coordinates of the particles m and m_1 in the buckled state are

$$dx_1,\ r\frac{\partial \eta}{\partial x}dx,\ \frac{\partial w}{\partial x}dx.$$

Therefore

$$(dx+du)^2=(dx_1)^2+\left(r\frac{\partial\eta}{\partial x}dx\right)^2+\left(\frac{\partial w}{\partial x}dx\right)^2,$$

whence

$$(dx_1)^2=\left\{\left(1+\frac{\partial u}{\partial x}\right)^2+r^2\left(\frac{\partial\eta}{\partial x}\right)^2+\left(\frac{\partial w}{\partial x}\right)^2\right\}(dx)^2$$

$$=\left(1+\frac{\partial u}{\partial x}\right)^2\left\{1+r^2\left(\frac{\partial\eta}{\partial x}\right)^2+\left(\frac{\partial w}{\partial x}\right)^2\right\}(dx)^2 \text{ approximately.}$$

Consequently, by obvious approximations,

$$dx_1=\left(1+\frac{\partial u}{\partial x}\right)\left\{1+\frac{1}{2}r^2\left(\frac{\partial\eta}{\partial x}\right)^2+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right\}dx,$$

and therefore

$$dx_1-dx=\left\{\frac{\partial u}{\partial x}+\frac{1}{2}r^2\left(\frac{\partial\eta}{\partial x}\right)^2+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right\}dx \quad . \quad . \quad (17.163)$$

Thus the work done by P_0 on a cylinder of length l is

$$W=2hP_0\int_0^{2\pi}\int_0^l\left\{\frac{\partial u}{\partial x}+\frac{1}{2}r^2\left(\frac{\partial\eta}{\partial x}\right)^2+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right\}dx\,r\,d\theta. \quad (17.164)$$

Now let us take the same values of u, η, w, as in equations (17.116). Then

$$a=\frac{\partial u}{\partial x}=-\frac{k}{r}A\cos m\theta\sin\frac{kx}{r},$$

$$\beta=\frac{\partial\eta}{\partial\theta}+\frac{w}{r}=\frac{1}{r}(mB+C)\cos m\theta\sin\frac{kx}{r},$$

$$c=r\frac{\partial\eta}{\partial x}+\frac{1}{r}\frac{\partial u}{\partial\theta}$$

$$=\frac{1}{r}(kB-mA)\sin m\theta\cos\frac{kx}{r},$$

$$c_1=\frac{\partial^2 w}{\partial x^2}=-\frac{k^2}{r^2}C\cos m\theta\sin\frac{kx}{r},$$

$$c_2=\frac{1}{r^2}\left(w+\frac{\partial^2 w}{\partial\theta^2}\right)$$

$$=\frac{1}{r^2}(1-m^2)C\cos m\theta\sin\frac{kx}{r},$$

$$\tau=\frac{1}{r}\frac{\partial^2 w}{\partial x\partial\theta}-\frac{\partial\eta}{\partial x}$$

$$=-\frac{k}{r^2}(mC+B)\sin m\theta\cos\frac{kx}{r},$$

$$\tau_0=0.$$

If V_1 denotes the strain energy per unit area of the tube the total potential energy of the internal and external forces is

$$V = \int_0^{2\pi} \int_0^l V_1 \, dx r d\theta - W. \quad . \quad . \quad . \quad . \quad (17.165)$$

Now

$$\int_0^{2\pi} \sin^2 m\theta \, d\theta = \int_0^{2\pi} \cos^2 m\theta \, d\theta = \pi.$$

Also, if the length l of the tube contains a whole number of half wave lengths; in short, if

$$\frac{kl}{r} = n\pi,$$

n being an integer, we find that

$$\int_0^l \sin^2 \frac{kx}{r} dx = \int_0^l \cos^2 \frac{kx}{r} dx = \tfrac{1}{2} l.$$

Moreover, even if n is not an integer provided only that it is a big number, then the last equation is still approximately true. For a long tube therefore with a large number of waves along a generating line equation (17.165) gives

$$\begin{aligned} \frac{2rV}{\pi h l E'} = {} & k^2A^2 + (mB+C)^2 - 2\sigma kA(mB+C) + \tfrac{1}{2}(1-\sigma)(kB-mA)^2 \\ & + f\left[\{k^4 + (m^2-1)^2 + 2\sigma k^2(m^2-1)\}C^2 + 2(1-\sigma)k^2(B+mC)^2\right] \\ & - \frac{k^2 P_0}{E'}(B^2+C^2), \quad . \quad . \quad . \quad . \quad . \quad (17.166) \end{aligned}$$

f being used, as in equation (17.123), to denote $\frac{1}{3}\frac{h^2}{r^2}$.

Now the conditions that V should have a minimum or maximum value for variations in A, B, C, are

$$\frac{\partial V}{\partial A} = 0, \quad \frac{\partial V}{\partial B} = 0, \quad \frac{\partial V}{\partial C} = 0,$$

which can be written in the forms

$$\{k^2 + \tfrac{1}{2}(1-\sigma)m^2\}A - \tfrac{1}{2}(1+\sigma)mkB - \sigma kC = 0 \quad . \quad (17.167)$$

$$\begin{aligned} -\tfrac{1}{2}(1+\sigma)mkA + \left\{m^2 + \tfrac{1}{2}(1-\sigma)k^2 + 2(1-\sigma)fk^2 - \frac{k^2P_0}{E'}\right\}B & \\ + \{m + 2(1-\sigma)fmk^2\}C = 0 \quad . \quad . \quad . \quad . \quad (17.168) \end{aligned}$$

$$\begin{aligned} -\sigma kA + \{m + 2(1-\sigma)fmk^2\}B + \left(1 - \frac{k^2P_0}{E'}\right) & \\ + f\{k^4 + (m^2-1)^2 + 2k^2(m^2-\sigma)\}C = 0 \quad . \quad . \quad . \quad (17.169) \end{aligned}$$

On eliminating the ratios A:B:C from these equations we find that the following determinant must be zero:—

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$$\begin{vmatrix} k^2+\frac{1}{2}(1-\sigma)m^2 & -\frac{1}{2}(1+\sigma)mk & -\sigma k \\ -\frac{1}{2}(1+\sigma)mk & \begin{array}{c} m^2+\frac{1}{2}(1-\sigma)k^2 \\ +2(1-\sigma)fk^2-\dfrac{k^2P_0}{E'} \end{array} & m+2(1-\sigma)fmk^2 \\ -\sigma k & m+2(1-\sigma)fmk^2 & \begin{array}{c} 1-\dfrac{k^2P_0}{E'} \\ +f\{k^4+(m^2-1)^2 \\ +2k^2(m^2-\sigma)\} \end{array} \end{vmatrix}$$

Neglecting squares and products of f and $\frac{P_0}{E'}$ this gives, after division by the factor $\frac{1}{2}(1-\sigma)$,

$$-k^2\frac{P_0}{E'}\{(m^2+k^2)^2+m^2+2(1+\sigma)k^2\}$$
$$+f\left\{\begin{array}{l}(m^2+k^2)^4+m^4+(4-2\sigma)m^2k^2+(5-4\sigma^2)k^4 \\ \quad -2m^6-(8-2\sigma)m^4k^2-(10-2\sigma^2)m^2k^4-2\sigma k^6\end{array}\right\}$$
$$+(1-\sigma^2)k^4=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (17.170)$$

Although the coefficient of f in this equation differs from the corresponding coefficient in equation (17.126) this difference is in terms that do not matter.

Equations (17.136) and (17.137) could be deduced from this last equation by exactly the same arguments as were used to deduce them from (17.120).

It is worth while to notice here that, in spite of the great difficulty, mentioned in Art. 211, in getting an accurate expression for the strain energy in a bent plate, the energy method has, nevertheless, given a result essentially the same as the equations of equilibrium gave. Moreover, this particular problem is a good test case for the validity of the energy method because the energy due to bending, which gives rise to the terms containing f, and the energy due to the strains in the middle surface, to which the term $(1-\sigma^2)k^4$ in (17.170) is due, have about equal importance in the final equation for P. It seems safe then in every case to use the expression for the energy given in (17.162).

CHAPTER XVIII

VIBRATIONS OF ROTATING DISKS.

325. The forces controlling the transverse vibrations of a rotating disk.

The problem before us in this chapter is to find the periods and modes of transverse vibrations of disks when they are rotating about their axes and when they are not rotating. We shall deal with disks of uniform thickness and also with disks whose thicknesses at radius r are proportional to $r^{-\beta}$. We shall first use accurate methods, and later, we shall give approximate methods of great simplicity which give results that are quite as good for practical problems as the very cumbersome accurate methods. By these approximate methods the periods of vibration of a turbine disk can be easily calculated.

When a rigid disk is not rotating it has definite modes of vibration due to its stiffness, the controlling force in these vibrations being the same as that which controls the vibrations of thin rods, which are investigated in Chapter 9. Moreover, just as a tightly stretched flexible string — a fiddle string for instance — can vibrate with the tension in the string as the controlling factor, so also can a tightly stretched membrane, such as a drum top, vibrate under the action of the tension. Again a flexible disk in rotation is in tension, and this tension is capable of producing vibrations when the disk has been disturbed from the plane state. When a rigid disk in rotation vibrates it is controlled both by the tensions and the stiffness. Since the tensions in the disk are proportional to the square of the angular velocity it is clear that, at low speeds, the main controlling force is the rigidity, whereas, at very high speeds, the main controlling force may be mainly the tensions due to rotation. We shall first find the periods of vibration of a disk which has a negligible rigidity. Afterwards, when the periods of vibration due to rigidity alone have also been found, a method will be given for finding the period of the rotating disk in which both rigidity and tensions are taken into account.

326. The equations of motion of a uniform disk.'

In equation (15.17) is given the expression which represents the transverse force, due to the stresses in the middle surface of a plate, on an element $dxdy$. A simplified form of this expression is used in (15.23), the simplification being due to the fact that the stresses in the middle surface are assumed in Chapter 15 to be in equilibrium. In a rotating disk these stresses are not in equilibrium and consequently we are not justified in using the simplified form. The more general form of (15.23) is, after division by $2h$,

$$\frac{\partial}{\partial x}\left(P'_1\frac{\partial w}{\partial x}\right)+\frac{\partial}{\partial y}\left(P'_2\frac{\partial w}{\partial y}\right)+\frac{\partial}{\partial x}\left(S'_3\frac{\partial w}{\partial y}\right)+\frac{\partial}{\partial y}\left(S'_3\frac{\partial w}{\partial x}\right)$$
$$=\frac{Eh^2}{3(1-\sigma^2)}\nabla_1{}^4 w-\frac{p}{2h}, \quad . \quad . \quad . \quad . \quad (18.1)$$

since $I = \frac{2}{3}h^3$.

In this equation P'_1, P'_2, are tensions in the middle surface across sections perpendicular respectively to OX and OY, and S'_3 is the shear stress in the middle surface on the same sections; w is the displacement in the direction OZ; p is the pressure acting on unit area of the plate reckoned positive in the same direction as w; $2h$ is the thickness of the plate. Now in the vibration problem we propose here to solve there is no actual pressure p but there is an acceleration $\frac{\partial^2 w}{\partial t^2}$; the inertia force per unit mass—that is, the product of mass per unit area and the *reversed* acceleration—which is the equivalent of a pressure in the equations of motion, is

$$-2h\varrho\frac{\partial^2 w}{\partial t^2},$$

ϱ being the mass per unit volume of the plate. This expression replaces p in (18.1).

Let P, Q, denote the mean radial and circumferential tensions in a rotating disk. These are called P′ and Q′ in (12.101) and (12.102).

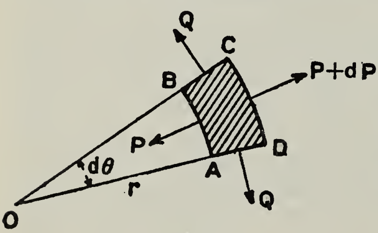

Fig. 174

To adapt equation (18.1) to polar coordinates it is best to find the component force in the z-direction on an element $dr \times rd\theta$ due to the tensions P and Q. Thus the force on the section AB (fig. 174) is $P \times 2hrd\theta$. The inclination of this force to the negative direction along the z-axis is $\frac{\partial w}{\partial r}$. Thus the component force in the direction OZ due to the tension on AB is

$$-2hrP\frac{\partial w}{\partial r}d\theta.$$

If we express this in the form $-Z_1 d\theta$, then the component force on CD due to $(P+dP)$ is clearly

$$\left(Z_1 + \frac{\partial Z_1}{\partial r}\,dr\right)d\theta\,.$$

Thus the resultant of the tensions across AB and CD is

$$\frac{\partial Z_1}{\partial r}\,dr\,d\theta = \frac{\partial}{\partial r}\left(2hrP\frac{\partial w}{\partial r}\right)drd\theta\,.$$

Again the force across AD due to Q is $2hdrQ$, and the component of this in the direction of OZ is

$$-\,2hdrQ\frac{\partial w}{r\partial\theta} = -\,Z_2\,dr \text{ say.}$$

The corresponding component force on BC is

$$\left(Z_2 + \frac{\partial Z_2}{\partial\theta}\,d\theta\right)dr\,.$$

Thus the resultant of the two forces Q on AD and BC is

$$\frac{\partial Z_2}{\partial\theta}\,dr\,d\theta = \frac{\partial}{\partial\theta}\left(\frac{2hQ}{r}\frac{\partial w}{\partial\theta}\right)dr\,d\theta.$$

If Q and h are functions of r only this can be written thus

$$\frac{2Qh}{r}\frac{\partial^2 w}{\partial\theta^2}\,.$$

Therefore the total force in the z-direction on the small element, due to the tensions P and Q, is

$$\left\{2\frac{\partial}{\partial r}\left(hrP\frac{\partial w}{\partial r}\right) + \frac{2Qh}{r}\frac{\partial^2 w}{\partial\theta^2}\right\}drd\theta.$$

Since the area is $rdrd\theta$ the force per unit area due to P and Q is

$$\frac{2}{r}\frac{\partial}{\partial r}\left(hrP\frac{\partial w}{\partial r}\right) + \frac{2Qh}{r^2}\frac{\partial^2 w}{\partial\theta^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (18.2)$$

This expression, when divided by $2h$, is the equivalent of the left hand side of (18.1). Therefore, equation (18.1), when expressed in polar coordinates, becomes

$$\frac{1}{hr}\frac{\partial}{\partial r}\left(hrP\frac{\partial w}{\partial r}\right) + \frac{Q}{r^2}\frac{\partial^2 w}{\partial\theta^2} = \frac{Eh^2}{3(1-\sigma^2)}\nabla_1{}^4 w + \varrho\frac{\partial^2 w}{\partial t^2} \quad . \quad (18.3)$$

where

$$\nabla_1{}^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} \quad . \quad . \quad . \quad . \quad . \quad (18.4)$$

The right hand side of (18.3) applies only to a uniform disk where h is not a function of r. The right hand side simplifies also in this case, the resulting equation being

$$\frac{1}{r}\frac{\partial}{\partial r}\left(rP\frac{\partial w}{\partial r}\right) + \frac{Q}{r^2}\frac{\partial^2 w}{\partial\theta^2} = \frac{Eh^2}{3(1-\sigma^2)}\nabla_1{}^4 w + \varrho\frac{\partial^2 w}{\partial t^2} \quad . \quad . \quad (18.5)$$

327. The vibrations due to the tensions set up by rotation in a uniform disk.*

The term involving ∇_1^4 in (18.5) represents the restoring force due to the rigidity of the disk. If this term is negligible we get

$$\frac{1}{r}\frac{\partial}{\partial r}\left(rP\frac{\partial w}{\partial r}\right)+\frac{Q}{r^2}\frac{\partial^2 w}{\partial \theta^2}=\varrho\frac{\partial^2 w}{\partial t^2}. \quad . \quad . \quad . \quad . \quad (18.6)$$

This is the equation which determines the modes of vibration due to the tensions P and Q. To find the normal modes let us first assume that

$$w=w_1 \sin p_1 t \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18.7)$$

Then (18.6) becomes, after division by $\sin p_1 t$,

$$\frac{1}{r}\frac{\partial}{\partial r}\left(rP\frac{\partial w_1}{\partial r}\right)+\frac{Q}{r^2}\frac{\partial^2 w_1}{\partial \theta^2}=-\varrho p_1{}^2 w_1 \quad . \quad . \quad (18.8)$$

It is easy to see that this has solutions of form

$$w_1=z\sin(n\theta+\alpha), \quad . \quad . \quad . \quad . \quad . \quad . \quad (18.9)$$

where z is a function of r only. With this substitution (18.8) becomes, after division by $\sin(n\theta+\alpha)$,

$$\frac{1}{r}\frac{d}{dr}\left(rP\frac{dz}{dr}\right)-\frac{Q}{r^2}n^2z+\varrho p_1{}^2z=0 \quad . \quad (18.10)$$

Now let us suppose that we are dealing with a complete disk rotating with angular velocity ω in which case we have, from (12.99) and (12.100),

$$P=A(a^2-r^2)\varrho\omega^2, \quad Q=(Aa^2-Br^2)\varrho\omega^2, \quad . \quad . \quad (18.11)$$

where

$$A=\tfrac{1}{8}(3+\sigma), \quad B=\tfrac{1}{8}(1+3\sigma). \quad . \quad . \quad . \quad . \quad (18.12)$$

The substitution of the values of P and Q from (18.11) in (18.10) gives

$$\frac{A}{r}\frac{d}{dr}\left\{(a^2-r^2)r\frac{dz}{dr}\right\}-\left(A\frac{a^2}{r^2}-B\right)n^2z+\frac{p_1{}^2}{\omega^2}z=0. \quad (18.13)$$

This equation can easily be integrated by a series of powers of r. Putting

$$z=\Sigma C_k\left(\frac{r}{a}\right)^k \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18.14)$$

in (18.13) we get

$$\sum C_k\left(\frac{r}{a}\right)^k\left[A\left\{k^2\frac{a^2}{r^2}-k(k+2)\right\}-An^2\frac{a^2}{r^2}+Bn^2+\frac{p_1{}^2}{\omega^2}\right]=0. \quad (18.15)$$

* The substance of this article, as well as of most of the work in this chapter on uniform disks, was first given in a paper by Lamb and Southwell, *The Vibrations of a Spinning Disk.* Proc. Roy. Soc. A, Vol. 99, 1921.

Equating to zero the coefficient of $\left(\frac{r}{a}\right)^{k-2}$ in the sum on the left of the last equation we get

$$C_{k-2}\left\{Bn^2+\frac{p_1^{\,2}}{\omega^2}-k(k-2)A\right\}+AC_k(k^2-n^2)=0 \qquad (18.16)$$

If we take $k=n$ then (18.16) gives

$$C_{n-2}=0 \; . \; . \; . \; . \; . \; . \; . \; . \; . \qquad (18.17)$$

Next, by putting $(n-2)$ for k in (18.16), we find

$$C_{n-4}=0 \, .$$

Likewise C_{n-6}, C_{n-8}, C_{n-10}, etc., are all zero. Moreover, for any value of k except $k=\pm n$, equation (18.16) gives

$$\frac{C_k}{C_{k-2}}=\frac{k(k-2)A-Bn^2-\dfrac{p_1^{\,2}}{\omega^2}}{(k^2-n^2)A} \; , \; . \; . \; . \; . \; . \qquad (18.18)$$

It is now clear that there is a value of z satisfying (18.13) which is expressible in a series of ascending powers of r starting with r^n. This series is

$$z=\left(\frac{r}{a}\right)^n\left\{C_n+C_{n+2}\frac{r^2}{a^2}+C_{n+4}\frac{r^4}{a^4}+\ldots\right\} \; . \; . \; . \qquad (18.19)$$

There is another series starting with r^{-n}, but as this is infinite at the centre of the disk it does not apply to our problem. Equation (18.18) shows that the coefficient C_m in this series vanishes (and consequently all the succeeding coefficients vanish also) provided that

$$\frac{p_1^{\,2}}{\omega^2}=m(m-2)A-Bn^2 \, . \; . \; . \; . \; . \qquad (18.20)$$

For example, the coefficient C_{n+2} vanishes provided that

$$\begin{aligned}\frac{p_1^{\,2}}{\omega^2}&=n(n+2)A-n^2B\\&=\tfrac{1}{8}n\{(n+2)(3+\sigma)-n(1+3\sigma)\}\\&=\tfrac{1}{4}n\{(n+3)-\sigma(n-1)\} \; . \; . \; . \; . \qquad (18.21)\end{aligned}$$

For this value of $p_1^{\,2}$ the expression for z reduces to one term.

In the same way we can make C_{n+4} vanish and find the corresponding value of p_1^2. It is clear then that, by putting m successively equal to $(n+2)$, $(n+4)$, $(n+6)$, etc., in (18.20), we find a succession of values of p^2 which make the series in (18.19) reduce to one, two, three, etc., terms respectively; that is, there is an infinite succession of values of $p_1^{\,2}$ corresponding to each of which there is a value of w which satisfies equation (18.6) and is finite over the whole disk. These values of p_1^2 must therefore give the normal modes of vibration of the disk.

To show more clearly the form of the solution let $(n+2s+2)$ be written for m in (18.20). Since m and n are integers s must also be an integer. Thus

$$\frac{p_1^2}{\omega^2}+n^2B=(n+2s+2)(n+2s)A \quad . \quad . \quad . \quad (18.22)$$

Substituting in (18.18) the value of p_1^2 from (18.22) the former equation becomes

$$\frac{C_k}{C_{k-2}}=\frac{k(k-2)-(n+2s+2)(n+2s)}{k^2-n^2}$$
$$=-\frac{(n-k+2s+2)(n+k+2s)}{(k-n)(k+n)} \quad . \quad . \quad . \quad (18.23)$$

Putting $(n+2)$, $(n+4)$, in succession for k in this we find

$$\frac{C_{n+2}}{C_n}=-\frac{2s(2n+2s+2)}{2(2n+2)}$$
$$=-\frac{s(n+s+1)}{1(n+1)},$$
$$\frac{C_{n+4}}{C_{n+2}}=-\frac{(s-1)(n+s+2)}{2(n+2)}.$$

Therefore

$$z=C_n\left(\frac{r}{a}\right)^n\left\{1-\frac{s(n+s+1)}{1.(n+1)}\frac{r^2}{a^2}\right.$$
$$\left.+\frac{(s-1)s(n+s+1)(n+s+2)}{1.2(n+1)(n+2)}\frac{r^4}{a^4}-\ldots\right\} \quad . \quad 18.24)$$

This series contains $(s+1)$ terms, which are alternately positive and negative. The sum of the series vanishes for s different values of r between 0 and a, as well as for $r=0$. It follows therefore that w is zero over the circumferences of s different circles for all values of t. These circles that have no motion are called *nodal circles*. Moreover, because w contains the factor $\sin(n\theta+\alpha)$, which vanishes along n different diameters of the disk, it follows that there are also n *nodal diameters.*

The following are particular solutions included in the general solution. The value of p_1^2 corresponding to w is given in each case.

$$\left.\begin{aligned} w&=C\left(\frac{r}{a}\right)^n\sin(n\theta+\alpha)\sin p_1t,\\ \frac{p_1^2}{\omega^2}&=\tfrac{1}{4}n\{(n+3)-\sigma(n-1)\},\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (18.25)$$

giving n nodal diameters
and no nodal circle;

$$\left.\begin{aligned} w &= C\left(\frac{r}{a}\right)^n\left(1-\frac{n+2}{n+1}\frac{r^2}{a^2}\right)\sin(n\theta+\alpha)\sin p_1 t, \\ \frac{p_1^2}{\omega^2} &= \tfrac{1}{4}\{n^2+9n+12-\sigma(n^2-3n-4)\}, \end{aligned}\right\} \quad (18.26)$$

giving n nodal diameters
and one nodal circle;

$$\left.\begin{aligned} w &= C\frac{r}{a}\sin(\theta+\alpha)\sin p_1 t, \\ \frac{p_1^2}{\omega^2} &= 1, \end{aligned}\right\} \quad (18.27)$$

giving one nodal diameter
and no nodal circle;

$$\left.\begin{aligned} w &= C\frac{r^2}{a^2}\sin(2\theta+\alpha)\sin p_1 t, \\ \frac{p_1^2}{\omega^2} &= \tfrac{1}{2}(5-\sigma), \end{aligned}\right\} \quad (18.28)$$

giving two nodal diameters
and no nodal circle.

$$\left.\begin{aligned} w &= C\left(1-2\frac{r^2}{a^2}\right)\sin p_1 t, \\ \frac{p_1^2}{\omega^2} &= 3+\sigma, \end{aligned}\right\} \quad (18.29)$$

giving no nodal diameter
and one nodal circle.

The displacement represented by w in (18.27) is that due to the rotation of the middle plane of the disk about a diameter without any bending of that surface. Moreover the period of oscillation is the same as that of rotation. Thus every particle returns to the same position after one revolution of the disk. This means that the axis of the disk is slightly inclined to the z-axis. If the disk is mounted on a shaft which constrains its axis to stay along the z-axis this vibration is not possible. In any case it is not a vibration in any real sense.

In the modes with no nodal diameters the centre of the disk is not at rest. This follows from the fact that z is not zero where $r=0$. These modes again are not possible for a disk mounted on a shaft. They would be possible only for a free disk. When there are no nodal diameters there must be at least one nodal circle. The case of one nodal circle is given in (18.29), the radius of this circle being $0{\cdot}707a$.

For the case of two nodal diameters and one nodal circle the nodal lines are shown in fig. 175.

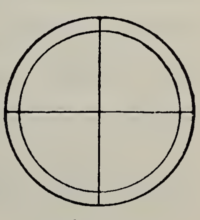

Fig. 175

The radius of the nodal circle is obtained by putting $n=2$ in (18.26) and equating w to zero. This gives

$$r=\sqrt{\frac{3}{4}}\,a=0{\cdot}866a \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18.30)$$

328. The rotation of the nodal diameters.

Since $\frac{\partial^2 w}{\partial t^2}$ is used in equation (18.3) for the acceleration of a particle of the disk it follows that w is the displacement of a particle which rotates with the disk; it is not the displacement of the particle which happens to be at the point whose coordinates are r, θ, referred to fixed axes. The angle θ must, in fact, be measured from a line which rotates with the disk. The nodal diameters are diameters passing through the same set of particles of the disk which have no motion parallel to OZ throughout the vibration. These nodal diameters rotate therefore with angular velocity ω in the same direction as the disk. We shall now show that it is possible for the nodal diameters to rotate with an angular velocity different form ω.

Instead of putting

$$w=z\sin(n\theta+a)\sin p_1 t \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18.31)$$

we might have put

$$w=z\sin(n\theta\pm p_1 t+a) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18.32)$$

in equation (18.6). We should then have arrived at the same equation (18.10) for z. Therefore the same values of z and the same values of p_1 can be used in (18.31) and (18.32). In the latter equation θ is still an angle measured from a diameter which rotates with the disk. The nodal diameters corresponding to (18.32) are those diameters for which

$$n\theta\pm p_1 t+a=m\pi, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18.33)$$

that is, for which

$$\frac{d\theta}{dt}=\pm\frac{p_1}{n}.$$

Thus the nodal diameters may rotate in either direction with an angular velocity $\frac{p_1}{n}$ relative to the disk. The actual angular velocities of the nodal diameters given by (18.32) are $\omega\pm\frac{p_1}{n}$. There are thus three possible angular velocities of the nodal diameters, namely, $\omega-\frac{p_1}{n}$, ω, $\omega+\frac{p_1}{n}$. In the case given in (18.27), when $p_1=\omega$, these possible angular velocities are 0, ω, 2ω. Thus it is possible for the single nodal diameter to be at rest. The three values of w for this case are

$$w = C\frac{r}{a}\sin(\theta + \omega t + a) \quad . \quad . \quad . \quad . \quad . \quad (18.34)$$

$$w = C\frac{r}{a}\sin(\theta + \alpha)\sin\omega t \quad . \quad . \quad . \quad . \quad . \quad (18.35)$$

$$w = C\frac{r}{a}\sin(\theta - \omega t + a) \quad . \quad . \quad . \quad . \quad . \quad . \quad (18.36)$$

When the single nodal diameter is at rest the disk rotates in a plane slightly inclined to the xy plane, in which case there is no real vibration at all.

In the general case it is possible for n nodal diameters to be at rest provided

$$\omega = \frac{p_1}{n}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18.37)$$

This last equation, combined with (18.22), gives

$$n^2(1 + B) = (n + 2s + 2)(n + 2s)A,$$

or $$n^2(9 + 3\sigma) = (n + 2s + 2)(n + 2s)(3 + \sigma),$$

whence $$(n + 2s + 1)^2 - 3n^2 = 1. \quad . \quad . \quad . \quad . \quad . \quad . \quad (18.38)$$

We know that one pair of integral solutions of this is

$$n = 1, \quad s = 0;$$

and it is shown in works on Algebra that the integral solutions of the equation

$$x^2 - Ny^2 = 1$$

can be written down when one pair of such integral values of x and y is known. In our case

$$\begin{aligned}\{(n + 2s + 1) + \sqrt{3}n\}&\{(n + 2s + 1) - \sqrt{3}n\} \\ &= (2 + \sqrt{3})(2 - \sqrt{3}) = 1 = 1^q \\ &= (2 + \sqrt{3})^q(2 - \sqrt{3})^q \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18.39)\end{aligned}$$

Now sets of integral values of $(n + 2s + 1)$ and n are obtained by equating the factors of the first and last members of the preceding equation provided that q is an integer. Thus we may take

$$n + 2s + 1 + \sqrt{3}n = (2 + \sqrt{3})^q,$$

$$n + 2s + 1 - \sqrt{3}n = (2 - \sqrt{3})^q;$$

whence

$$\left.\begin{aligned} n + 2s + 1 &= \tfrac{1}{2}\{(2 + \sqrt{3})^q + (2 - \sqrt{3})^q\}, \\ \text{and} \quad n &= \frac{1}{2\sqrt{3}}\{(2 + \sqrt{3})^q - (2 - \sqrt{3})^q\}, \end{aligned}\right\} \quad . \quad (18.40)$$

Corresponding pairs of values of n and s derived from these equations are given in the following table

n	1	4	15	56
s	0	1	5	20

Thus if there are four nodal diameters and one nodal circle the nodal diameters may be at rest and the period of vibration is

$$\frac{2\pi}{p_1}=\frac{2\pi}{n\omega}=\frac{\pi}{2\omega}, \qquad (18.41)$$

which is a quarter of the period of rotation of the disk. In fact, while the disk makes one revolution the nodal diameters rotate relatively to the disk through an angle equal to twice the angle between two consecutive nodal diameters.

329. The condition of convergence for the series for z.

It might seem as if there were no necessity that m in equation (18.20) should be an integer. If, however, m were not an integer then s would not be an integer and the series (18.24) would not terminate, and it can be shown that the series is divergent when $r=a$ if it does not terminate. Thus m must be an integer in order that the series for z should be convergent.

Let

$$c^2=\frac{B}{A}n^2+\frac{p_1{}^2}{A\omega^2}.$$

Then (18.18) becomes

$$\frac{C_k}{C_{k-2}}=\frac{k(k-2)-c^2}{k^2-n^2}$$

$$=\frac{k-2}{k}\,\frac{1-\dfrac{c^2}{k(k-2)}}{1-\dfrac{n^2}{k^2}}$$

$$=\frac{k-2}{k}\left\{1-\frac{c^2-n^2}{k^2}+\text{etc.}\right\}. \qquad (18.42)$$

Because there is no term containing k^{-1} inside the brackets in the last line in this equation it can be shown that

$$\frac{C_k}{\dfrac{1}{k}}=\text{a finite quantity}$$

for all values of k. Therefore the series

$$C_n+C_{n+2}+\dots+C_k+C_{k+2}+\dots$$

is divergent because the series

$$\frac{1}{n}+\frac{1}{n+2}+\ldots+\frac{1}{k}+\frac{1}{k+2}+\ldots$$

is divergent.

It follows therefore that z would be infinite at the rim of the disk if s were not an integer. Consequently p_1 can have no values except those derived from integral values of s.

330. The vibrations due to tensions set up by rotation in a disk of variable thickness.

When the rigidity is negligible equation (18.3) becomes

$$\frac{1}{hr}\frac{\partial}{\partial r}\left(hrP\frac{\partial w}{\partial r}\right)+\frac{Q}{r^2}\frac{\partial^2 w}{\partial\theta^2}=\varrho\frac{\partial^2 w}{\partial t^2}. \quad . \quad . \quad . \quad . \quad (18.43)$$

With the same substitution as for the uniform disk, namely,

$$w=z\sin p_1 t\sin(n\theta+\alpha), \quad . \quad . \quad . \quad . \quad . \quad (18.44)$$

this becomes, after division by $\sin p_1 t\sin(n\theta+\alpha)$,

$$\frac{1}{hr}\frac{d}{dr}\left(hrP\frac{dz}{dr}\right)-\frac{n^2Q}{r^2}z+\varrho p_1{}^2z=0. \quad . \quad . \quad . \quad (18.45)$$

This is the correct equation when h is a function of r but not of θ, in which case P and Q are also functions of r.

Now let us take

$$h=c\left(\frac{r}{a}\right)^{-\beta}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18.46)$$

which is the usual form for a turbine disk. We found in Art. 222 that the mean tensions for this shape of disk are, for a complete disk,

$$P=\varrho\omega^2a^2H\left\{\left(\frac{r}{a}\right)^{q+\beta-1}-\left(\frac{r}{a}\right)^2\right\}, \quad . \quad . \quad . \quad (18.47)$$

$$Q=\varrho\omega^2a^2H\left\{q\left(\frac{r}{a}\right)^{q+\beta-1}-b\left(\frac{r}{a}\right)^2\right\}, \quad . \quad . \quad . \quad (18.48)$$

where

$$H=\frac{3+\sigma}{8-(3+\sigma)\beta}, \; b=\frac{1+3\sigma}{3+\sigma}, \quad . \quad . \quad . \quad . \quad (18.49)$$

and q is the greater root of the equation

$$q^2+\beta q-1-\sigma\beta=0; \quad . \quad . \quad . \quad . \quad . \quad . \quad (18.50)$$

that is,

$$q=\sqrt{1+\sigma\beta+\tfrac{1}{4}\beta^2}-\tfrac{1}{2}\beta. \quad . \quad . \quad . \quad . \quad . \quad (18.51)$$

Now let

$$\eta=\frac{r}{a}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18.52)$$

so that $\qquad dr=a\,d\eta;$

then equation (18.45) becomes

$$\frac{1}{\eta h a^2}\frac{d}{d\eta}\left(h\eta P\frac{dz}{d\eta}\right)-\frac{n^2 Q}{a^2\eta^2}z+\varrho p_1{}^2 z=0. \quad . \quad . \quad . \quad (18.53)$$

Substituting for P, Q, and h, in this last equation we get

$$\frac{\eta^{\beta-1}}{ca^2}\frac{d}{d\eta}\left\{\varrho\omega^2 a^2 c H(\eta^q-\eta^{3-\beta})\frac{dz}{d\eta}\right\}$$
$$-n^2\varrho\omega^2 Hz(q\eta^{q+\beta-3}-b)+\varrho p_1{}^2 z=0,$$

whence

$$\eta^{\beta-1}\frac{d}{d\eta}\left\{(\eta^q-\eta^{3-\beta})\frac{dz}{d\eta}\right\}-n^2 z(q\eta^{q+\beta-3}-b)+\frac{p_1{}^2}{H\omega^2}z=0. \quad (18.54)$$

The method of solution of this equation in series is exactly the same as that used for equation (18.13). Thus putting

$$z=\Sigma C_k\eta^k$$

in (18.54) and writing γ for $(3-q-\beta)$ we get

$$\sum C_k\eta^k\left[\begin{matrix}k(k+q-1)\eta^{-\gamma}-k(k+2-\beta)\\ -n^2 q\eta^{-\gamma}+n^2 b+\dfrac{p_1{}^2}{H\omega^2}\end{matrix}\right]=0, \quad (18.55)$$

which shows that z is expressible in powers of η^γ if γ is positive. If γ is negative z is expressible in powers of $\eta^{-\gamma}$. For a turbine disk γ is more likely to be positive than negative. We shall first work on the assumption that γ is positive and refer later to the other case.

By equating to zero the coefficient of $\eta^{k-\gamma}$ in (18.55) we get

$$\{k(k+q-1)-n^2 q\}C_k=\left\{(k-\gamma)(k+2-\gamma-\beta)-n^2 b-\frac{p_1{}^2}{H\omega^2}\right\}C_{k-\gamma} \quad (18.56)$$

whence

$$\frac{C_k}{C_{k-\gamma}}=\frac{(k-\gamma)(k+2-\gamma-\beta)-n^2 b-\dfrac{p_1{}^2}{H\omega^2}}{k(k+q-1)-n^2 q}. \quad . \quad . \quad (18.57)$$

Equation (18.56) shows that $C_{k_1-\gamma}$ is zero if

$$k_1(k_1+q-1)-n^2 q=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (18.58)$$

which equation gives the possible values of k at which the series can begin. Now the equation for k_1 has one negative and one positive root, and the negative root is inadmissible because this would make z infinite at the centre of the disk. Therefore, taking the positive root,

$$k_1=\frac{\sqrt{4n^2 q+(1-q)^2}+1-q}{2}. \quad . \quad . \quad . \quad . \quad (18.59)$$

The solution we have now got is

$$z=\eta^{k_1}\{C_{k_1}+\eta^\gamma C_{k_1+\gamma}+\eta^{2\gamma}C_{k_1+2\gamma}+\ldots\ldots\}. \quad . \quad . \quad (18.60)$$

This series has $s+1$ terms provided $C_{k_1+s\gamma+\gamma}$ vanishes, and equation (18.57) shows that this coefficient vanishes if the numerator of the fraction in that equation is zero when $k_1+s\gamma+\gamma$ is subsituted for k; that is, if

$$(k_1+s\gamma)(k_1+s\gamma+2-\beta)-n^2b-\frac{p_1^{\,2}}{H\omega^2}=0,$$

that is, if

$$\frac{p_1^{\,2}}{H\omega^2}=(k_1+s\gamma)(k_1+s\gamma+2-\beta)-n^2b. \quad . \quad . \quad . \quad (18.61)$$

Now by (18.57)

$$\frac{C_{k_1+m\gamma+\gamma}}{C_{k_1+m\gamma}}=\frac{(k_1+m\gamma)(k_1+m\gamma+2-\beta)-n^2b-\dfrac{p_1^{\,2}}{H\omega^2}}{(k_1+m\gamma+\gamma)(k_1+m\gamma+\gamma+q-1)-n^2q}. \quad . \quad (18.62)$$

When the value of p_1^2 from (18.61) is substituted in this equation the numerator on the right becomes

$$(k_1+m\gamma)(k_1+m\gamma+2-\beta)-(k_1+s\gamma)(k_1+s\gamma+2-\beta)$$
$$=-(s-m)\gamma\{2k_1+2-\beta+(s+m)\gamma\}.$$

Also, by means of (18.58), the denominator becomes

$$(k_1+m\gamma+\gamma)(k_1+m\gamma+\gamma+q-1)-k_1(k_1+q-1).$$
$$=(m+1)\gamma\{2k_1+q-1+m\gamma+\gamma\}.$$

Therefore

$$\frac{C_{k_1+m\gamma+\gamma}}{C_{k_1+m\gamma}}=-\frac{(s-m)(2k_1+2-\beta+m\gamma+s\gamma)}{(m+1)(2k_1+q-1+m\gamma+\gamma)}$$
$$=-\frac{(s-m)(2k_1+2-\beta+m\gamma+s\gamma)}{(m+1)(2k_1+2-\beta+m\gamma)}. \quad . \quad (18.63)$$

It is easy to see from this form of the ratio of the coefficients that the terms in the series for z are alternately positive and negative, Moreover z vanishes for s different values of η between 0 and 1 as well as when $\eta=0$ Consequently there are s nodal circles; and the factor $\sin(n\theta+\alpha)$ introduces n nodal diameters.

Equation (18.61), giving p_1 for the mode with s nodal circles and n nodal diameters, can be written thus:—

$$\frac{p_1^{\,2}}{\omega^2}=\frac{3+\sigma}{8-(3+\sigma)\beta}\left\{(k_1+s\gamma)(k_1+s\gamma+2-\beta)-\frac{1+3\sigma}{3+\sigma}n^2\right\}, \quad (18.64)$$

where γ denotes $(3-q-\beta)$, and q and k_1 are given by (18.51) and (18.59).

As a particular example we take the case of two nodal diameters and no nodal circle, assuming $\sigma=\frac{1}{4}$ and $\beta=1$. Here

$$n=2\,,\quad s=0;$$

$$q=\sqrt{\tfrac{3}{2}}-\tfrac{1}{2}=0\cdot7247\,,$$

$$k_1=\frac{\sqrt{16q+(1-q)^2}+1-q}{2}=1\cdot846\,,$$

$$\gamma=3-q-\beta=1\cdot2753.$$

Therefore

$$\frac{p_1^{\,2}}{\omega^2}=\tfrac{13}{19}\{(1\cdot846)(2\cdot846)-\tfrac{28}{13}\}$$

$$=2\cdot121 \quad \ldots \quad (18.65)$$

whence

$$\frac{p_1}{\omega}=1\cdot456 \quad \ldots \quad (18.66)$$

The corresponding result for a uniform disk is, by (18.28),

$$\frac{p_1}{\omega}=1\cdot541 \quad \ldots \quad (18.67)$$

These two values of p_1 do not differ so much as might be expected from the great difference in the forms of the disks.

Let us next take the case of two nodal diameters and one nodal circle with $\sigma=\frac{1}{4}$ and $\beta=1$ as before. Now

$$n=2\,,\quad s=1\,,$$
$$q=0\cdot7247\,,$$
$$k_1=1\cdot846,$$
$$\gamma=1\cdot2753.$$

Therefore

$$\frac{p_1^{\,2}}{\omega}=\tfrac{13}{19}\{(3\cdot121)(4\cdot121)-\tfrac{28}{13}\}=7\cdot327,$$

whence

$$\frac{p_1}{\omega}=2\cdot706 \quad \ldots \quad (18.68)$$

Also the series for z contains two terms, the ratio of the coefficients being obtained by putting $m=0$ in (18.63). Thus

$$\frac{C_{k_1+\gamma}}{C_{k_1}}=-\frac{2k_1+2-\beta+\gamma}{2k_1+q-1+\gamma}=\frac{5\cdot967}{4\cdot692}=1\cdot272. \quad \ldots \quad (18.69)$$

Thus the value of z is

$$z=C\eta^{1\cdot846}(1-1\cdot272\,\eta^{1\cdot275}). \quad \ldots \quad (18.70)$$

The radius of the nodal circle is given by

$$\left(\frac{r}{a}\right)^{1\cdot275}=\eta^{1\cdot275}=\frac{1}{1\cdot272}\,,$$

whence

$$r=0\cdot828a. \quad \ldots \quad (18.71)$$

This should be compared with (18.30).

331. The case where γ is negative.

The preceding work is all based on the assumption that γ is positive. Let us now find what happens when γ is negative.

If

$$8-(3+\sigma)\beta<0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (18.72)$$

the coefficient H which occurs in P and Q in (18.47) and (18.48) is negative. But this does not make P and Q negative because the other factors in P and Q change sign at the same time as H changes sign. Suppose β is given by the equation

$$8-(3+\sigma)\beta=0.$$

Then

$$q=\sqrt{1+\sigma\beta+\tfrac{1}{4}\beta^2}-\tfrac{1}{2}\beta \quad . \quad . \quad . \quad . \quad . \quad (18.73)$$

$$=\frac{5+3\sigma}{3+\sigma}-\frac{4}{3+\sigma}$$

$$=\frac{1+3\sigma}{3+\sigma}=b. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18.74)$$

Also, for this value of β,

$$\gamma=3-\beta-q=0.$$

The general value of γ is given by

$$\gamma=3-\tfrac{1}{2}\beta-\sqrt{1+\sigma\beta+\tfrac{1}{4}\beta^2}, \quad . \quad . \quad . \quad . \quad (18.75)$$

which clearly decreases as β increases. Then for all values of β greater than $\dfrac{8}{3+\sigma}$ both and H are negative.

When γ is negative we may still assume, as the solution of (18.54),

$$z=\Sigma C_k\eta^k, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18.76)$$

and we shall get the same equation (18.56) for the relation between the coefficients. Now just as when γ was positive the normal modes of vibration are represented by a terminating series of powers of η^γ or $\eta^{-\gamma}$. When γ is negative $(k-\gamma)$ is greater than k and consequently $C_{k-\gamma}$ is the coefficient of a higher power of η than C_k. Let η^{k_1} be the *highest* power of η in the series for z. Then, since $C_{k_1-\gamma}$ is the coefficient of a higher power than C_{k_1} the former coefficient must be zero. Thus (18.56) gives, since C_{k_1} is not zero,

$$k_1(k_1+q-1)-n^2q=0. \quad . \quad . \quad . \quad . \quad . \quad (18.77)$$

Now suppose the series arranged in powers of η^γ, which is now a descending series. This series is identical in form with (18.60). Also suppose the series has $(s+1)$ terms, which, as we can prove, corresponds to s nodal circles. Then the coefficient of $\eta^{k_1+s\gamma+\gamma}$ must vanish, while the coefficient of $\eta^{k_1+s\gamma}$ does not vanish. That is, putting $(k_1+s\gamma+\gamma)$ for k in (18.56), we get, since $C_{k_1+s\gamma+\gamma}$ is zero,

$$(k_1+s\gamma)(k_1+2+s\gamma-\beta)-n^2b-\frac{p_1^{\,2}}{H\omega^2}=0. \quad . \quad . \quad (18.78)$$

Equations (18.77) and (18.78), which determine k_1 and p_1, are identical with the corresponding equations that we found when γ was positive. There is, however, one very important difference between the two cases when γ is positive and when γ is negative. When γ is negative the series (18.60) is a series arranged in descending powers of η, and the condition that z must contain no negative powers of r puts a limit to the number of terms in the series. Thus if there are s nodal circles the index of the last power in the series is $(k_1 + s\gamma)$, and this must be a positive quantity. That is,

$$k_1 + s\gamma > 0 , \qquad (18.79)$$

whence it follows that the number of nodal circles cannot exceed the greatest integer in $\frac{k_1}{-\gamma}$. When γ is positive there is no limit to the possible number of nodal circles. It is therefore rather surprising to find that, when β exceeds the value $\frac{8}{3+\sigma}$, there is a definite upper limit—depending on the values of n and β—to the number of possible nodal circles. There is still another condition that may limit the number of nodal circles, namely the condition that p_1 must be real.

The value of p_1 for no nodal circle is given by

$$\begin{aligned} -\frac{p_1^2}{H\omega^2} &= n^2 b - k_1(k_1 + 2 - \beta) \\ &= n^2 b - k_1(k_1 + \gamma + q - 1) \\ &= n^2 b - k_1\gamma - k_1(k_1 + q - 1) . \qquad (18.80) \end{aligned}$$

By means of (18.77) this reduces to

$$-\frac{p_1^2}{H\omega^2} = n^2(b - q) - k_1\gamma \qquad (18.81)$$

Now by (18.74) we know that $q = b$ when γ is zero. Moreover

$$\begin{aligned} \frac{dq}{d\beta} &= \frac{1}{2}\frac{\sigma + \frac{1}{2}\beta}{\sqrt{1 + \sigma\beta + \frac{1}{4}\beta^2}} - \frac{1}{2} , \\ &< \frac{1}{2}\frac{\sigma + \frac{1}{2}\beta}{\sqrt{\sigma^2 + \sigma\beta + \frac{1}{4}\beta^2}} - \frac{1}{2} , \end{aligned}$$

that is,

$$< \tfrac{1}{2} - \tfrac{1}{2} .$$

Thus $\frac{dq}{d\beta}$ is negative, and consequently q decreases as β increases. It follows that b is greater than q when γ is negative. This shows that the right hand side of equation (18.81) is positive, and since H is negative p_1 is consequently real. The only power of η occurring in z is η^{k_1}, and there is certainly one positive value of k_1 satisfying (18.77). We see then that both conditions for the possibility of vibrations with no nodal circle are satisfied.

The following sums up our results:—

If $(k_1+s\gamma)$ is positive, and if the value of p_1, given by the equation

$$-\frac{p_1^2}{H\omega^2}=n^2b-(k_1+s\gamma)(k_1+s\gamma+2-\beta)\ . \quad . \quad . \quad (18.22)$$

is real, then $\frac{2\pi}{p_1}$ is the period of vibration of the disk with n nodal meters and s nodal circles; q is found from (18.51), k_1 from (18.59), and γ from (18.75).

332. Vibrations of a uniform disk controlled by rigidity.

When P and Q are zero or negligible equation (18.5) becomes

$$\frac{Eh^2}{3(1-\sigma^2)}\nabla_1{}^4 w+\varrho\frac{\partial^2 w}{\partial t^2}=0. \quad . \quad . \quad . \quad . \quad (18.83)$$

Assuming, for a normal mode,

$$w=w_2\sin p_2 t \ . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18.84)$$

equation (18.83) becomes, after division by $\sin p_2 t$.

$$\frac{Eh^2}{3(1-\sigma^2)}\nabla_1{}^4 w_2-\varrho p_2{}^2 w_2=0\,,$$

or

$$\nabla_1{}^4 w_2=k^4 w_2\,, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18.85)$$

where

$$k^4=\frac{3(1-\sigma^2)\varrho p_2{}^2}{Eh^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18.86)$$

Now suppose w_3 is any function of x and y satisfying the equation

$$\nabla_1{}^2 w_3=k^2 w_3\,; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18.87)$$

then

$$\nabla_1{}^4 w_3=\nabla_1{}^2(\nabla_1{}^2 w_3)=\nabla_1{}^2(k^2 w_3)=k^4 w_3.$$

Therefore

$$w_2=w_3$$

is a solution of (18.85).

Likewise, if w_4 is any solution of

$$\nabla_1{}^2 w_4=-k^2 w_4\,, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18.88)$$

then

$$\nabla_1{}^4 w_4=\nabla_1{}^2(-k^2 w_4)=k^4 w_3.$$

Consequently

$$w_2=w_4$$

is a solution of (18.85).

The complete solution of (18.85) is

$$w_2=w_3+w_4\,,$$

w_3 and w_4 being the complete solutions of (18.87) and (18.88). Thus any solution of

$$\frac{\partial^2 w_2}{\partial r^2}+\frac{1}{r}\frac{\partial w_2}{\partial r}+\frac{1}{r^2}\frac{\partial^2 w_2}{\partial \theta^2}=\pm k^2 w_2 \quad . \quad . \quad . \quad . \ (18.89)$$

is a solution of (18.85).

Now in (18.89) put

$$w_2 = z\sin(n\theta+\alpha) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \ (18.90)$$

z being a function of r only. Then we get

$$\frac{d^2z}{dr^2}+\frac{1}{r}\frac{dz}{dr}-\frac{n^2z}{r^2}=\pm k^2 z \quad . \quad . \quad . \quad . \quad . \quad . \ (18.91)$$

The equation

$$\frac{d^2z}{dr^2}+\frac{1}{r}\frac{dz}{dr}+\left(k^2-\frac{n^2}{r^2}\right)z=0 \quad . \quad . \quad . \quad . \quad . \ (18.92)$$

is the equation for Bessel functions of order n. If we attempt to solve this equation by means of a series we find

$$z = AJ_n(kr)+BJ_{-n}(kr) \quad . \quad . \quad . \quad . \quad . \quad . \ (18.93)$$

where $J_n(x)$ is a function defined by the equation*

$$J_n(x)=\frac{x^n}{2^n\Gamma(n+1)}\left\{1-\frac{x^2}{2(2n+2)}+\frac{x^4}{2.4(2n+2)(2n+4)}\cdots\right\} \ (18.94)$$

$\Gamma(n+1)$ being the gamma function of $(n+1)$, which reduces to $\lfloor n$ when n is a positive integer.

When n is an integer it can be shown that

$$J_{-n}(x)=(-1)^n J_n(x) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \ (18.95)$$

In the disk problem n is certainly an integer and consequently (18.93) really only contains one function $J_n(kr)$. The reason why the series method fails to give the second function when n is an integer is because this second function involves $\log_e r$, which cannot be expressed in powers of r.

When n is an integer one form of the second function which replaces $J_{-n}(kr)$ in (18.93) is $Z_n(kr)$, this function being defined by the equation

$$Z_n(x)=J_n(x)\log_e x-\tfrac{1}{2}(\tfrac{1}{2}x)^{-n}\sum_{m=0}^{m=n-1}\frac{\lfloor n-m-1}{\lfloor m}(\tfrac{1}{2}x)^{2m}$$

$$-\tfrac{1}{2}(\tfrac{1}{2}x)^{n}\sum_{m=0}^{m=\infty}\frac{(-1)^m}{\lfloor n+m\,\lfloor m}(\tfrac{1}{2}x)^{2m}(s_m+s_{n+m}) \quad . \quad . \ (18.96)$$

where

$$s_m=\tfrac{1}{1}+\tfrac{1}{2}+\tfrac{1}{3}+\ldots+\frac{1}{m};\ s_0=0; \quad . \quad . \quad . \ (18.97)$$

$$\lfloor 0 = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \ (18.98)$$

* For the necessary theory of Bessel functions see Appendix A.

When $n=0$ it is to be understood that there are no terms in the sum from $m=0$ to $m=n-1$; and when $n=1$ there is one term, namely $\frac{1}{x}$.

The function defined by (18.96) is Neumann's * form of the Bessel function of the n^{th} oder of the second kind. This form is the most convenient for our present problem. Nevertheless, in order to make use of published tables, another function $Y_n(x)$—Weber's† form of the second function—is defined by the following equation:—

$$\frac{\pi}{2} Y_n(x) = Z_n(x) - (\log_e 2 - \gamma) J_n(x)$$

where γ denotes Euler's constant 0·577216. . . . .

We shall in future write λ for the constant $(\log_e 2 - \gamma)$. Then

$$Y_n(x) = \frac{2}{\pi} Z_n(x) - \frac{2\lambda}{\pi} J_n(x) \quad . \quad . \quad . \quad . \quad . \quad (18.99)$$

Thus when n is an integer the complete solution of (18.88) is

$$z = AJ_n(kr) + BZ_n(kr). \quad . \quad . \quad . \quad . \quad . \quad (18.100)$$

The equation

$$\frac{d^2z}{dr^2} + \frac{1}{r}\frac{dz}{dr} - \left(k^2 + \frac{n^2}{r^2}\right) z = 0, \quad . \quad . \quad . \quad . \quad (18.101)$$

obtained by taking the lower sign on the right of (18.91), differs from (18.92) only in having ik for k, where i denotes $\sqrt{-1}$. We could therefore get the solution of (18.101) by writing ikr for kr in the solution of (18.92); but this method has the disadvantage that it introduces imaginary quantities into the solution of an equation with real coefficients. To avoid this certain functions of a real variable are defined below.

Let

$$I_n(x) = i^{-n} J_n(ix)$$

$$= \frac{x_n}{2^n \lfloor n} \left\{ 1 + \frac{x^2}{2(2n+2)} + \frac{x^4}{2.4(2n+2)(2n+4)} + \dots \right\} \quad (18.102)$$

Then one solution of (18.101) is

$$z = AI_n(kr) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18.103)$$

Let us also put

* Carl Gottfried Neumann, *Theorie der Besselschen Functionen* (Leipzig, 1867).

† H. Weber, *Ueber die Besselschen Functionen und ihre Anwendungen auf die der elektrischen Ströme. Journal für Mathematik,* LXXV, 1873.

$$H_n(x) = i^{-n} Z_n(ix) - I_n(x)\log i$$
$$= I_n(x)\log x - \tfrac{1}{2}\sum_{m=0}^{m=n-1}\frac{\lfloor n-m-1}{\lfloor m}(-1)^{m-n}(\tfrac{1}{2}x)^{2m-n}$$
$$-\tfrac{1}{2}(\tfrac{1}{2}x)^n \sum_{m=0}^{m=8}\frac{s_m + s_{n+m}}{\lfloor n+m \lfloor m}(\tfrac{1}{2}x)^{2m}. \quad . \quad . \quad . \quad (18.104)$$

Then the complete solution of (18.101) is

$$z = CI_n(kr) + DH_n(kr). \quad . \quad . \quad . \quad . \quad . \quad (18.105)$$

Another form of the second solution of (18.101), the numerical values of which have been tabulated, is

$$K_n(x) = (-1)^{n+1}\{H_n(x) - (\log 2 - \gamma)I_n(x)\}$$
$$= (-1)^{n+1}\{H_n(x) - \lambda I_n(x)\}, \quad . \quad . \quad . \quad . \quad (18.106)$$

x being written for kr.

Thus one solution of (18.83) giving a normal mode is

$$w = \{AJ_n(kr) + BZ_n(kr)\}\sin(n\theta + \alpha)\sin p_2 t$$
$$+ \{CI_n(kr) + DH_n(kr)\}\sin(n\theta + \beta)\sin p_2 t \quad . \quad . \quad (18.107)$$

For a disk with a central hole there are two boundary conditions at the hole, and two more at the outer rim—the usual conditions for a bent plate (see Chap. XIV). If there is no central hole, and if the disk is free at the centre, there are no necessary boundary conditions at the centre, but there are two at the rim. For a complete disk whose centre is fixed in any way there are two conditions at the centre just as when there is a hole. In any case, however, the greatest number of boundary conditions is four. These boundary conditions have to determine, among other things, the yet undetermined constant p_2, from which the period of the particular mode of vibration is got. There are thus three equations left to determine the constants of integration. If $\alpha = \beta$ these three equations will determine the three ratios existing among the four constants A, B, C, D. It follows that, in any normal mode, α and β must be equal. Therefore we may write

$$w = \{AJ_n(kr) + BZ_n(kr) + CI_n(kr) + DH_n(kr)\}\sin(n\theta + \alpha)\sin p_2 t \quad (18.108)$$

It is clear that w is zero where $\sin(n\theta + \alpha) = 0$, that is, along n different diameters of the disk. There are thus n nodal diameters in the mode of vibration indicated by (18.108).

333. Oscillations symmetrical about the axis of a free disk.

If we put $n = 0$ in (18.108) and amalgamate $\sin\alpha$ in the constants A, B, C, D, we get

$$w = \{AJ_0(kr) + BZ_0(kr) + CI_0(kr) + DH_0(kr)\}\sin p_2 t \ . \quad (18.109)$$

This is the equation for oscillations symmetrical about the axis of a disk.

If the disk is free, that is, not held at any point, the conditions at the rim are the boundary conditions for a free plate. These conditions are given in (14.46) and (14.47) for any plate. In the case of axial symmetry these can be reduced, by means of (14.64), (14.66), (14.68), to the following:—

$$\left.\begin{aligned}&\frac{\partial^2 w}{\partial r^2}+\frac{\sigma}{r}\frac{\partial w}{\partial r}=0 \quad . \quad . \\ &\frac{\partial}{\partial r}\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right)\right\}=0\end{aligned}\right\}\text{where } r=a \quad \begin{aligned}&\ldots\ldots\ldots (18.110)\\ &\ldots\ldots\ldots (18.111)\end{aligned}$$

a being the radius of the disk.

If the disk had a central hole of radius b the same two conditions would have to be satisfied at the edge of the hole as well as at the outer rim.

It is impossible to make both w and the shearing force F_1 finite at the centre of the disk unless B and D are both zero, for the important terms in w and F_1 when $r=0$ are

$$w=(\mathrm{B}\log kr+\mathrm{D}\log kr)\sin p_2 t \quad . \quad . \quad . \quad . \quad (18.112)$$

$$\mathrm{F}_1=\mathrm{E'I}\frac{\partial}{\partial r}\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right)\right\}$$

$$=\frac{\mathrm{E'I}k^2}{r}(\mathrm{D}-\mathrm{B})\sin p_2 t. \quad . \quad . \quad . \quad . \quad . \quad (18.113)$$

If B and D are not both zero either w or F_1 will be infinite where $r=0$. Then for a free disk B and D must be both zero. Therefore

$$w=\left\{\mathrm{AJ}_0(kr)+\mathrm{CI}_0(kr)\right\}\sin p_2 t$$

$$=\mathrm{A}\left\{\mathrm{J}_0(kr)+c\mathrm{I}_0(kr)\right\}\sin p_2 t, \quad . \quad . \quad . \quad . \quad (18.114)$$

Ac being written for C.

Now the condition (18.111) can be written

$$\frac{\partial}{\partial r}\left\{\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}\right\}=0 \text{ where } r=a \quad . \quad . \quad . \quad (18.115)$$

If we write

$$u=\mathrm{J}_0(kr), \quad v=\mathrm{I}_0(kr),$$

then

$$w=\mathrm{A}(u+cv)\sin p_2 t \quad . \quad . \quad . \quad . \quad . \quad . \quad (18.116)$$

Therefore the boundary condition (18.111) becomes

$$\mathrm{A}\sin p_2 t\frac{d}{dr}\left\{\frac{d^2u}{dr^2}+\frac{1}{r}\frac{du}{dr}+c\left(\frac{d^2v}{dr^2}+\frac{1}{r}\frac{dv}{dr}\right)\right\}=0$$

$$\text{where } r=a. \quad \cdot \quad (18.117)$$

Now the differential equation from which u is found is

$$\frac{d^2u}{dr^2}+\frac{1}{r}\frac{du}{dr}=-k^2u; \quad . \quad . \quad . \quad . \quad . \quad . \quad (18.118)$$

and the equation from which v is found is

$$\frac{d^2v}{dr^2}+\frac{1}{r}\frac{dv}{dr}=k^2v \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18.119)$$

Consequently the condition (18.117) reduces to

$$\frac{d}{dr}(u-cv)=0 \text{ where } r=a;$$

that is,

$$J'_0(ka)-cI'_0(ka)=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (18.120)$$

the dashes indicating differentiations with respect to ka.

Now it is proved in the appendix, equations (A.26) and (A.28), that

$$\left.\begin{aligned} J'_0(x)&=-J_1(x)\\ I'_0(x)&=I_1(x)\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18.121)$$

Therefore (18.120) becomes

$$J_1(ka)+cI_1(ka)=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (18.122$$

Again condition (18.110) can be written thus

$$\frac{\partial^2w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}=(1-\sigma)\frac{1}{r}\frac{\partial w}{\partial r},$$

which is equivalent to

$$\frac{d^2u}{dr^2}+\frac{1}{r}\frac{du}{dr}+c\left(\frac{d^2v}{dr^2}+\frac{1}{r}\frac{dv}{dr}\right)-(1-\sigma)\frac{1}{r}\left(\frac{du}{dr}+c\frac{dv}{dr}\right)=0 \quad (18.123)$$

By (18.118) and (18.119) this becomes

$$k^2(u-cv)+\frac{1-\sigma}{r}\left(\frac{du}{dr}+c\frac{dv}{dr}\right)=0 \text{ where } r=a.$$

Now using (18.121) we get

$$ka\,J_0(ka)-cka\,I_0(ka)-(1-\sigma)\{J_1(ka)-cI_1(ka)\}=0 \quad (18.124)$$

Equating the two values of $-c$ from (18.122) and (18.124) we get

$$\frac{J_1(ka)}{I_1(ka)}=-\frac{ka\,J_0(ka)-(1-\sigma)J_1(ka)}{ka\,I_0(ka)-(1-\sigma)I_1(ka)} \quad . \quad . \quad . \quad (18.125)$$

If we write b for ka this last equation gives

$$2(1-\sigma)=b\left\{\frac{I_0(b)}{I_1(b)}+\frac{J_0(b)}{J_1(b)}\right\} \quad . \quad . \quad . \quad . \quad (18.126)$$

This equation determines b, and then (18.86) gives

$$p_2{}^2=\frac{Eh^2k^4}{3\varrho(1-\sigma^2)}=\frac{Eh^2b^4}{3\varrho a^4(1-\sigma^2)}, \quad . \quad . \quad . \quad (18.127)$$

and the corresponding period of vibration is

$$t=\frac{2\pi}{p_2}=2\pi\frac{a^2}{hb^2}\sqrt{\frac{3(1-\sigma^2)\varrho}{E}} \quad . \quad . \quad . \quad . \quad (18.128)$$

Equatio. (18.126) has an infinite number of roots. Corresponding to any particular value of b given by (18.126) there is a value of c given by (18.122). Thus

$$c = -\frac{J_1(b)}{I_1(b)} \quad . \quad . \quad . \quad . \quad (18.129)$$

For all values of b except the smallest the vibrating disk has nodal circles. If the roots of (18.126) in ascending order of magnitude are b_1, b_2, b_3, etc., then in the mode of vibration corresponding to b_m the disk has m nodal circles. If c_m corresponds to b_m the radii of the nodal circles are obtained from

$$w = 0,$$

that is, from

$$J_0(kr) + c_m I_0(kr) = 0. \quad . \quad . \quad . \quad . \quad (18.130)$$

If $\sigma = \frac{1}{4}$ equation (18.126) is

$$b\left\{\frac{I_0(b)}{I_1(b)} + \frac{J_0(b)}{J_1(b)}\right\} = 1{\cdot}5 \quad . \quad . \quad . \quad . \quad (18.131)$$

Let

$$F(b) = b\left\{\frac{I_0(b)}{I_1(b)} + \frac{J_0(b)}{J_1(b)}\right\}. \quad . \quad . \quad . \quad . \quad (18.132)$$

From the series for the Bessel functions we find that $F(b) = 4$ when $b = 0$. The following table gives values of $F(b)$ calculated from tables of values of J_0, J_1, I_0, I_1.

b	0	2	2·98	2·982	3·832	6·18	6·20
$F(b)$	4	3·65	1·5082	1·4985	∞	1·63003	1·40299

By interpolation from this table we find that the first and second roots of (18.131) are

$$k_1 a = b_1 = 2{\cdot}9816, \quad . \quad . \quad . \quad . \quad (18.133)$$

$$k_2 a = b_2 = 6{\cdot}1915 \quad . \quad . \quad . \quad . \quad (18.134)$$

The corresponding periods of vibration are got by substituting these values of b in (18.128).

In the first normal mode of vibration, the mode to which b_1 applies, there is a nodal circle whose radius is the value of r determined by the equation

$$J_0(k_1 r) - \frac{J_1(b_1)}{I_1(b_1)} I_0(k_1 r) = 0,$$

that is, by

$$J_0\left(\frac{b_1 r}{a}\right) = \frac{J_1(b_1)}{I_1(b_1)} I_0\left(\frac{b_1 r}{a}\right)$$
$$= \frac{J_1(2{\cdot}9816)}{I_1(2{\cdot}9816)} I_0\left(\frac{b_1 r}{a}\right)$$
$$= 0{\cdot}08930\, I_0\left(\frac{b_1 r}{a}\right). \quad . \quad . \quad . \quad . \quad . \quad (18.135)$$

Now

$$\frac{J_0(2{\cdot}04)}{I_0(2{\cdot}04)} = 0{\cdot}08569\,;$$
$$\frac{J_0(2{\cdot}02)}{I_0(2{\cdot}02)} = 0{\cdot}09187\,.$$

By interpolation

$$\frac{b_1 r}{a} = 2{\cdot}0283\,,$$

whence

$$\frac{r}{a} = \frac{2{\cdot}0283}{2{\cdot}9816} = 0{\cdot}6802 \quad . \quad . \quad . \quad . \quad . \quad (18{\cdot}136)$$

Thus the radius of the nodal circle in the first normal mode is just over two thirds of the radius of the disk. The nodal circle is shown in fig. 176(a).

Again for the second normal mode the radii of the nodal circles are given by

$$J_0\left(\frac{b_2 r}{a}\right) = \frac{J_1(b_2)}{I_1(b_2)} I_0\left(\frac{b_2 r}{a}\right)$$
$$= \frac{J_1(6{\cdot}20)}{I_1(6{\cdot}20)} I_0\left(\frac{b_2 r}{a}\right) \text{ nearly}$$
$$= -0{\cdot}00315\, I_0\left(\frac{b_2 r}{a}\right) \quad . \quad . \quad . \quad (18.137)$$

A first approximation is got by taking

$$J_0\left(\frac{b_2 r}{a}\right) = 0\,,$$

whence

$$\frac{b_2 r}{a} = 2{\cdot}41 \quad \text{or} \quad 5{\cdot}52 \text{ nearly} \quad . \quad . \quad . \quad (18.138)$$

Assume 2·42 as a first approximation for the smaller nodal circle. Then a second approximation is given by

$$J_0\left(\frac{b_2 r}{a}\right) = -\,0{\cdot}00315\, I_0(2{\cdot}42)$$
$$= -\,0{\cdot}00975.$$

Now
$$J_0(2{\cdot}44) = -0{\cdot}00785$$
$$= -0{\cdot}01812$$

Therefore
$$\frac{b_2 r}{a} = 2{\cdot}422 \text{ approximately,}$$

whence
$$\frac{r}{a} = \frac{2{\cdot}422}{b_2} = \frac{2{\cdot}422}{6{\cdot}19} = 0{\cdot}391 \quad . \quad . \quad (18{\cdot}139)$$

For the second nodal circle the same method gives
$$\frac{r}{a} = 0{\cdot}842 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18.140)$$

First Mode

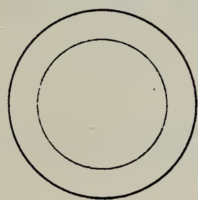

Fig. 176(a)

Second Mode

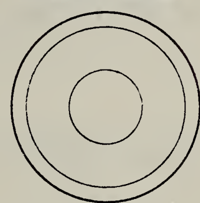

Fig. 176(b)

The nodal circles for the second mode are shown in fig. 176(b).

The other symmetrical modes of the free disk can be investigated in the same way.

334. Symmetrical oscillations of a disk clamped at the centre.

In this case there are two conditions at the centre which are equivalent to boundary conditions. They are

$$\left.\begin{aligned} w &= 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18.141)\\ \frac{\partial w}{\partial r} &= 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18.142)\end{aligned}\right\} \text{where } r = 0$$

The two conditions (18.110), (18.111), are also true. We therefore need all the four constants A, B, C, D, in (18.109),

Let $C = cA$, $D = dB$. Then (18.109) becomes

$$w = \left[A\left\{J_0(kr) + cI_0(kr)\right\} + B\left\{Z_0(kr) + dH_0(kr)\right\}\right]\sin p_2 t \quad (18.143)$$

The two conditions (18.141) and (18.142) give

$$A\left\{J_0(0) + cI_0(0)\right\} + B\left\{Z_0(0) + dH_0(0)\right\} = 0, \quad . \quad (18.144)$$

and

$$A\left\{J'_0(0) + cI'_0(0)\right\} + B\left\{Z'_0(0) + dH'_0(0)\right\} = 0. \quad (18.145)$$

Now

$$\left.\begin{array}{ll} J_0(0)=1, & I_0(0)=1, \\ J'_0(0)=0, & I'_0(0)=0. \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad (18.146)$$

Moreover, when kr is very small,

$$Z_0(kr)=J_0(kr)\log kr = \log kr \text{ approximately}, \quad . \quad . \quad (18.147)$$

$$H_0(kr)=I_0(kr)\log kr \doteq \log kr \text{ approximately} \quad . \quad . \quad (18.148)$$

Also

$$Z'_0(kr)=\frac{1}{kr}, \quad H'_0(kr)=\frac{1}{kr}. \quad . \quad . \quad . \quad . \quad . \quad (18.149)$$

Thus the coefficient of B in (18.144) and (18.145) is infinite unless $d=-1$. These two conditions can be satisfied by taking either

$$B=0 \quad \text{or } d=1.$$

Whichever alternative we choose the terms containing B disappear from (18.144) and (18.145). Then equation (18.144) gives

$$A(1+c)=0,$$

whence

$$c=-1.$$

Now we have still to satisfy two conditions at the rim, and we should not have enough available constants if we took $B=0$. We must therefore take $d=-1$. Then

$$w=\left[A\{J_0(kr)-I_0(kr)\}+B\{Z_0(kr)-H_0(kr)\}\right]\sin p_2 t. \quad (18.150)$$

Let us now put

$$\left.\begin{array}{ll} u=J_0(kr), & v=I_0(kr), \\ u_1=Z_0(kr), & v_1=H_0(kr). \end{array}\right\} \quad . \quad . \quad . \quad . \quad (18.151)$$

Then

$$w=\{A(u-v)+B(u_1-v_1)\}\sin p_2 t \quad . \quad . \quad . \quad (18.152)$$

Now because u and u_1 both satisfy (18.118), and because v and v_1 both satisfy (18.119), the two conditions (18.110) and (18.111) reduce to

$$k^2\{A(u+v)+B(u_1+v_1)\}+\frac{1-\sigma}{r}\left\{A\left(\frac{du}{dr}-\frac{dv}{dr}\right)+B\left(\frac{du_1}{dr}-\frac{dv_1}{dr}\right)\right\}=0 \quad (18.153)$$

and

$$A\left(\frac{du}{dr}+\frac{dv}{dr}\right)+B\left(\frac{du_1}{dr}+\frac{dv_1}{dr}\right)=0 \quad . \quad . \quad . \quad (18.154)$$

both to be true when $r=a$.

Now it is shown in the appendix that

$$\left.\begin{array}{l} J'_0(x)=-J_1(x), \\ Z'_0(x)=-Z_1(x), \\ I'_0(x)=I_1(x); \\ H'_0(x)=H_1(x). \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (18.155)$$

Therefore (18.153) and (18.154) become

$$A ka\{J_0(ka)+I_0(ka)\}-(1-\sigma)A\{J_1(ka)+I_1(ka)\}$$
$$+B ka\{Z_0(ka)+H_0(ka)\}-(1-\sigma)B\{Z_1(ka)+H_1(ka)\}=0, \quad (18.156)$$

and

$$A\{I_1(ka)-J_1(ka)\}+B\{H_1(ka)-Z_1(ka)\}=0. \quad . \ (18.157)$$

Equating the values of $-\frac{B}{A}$ given by the last two equations we get, writing b for (ka),

$$\frac{b\{J_0(b)+I_0(b)\}-(1-\sigma)\{J_1(b)+I_1(b)\}}{b\{Z_0(b)+H_0(b)\}-(1-\sigma)\{Z_1(b)+H_1(b)\}}=\frac{I_1(b)-J_1(b)}{H_1(b)-Z_1(b)} \quad (18.158)$$

This equation has to be solved for b. Tables of the values of the functions involved are needed to do this.

Since the necessary tables of the functions $Y_n(x)$ and $K_n(x)$ are available we shall put the equation for b in terms of these functions. From equation (18.99) we get

$$Z_0(x)=\frac{\pi}{2}Y_0(x)+\lambda J_0(x), \quad . \ . \ . \ . \ (18.159)$$

and

$$Z_1(x)=\frac{\pi}{2}Y_1(x)+\lambda J_1(x). \quad . \ . \ . \ . \ (18.160)$$

Also, from equation (18.106), we get

$$H_0(x)=-K_0(x)+\lambda I_0(x), \quad . \ . \ . \ . \ (18.161)$$

and

$$H_1(x)=K_1(x)+\lambda I_1(x). \quad . \ . \ . \ . \ . \ (18.162)$$

Now when the Z and H functions are replaced by the Y and K functions equation (18.158) becomes [I_0 being written for $I_0(b_1)$ etc.],

$$\frac{b(J_0+I_0)-(1-\sigma)(J_1+I_1)}{b\left\{\frac{\pi}{2}Y_0+\lambda J_0+\lambda I_0-K_0\right\}-(1-\sigma)\left\{\frac{\pi}{2}Y_1+\lambda J_1+K_1+\lambda I_1\right\}}$$
$$=\frac{I_1-J_1}{K_1-\frac{\pi}{2}Y_1+\lambda I_1-\lambda J_1},$$

which can be reduced to

$$\frac{b(J_0+I_0)-(1-\sigma)(J_1+I_1)}{b\left(\frac{\pi}{2}Y_0-K_0\right)-(1-\sigma)\left(\frac{\pi}{2}Y_1+K_1\right)}=\frac{I_1-J_1}{K_1-\frac{\pi}{2}Y_1} \quad . \ (18.163)$$

If we denote the two expressions on the left and right sides of the last equation by L(b) and R(b), we find from tables, taking $\sigma = \frac{1}{4}$,

$$L(1{\cdot}92) - R(1{\cdot}92) = 0{\cdot}080,$$
$$L(1{\cdot}94) - R(1{\cdot}94) = -0{\cdot}115,$$

whence by interpolation

$$L(1{\cdot}9282) - R(1{\cdot}9282) = 0 \text{ approximately.}$$

Therefore

$$ka = b = 1{\cdot}928 \quad \ldots\ldots\ldots \quad (18.164)$$

is the least root of (15·163). If $\sigma = 0{\cdot}3$ the least root is 1·937.

Corresponding to the least root that we have just found there is no nodal circle. The whole middle surface of the disk except its centre oscillates from one side to the other of the plane touching this surface at the centre.

The period of vibration is, by (18.128),

$$\frac{2\pi}{p_2} = \frac{2\pi a^2}{hb^2}\sqrt{\frac{3(1-\sigma^2)\varrho}{E}}$$
$$= \frac{2\pi a^2}{(1{\cdot}928)^2 h}\sqrt{\frac{45\varrho}{16E}} \quad \ldots\ldots \quad (18.165)$$

when $\sigma = \frac{1}{4}$.

The periods of the other normal modes—the modes with one, two, or more nodal circles—can be got by finding the other roots of (15.163), of which there is an infinite number, successive roots differing by approximately π.

The shear force F_1 across a circular section in the direction of the z-axis is

$$F_1 = \frac{2}{3}\frac{Eh^3}{1-\sigma^2}\frac{\partial}{\partial r}\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right)\right\} \quad \ldots \quad (18.166)$$

Now when r is very small the most important terms in F_1 are the terms involving $\log(kr)$ or negative powers of r, unless these terms vanish; but we shall see they do not vanish. For, taking only the terms involving $\log r$ in (18.150), we get

$$w = B\left\{Z_0(kr) - H_0(kr)\right\}\sin p_2 t$$
$$= B\left\{J_0(kr) - I_0(kr)\right\}\log kr \sin p_2 t$$
$$= -2B\left\{\frac{k^2r^2}{2^2} + \frac{k^6r^6}{2^2.4^2.6^2} + \ldots\right\}\log kr \sin p_2 t.$$

Thus the important term in w when r is small is

$$w = -\tfrac{1}{2}Bk^2r^2 \log r \sin p_2 t \quad \ldots\ldots \quad (18.167)$$

Now

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left\{r\frac{d(r^2\log r)}{dr}\right\}\right] = \frac{4}{r} \quad \ldots\ldots \quad (18.168)$$

Therefore

$$F_1 = -\frac{4}{3}\frac{EBk^2h^3}{1-\sigma^2}\frac{1}{r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18.169)$$

is the approximate value of the shear force when r is small. But this is infinite at the centre of the disk. This result could easily be seen without any elaborate analysis, for it is obvious that the assumptions in this problem mean that an infinitely small circle at the centre of the disk is held fixed while the rest of the disk oscillates. The shear force F_1 round the rim of this infinitesimal circle supplies the action needed to change the momentum of the rest of the disk. Thus $2\pi r F_1$ is the force supplying momentum to the disk, and this must be finite; a result which is in agreement with (18.169).

Since an infinite shear force is impossible the actual conditions we have assumed in this problem cannot exist in practice. If, however, a disk were held fixed over a circle whose radius were much smaller than the radius of the disk itself the results that we have worked out would apply with fair accuracy to such a disk.

335. Free disk with nodal diameters.

In a disk which is free at the centre both w and F are clearly finite at the centre. These two conditions require that the coefficients B and D in (18.107) should be zero. Then

$$w = \{AJ_n(kr) + CI_n(kr)\}\sin(n\theta+\alpha)\sin p_2 t. \quad . \quad . \quad (18.170)$$

The boundary conditions are those given in (14.46) and (14.47). Now for a disk the bending moment across a circumferential section is, by (14.57),

$$M_1 = E'I\left\{\frac{\partial^2 w}{\partial r^2} + \sigma\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2}\right)\right\}. \quad . \quad . \quad . \quad (18.171)$$

Also the shear force across the same section is, by (14.59),

$$F_1 = -E'I\frac{\partial}{\partial r}\left\{\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2}\right\}; \quad . \quad . \quad . \quad (18.172)$$

and the torque across an element of area perpendicular to the radius through the area is

$$Q = \frac{EI}{1+\sigma}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w}{\partial\theta}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (18.173)$$

Therefore (14.46) and (14.47) give

$$\frac{\partial^2 w}{\partial r^2} + \sigma\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2}\right) = 0, \quad . \quad . \quad . \quad (18.174)$$

and

$$\frac{\partial}{\partial r}\left\{\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2}\right\} + (1-\sigma)\frac{\partial}{r\partial\theta}\left\{\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w}{\partial\theta}\right)\right\} = 0, \quad (18.175)$$

both to be true when $r = a$.

We may write these two equations thus

$$\nabla_1^2 w-(1-\sigma)\left(\frac{1}{r}\frac{\partial w}{\partial r}+\frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right)=0, \quad . \quad . \quad . \quad (18.176)$$

$$\frac{\partial}{\partial r}(\nabla_1^2 w)+(1-\sigma)\frac{1}{r}\frac{\partial}{\partial \theta}\left\{\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w}{\partial \theta}\right)\right\}=0. \quad . \quad . \quad . \quad (18.177)$$

Now $J_n(kr)\sin(n\theta+\alpha)$ and $I_n(kr)\sin(n\theta+\alpha)$ are respectively the solutions of the equations

$$\nabla_1^2 w=-k^2 w, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18.178)$$

and

$$\nabla_1^2 w=k^2 w. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18.179)$$

Therefore, if we write cA for C, the boundary conditions (18.176) and (18.177) can be written thus:—

$$k^2\{J_n(ka)-cI_n(ka)\}-(1-\sigma)\frac{n^2}{a^2}\{J_n(ka)+cI_n(ka)\}$$

$$+(1-\sigma)\frac{k}{a}\{J'_n(ka)+cI'_n(ka)\}=0, \quad . \quad . \quad . \quad (18.180)$$

and

$$k^3\{J'_n(ka)-cI'_n(ka)\}+(1-\sigma)\frac{kn^2}{a^2}\{J'_n(ka)+cI'_n(ka)\}$$

$$-(1-\sigma)\frac{n^2}{a^3}\{J_n(ka)+cI_n(ka)\}=0 \quad . \quad . \quad . \quad (18.181)$$

Equating the values of c given by these two equations we get

$$\frac{k^2a^2J_n(ka)+(1-\sigma)\{kaJ'_n(ka)-n^2J_n(ka)\}}{k^2a^2I_n(ka)-(1-\sigma)\{kaI'_n(ka)-n^2I_n(ka)\}}=c$$

$$=\frac{k^3a^3J'_n(ka)+(1-\sigma)n^2\{kaJ'_n(ka)-J_n(ka)\}}{k^3a^3I'_n(ka)-(1-\sigma)n^2\{kaI'_n(ka)-I_n(ka)\}}. \quad . \quad . \quad (18.182)$$

But, by (A.74) and (A.75) in the Appendix, we have

$$kaJ'_n(ka)=nJ_n(ka)-kaJ_{n+1}(ka). \quad . \quad . \quad . \quad (18.183)$$

and

$$kaJ'_n(ka)=-nJ_n(ka)+kaJ_{n-1}(ka). \quad . \quad . \quad (18.184)$$

Either of these can be substituted for $kaJ'_n(ka)$ in the boundary conditions.

Also, by putting ix for x in the last two equations and using (A.59) we get

$$kaI'_n(ka)=nI_n(ka)+kaI_{n+1}(ka), \quad . \quad . \quad . \quad (18.185)$$

and

$$kaI'_n(ka)=-nI_n(ka)+kaI_{n-1}(ka). \quad . \quad . \quad (18.186)$$

When equations (18.183) to (18.185) have been used to eliminate $J'_n(ka)$ and $I'_n(ka)$ from (18.182) the values of ka satisfying the resulting equation can be found by means of tables of Bessel functions.

The particular case of $n=0$ we have already worked out. If $n=1$ equation (18.182) can be reduced to

$$b\left\{\frac{J_1(b)}{J_2(b)}+\frac{I_1(b)}{I_2(b)}\right\}=2(1-\sigma), \quad . \quad . \quad . \quad . \quad (18.187)$$

b being written for ka.

If $\sigma=0{\cdot}25$ and if $F(b)$ be written for the left hand side of the last equation, we get

$$F(b)=1{\cdot}50.$$

Now from tables of Bessel functions

$$F(b)=1{\cdot}734 \text{ when } b=4{\cdot}50,$$
$$F(b)=0{\cdot}464 \text{ when } b=4{\cdot}60.$$

By interpolation we find, as the first root of (18.187),

$$ka=b=4{\cdot}518. \quad . \quad . \quad . \quad . \quad . \quad . \quad (18.188)$$

When $n=1$ the two equations (18.182) become

$$c=\frac{b^2J_1(b)+(1-\sigma)\{bJ'_1(b)-J_1(b)\}}{b^2I_1(b)-(1-\sigma)\{bI'_1(b)-I_1(b)\}}, \quad . \quad . \quad . \quad (18.189)$$

$$c=\frac{b^3J'_1(b)+(1-\sigma)\{bJ'_1(b)-J_1(b)\}}{b^3I'(b)-(1-\sigma)\{bI'_1(b)-I_1(b)\}}. \quad . \quad . \quad . \quad (18.190)$$

By combining these we get

$$c=\frac{b^3J'_1(b)-b^2J_1(b)}{b^3I'_1(b)-b^2I_1(b)}$$
$$=\frac{bJ'_1(b)-J_1(b)}{bI'_1(b)-I_1(b)}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18.191)$$

whence we get, by means of (18.183) and (18.185),

$$c=-\frac{J_2(b)}{I_2(b)}=-\frac{J_2(ka)}{I_2(ka)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18.192)$$

Therefore the expression for the deflexion is

$$w=A\left\{J_1(kr)-\frac{J_2(ka)}{I_2(ka)}I_1(kr)\right\}\sin(\theta+\alpha)\sin p_2t$$
$$=G\left\{\frac{J_1(kr)}{J_2(ka)}-\frac{I_1(kr)}{I_2(ka)}\right\}\sin(\theta+\alpha)\sin p_2t, \quad . \quad . \quad . \quad (18.193)$$

where

$$G=AJ_2(ka).$$

Substituting the numerical values of $J_2(4{\cdot}518)$ and $I_2(4{\cdot}518)$ we get

$$w=G\left\{\frac{J_1(kr)}{0{\cdot}2119}-\frac{I_1(kr)}{10{\cdot}84}\right\}\sin(\theta+\alpha)\sin p_2t. \quad . \quad . \quad (18.194)$$

This value of w is zero when

$$kr=3{\cdot}530, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18.195)$$

and since $ka=4{\cdot}518$ approximately there is a nodal circle of radius

$$r=\frac{3{\cdot}530}{4{\cdot}518}a=0{\cdot}781\,a. \quad . \quad . \quad . \quad . \quad . \quad (18.196)$$

What is remarkable in the results we have just got is the fact that the slowest vibration of a disk with one nodal diameter is one in which there is a nodal circle. There is no mode with one nodal diameter and no nodal circle. It is easy to see the physical reason for this. A free disk vibrating with one nodal diameter and no nodal circle would be like a free rod vibrating in an unsymmetrical mode with a node at the middle point and no other nodes. This is clearly impossible.

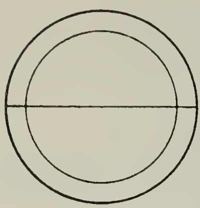

Fig. 177

The nodal lines for the first mode are shown in fig. 177.

The mode of vibration in which the frequency of a free disk is least is the one with two nodal diameters and no nodal circle. When $n=2$ equation (18.182) becomes, with b for ka,

$$\frac{b^2 J_2 + (1-\sigma)\{bJ'_2 - 4J_2\}}{b^2 I_2 - (1-\sigma)\{bI'_2 - 4I_2\}} = \frac{b^3 J'_2 + 4(1-\sigma)\{bJ'_2 - J_2\}}{b^3 I'_2 - 4(1-\sigma)\{bI'_2 - I_2\}} \quad . \quad (18.197)$$

Now

$$bJ'_2 = bJ_1 - 2J_2$$

and

$$bI'_2 = bI_1 - 2I_2.$$

Also

$$bJ_2 = 2J_1 - bJ_0$$

and

$$bI_2 = bI_0 - 2I_1.$$

Using the last four equations to eliminate J_2 and J'_2 from (18.197), and putting 0·25 for σ, we deduce from this equation

$$\frac{(11b^2-36)J_1 - 2b(2b^2-9)J_0}{4(b^4-b^2-18)J_0 + 4b(2b^2+9)J_0}$$
$$= \frac{2b(2b^2+9)I_0 - (11b^2+36)I_1}{4(b^4+b^2-18)I_0 - 4b(2b^2-9)I_0}. \quad . \quad (18.198)$$

The two smallest roots of this equation are

$$b_1 = 2{\cdot}3475, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18.199)$$

and

$$b_2 = 5{\cdot}9405. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18.200)$$

Kirchhoff* worked out the problem of the vibrating free uniform disk, and he gave a table of values of $\log(\frac{1}{2}ka)^4$. The following table of values of $(ka)^4$ is built up from Kirchhoff's table. It is useful to repeat here that, for values of n greater than one, the results are the same as for a disk clamped at the centre.

* *Gesammelte Abhandlungen* von G. Kirchhoff, Leipzig 1882.

Values of $(ka)^4$ for a free uniform disk with n nodal diameters and s nodal circles

	s	$n=0$	$n=1$	$n=2$	$n=3$
$\sigma=\frac{1}{4}$	0	—	—	30·373	162·42
	1	79·030	416·53	1245·2	2825·2
	2	1469·6	3567·9	—	—
$\sigma=\frac{1}{3}$	0	—	—	27·574	149·37
	1	82·376	420·96	1241·9	2799·6
	2	1483·3	3583·8	—	—

336. Disk clamped at the rim.

Since the centre is free the two Bessel functions that are infinite at the centre are not needed. That is, an adequate solution of the differential equation is, as for the free disk,

$$w = A\{J_n(kr) + cI_n(kr)\}\sin(n\theta + a)\sin p_2 t \quad . \quad . \quad (18.201)$$

The boundary conditions are

$$w = 0, \frac{\partial w}{\partial r} = 0, \text{ where } r = a \quad . \quad . \quad . \quad . \quad . \quad . \quad (18.202)$$

These become

$$J_n(ka) + cI_n(ka) = 0,$$
$$J'_n(ka) + cI'_n(ka) = 0,$$

whence we get, on eliminating c,

$$\frac{J'_n(ka)}{J_n(ka)} = \frac{I'_n(ka)}{I_n(ka)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (18.203)$$

By means of (18.184) and (18.186) this equation can be put into the form

$$\frac{J_{n-1}(ka)}{J_n(ka)} = \frac{I_{n-1}(ka)}{I_n(ka)}. \quad . \quad . \quad . \quad . \quad . \quad (18.204)$$

If we use (18.183) and (18.185) we transform (18.203) into

$$\frac{J_{n+1}(ka)}{J_n(ka)} + \frac{I_{n+1}(ka)}{I_n(ka)} = 0. \quad . \quad . \quad . \quad . \quad (18.205)$$

Either of the last two equations gives ka when n is not zero. It is best to use (18.205) when $n = 0$.

A few solutions of (18.203) are contained in the following table.

Values of $(ka)^4$ for a disk clamped at the rim vibrating with n nodal diameters and s nodal circles.

s	$n=0$	$n=1$	$n=2$
0	104·2	450·4	1214
1	1582	—	—
2	7902	—	—

337. Disk supported without clamping at the rim.

Here again

$$w = A\{J_n(kr) + cI_n(kr)\} \sin(n\theta + a) \sin p_2 t \quad . \quad . \quad (18.206)$$

The boundary conditions are now

$$w = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18.207)$$

and

$$\frac{\partial^2 w}{\partial r^2} + \sigma\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right) = 0 \quad . \quad . \quad . \quad . \quad (18.208)$$

where $r = a$.

The second of the preceding conditions can be written

$$\nabla_1{}^2 w - (1-\sigma)\left\{\frac{1}{r}\frac{\partial w}{\partial r} - \frac{n^2}{r^2} w\right\} = 0 \quad . \quad . \quad . \quad . \quad (18.209)$$

Now let

$$w_1 = J_n(kr) \sin(n\theta + a), \quad . \quad . \quad . \quad . \quad . \quad . \quad (18.210)$$

$$w_2 = I_n(kr) \sin(n\theta + a). \quad . \quad . \quad . \quad . \quad . \quad . \quad (18.211)$$

Then

$$\nabla_1{}^2 w_1 = -k^2 w_1, \quad . \quad . \quad . \quad . \quad . \quad . \quad (18.212)$$

$$\nabla_1{}^2 w_2 = k^2 w_2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (18.213)$$

Therefore the condition (18.209) becomes

$$(1-\sigma)\left\{\frac{k}{a} J'_n(ka) - \frac{n^2}{a^2} J_n(ka)\right\} + k^2 J_n(ka)$$

$$+ c(1-\sigma)\left\{\frac{k}{a} I'_n(ka) - \frac{n^2}{a^2} I_n(ka)\right\} - ck^2 I_n(ka) = 0. \quad . \quad (18.214)$$

Also condition (18.207) is

$$J_n(ka) + cI_n(ka) = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (18.215)$$

Eliminating c from the last two equations we get

$$(1-\sigma)\left\{\frac{I'_n(ka)}{I_n(ka)} - \frac{J'_n(ka)}{J_n(ka)}\right\} = 2ka, \quad . \quad . \quad . \quad (18.216)$$

which, by means of (18.183) to (18.186), can be put in either of the forms

$$\frac{I_{n-1}(ka)}{I_n(ka)} - \frac{J_{n-1}(ka)}{J_n(ka)} = \frac{2ka}{1-\sigma}, \quad . \quad . \quad . \quad . \quad (18.217)$$

or

$$\frac{I_{n+1}(ka)}{I_n(ka)} + \frac{J_{n+1}(ka)}{J_n(ka)} = \frac{2ka}{1-\sigma}. \quad . \quad . \quad . \quad . \quad (18.218)$$

When $n = 0$ and $\sigma = 0{\cdot}25$ the first root of (18.218) is approximately

$$ka = 2{\cdot}204,$$

from which

$$k^4 a^4 = 23{\cdot}60. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18.219)$$

338. Vibrations of a disk of variable thickness controlled by rigitidy.

The equation connecting pressure with deflexion in a plate of variable thickness is (14.35), which is repeated here:—

$$\frac{E}{1-\sigma^2}\nabla_1^2(I\nabla_1^2 w)=p. \quad . \quad . \quad . \quad . \quad . \quad (18.220)$$

Now putting, for the vibrating disk,

$$p=-2\varrho h\frac{\partial^2 w}{\partial t^2}$$

we get

$$\frac{E}{1-\sigma^2}\nabla_1^2(I\nabla_1^2 w)+2\varrho h\frac{\partial^2 w}{\partial t^2}=0 \quad . \quad . \quad . \quad (18.221)$$

Let us take, as in (18.46),

$$h=c\left(\frac{r}{a}\right)^{-\beta}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (18.222)$$

Then, since

$$I=\tfrac{2}{3}h^3, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18.223)$$

equation (18.221) gives

$$\nabla_1^2\left\{\left(\frac{r}{a}\right)^{-3\beta}\nabla_1^2 w\right\}+\frac{3\varrho(1-\sigma^2)}{Ec^2}\left(\frac{r}{a}\right)^{-\beta}\frac{\partial^2 w}{\partial t^2}=0. \quad . \quad (18.224)$$

If we next put

$$w=z\sin(n\theta+a)\sin p_2 t \quad . \quad . \quad . \quad . \quad . \quad (18.225)$$

equation (18.224) becomes, after division by $\sin(n\theta+a)\sin p_2 t$,

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{n^2}{r^2}\right)\left\{\left(\frac{r}{a}\right)^{-3\beta}\left(\frac{d^2z}{dr^2}+\frac{1}{r}\frac{dz}{dr}-\frac{n^2z}{r^2}\right)\right\}-\frac{\mu}{a^4}\left(\frac{r}{a}\right)^{-\beta}z=0,$$

where

$$\mu=\frac{3\varrho p_2^{\,2}a^4(1-\sigma^2)}{Ec^2}. \quad . \quad . \quad . \quad . \quad . \quad (18.226)$$

Let us next put $a\eta$ for r; then the equation for z becomes

$$\left(\frac{d^2}{d\eta^2}+\frac{1}{\eta}\frac{d}{d\eta}-\frac{n^2}{\eta^2}\right)\left\{\eta^{-3\beta}\left(\frac{d^2z}{d\eta^2}+\frac{1}{\eta}\frac{dz}{d\eta}-\frac{n^2z}{\eta^2}\right)\right\}-\mu\eta^{-\beta}z=0.. \quad (18.227)$$

The solution of this equation can be expressed by means of a series of powers of η for nearly any value of n. There are exceptional cases that do not yield to a mere series of powers; this we should expect because the simpler problem of a vibrating uniform disk led to Bessel functions some of which involve log r.

When we put, in (18.227),

$$z=\Sigma C_k\eta^k \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18.228)$$

we get, on writing l for $(2+3\beta)$ and m for $(4+2\beta)$,

$$\Sigma C_k \eta^{k-\beta}\left[(k^2-n^2)\{(k-l)^2-n^2\}\eta^{-m}-\mu\right]=0. \quad . \quad (18.229)$$

Equating to zero the coefficient of $\eta^{k-\beta-m}$ in this we get

$$(k^2-n^2)\{(k-l)^2-n^2\}C_k-\mu C_{k-m}=0. \quad . \quad . \quad (18.230)$$

Now this shows that C_{k-m} is zero for any arbitrary value of C_k provided that

$$(k^2-n^2)\{(k-l)^2-n^2\}=0,$$

that is, provided that

$$k=n,\ -n,\ l+n \text{ or } l-n. \quad . \quad . \quad . \quad . \quad (18.231)$$

Thus in general we get four series starting with the several values of k given by (18.231). In any one of these series the indices of the powers of η form an arithmetical progression with common difference m. Moreover the ratio of successive coefficients is, by (18.230),

$$\frac{C_k}{C_{k-m}}=\frac{\mu}{(k^2-n^2)\{(k-l)^2-n^2\}}$$

$$=\frac{\mu}{(k-n)(k+n)(k-l-n)(k-l+n)}. \quad . \quad . \quad (18.232)$$

If $n=0$ equation (18.231) gives

$$k=0,\ 0,\ l,\ l. \quad . \quad . \quad . \quad . \quad . \quad . \quad (18.233)$$

This indicates that there are only two simple power series and these begin with 1 and n^l respectively. The other two series begin with $\log\eta$, and $\eta^l\log\eta$. Now, for a disk clamped at the centre, z and $\frac{dz}{d\eta}$ must be zero when η is zero. This requires that the coefficients of the two series beginning with 1 and $\log\eta$ respectively must vanish. This leaves only the series beginning with η^l and $\eta^l\log\eta$. For the series beginning with η^l we get, from (18.232),

$$\frac{C_{l+m}}{C_l}=\frac{\mu}{(l+m)^2m^2}$$

$$\frac{C_{l+2m}}{C_{l+m}}=\frac{\mu}{(l+2m)^2(2m)^2}.$$

Therefore the series is

$$z_1=\eta^l\left\{1+\frac{\mu\eta^m}{m^2(l+m)^2}+\frac{\mu^2\eta^{2m}}{m^2(2m)^2(l+m)^2(l+2m)^2}+\ldots\right\} \quad (18.234)$$

By a method exactly like the one used in the appendix to get $Z_0(x)$ it can be shown that a second solution of the equation (18.227) when $n=0$ is

$$z_2=z^2\log\eta-2\eta^l\sum_{m=1}^{m=\infty}H_q\frac{\mu^q\eta^{qm}}{m^2(2m)^2\ldots(qm)^2(l+m)^2(l+2m)^2\ldots(l+qm)^2} \quad (18.235)$$

where

$$H_q = \frac{1}{m}\left\{1 + \tfrac{1}{2} + \tfrac{1}{3} + \ldots + \frac{1}{q}\right\}$$

$$+ \frac{1}{l+m} + \frac{1}{l+2m} + \ldots + \frac{1}{l+mq}. \quad . \quad . \quad (18.236)$$

The solution of (18.227) applicable to a disk with a fixed centre vibrating with no nodal diameters is therefore

$$z = Az_1 + Bz_2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (18.237)$$

For a complete disk vibrating with n nodal diameters we need only two of the series starting with powers of η whose indices are given by (18.231). We have to decide which of these series apply to the complete disk. Since the disk has an infinite thickness at the centre we cannot now reject a solution which gives an infinite bending moment at the centre, for an infinite bending moment is not unreasonable where the thickness is infinite. It is, however, safe to assume that in any possible vibrations the energy must be finite. If then two out of the four series give vibrations with finite energy while the other two give vibrations with infinite energy we must choose the first two series for the complete disk.

Now suppose η^{k_1} is the first term in one of the series for z. By (14.154) the elastic energy in the disk is

$$\iint \tfrac{1}{2} E'I\left[(\nabla_1{}^2 w)^2 - 2(1-\sigma)\left\{\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\right\}\right] dx dy \quad (18.238)$$

The term contributed to this integral by η^{k_1} is proportional to

$$\int_0^1 \eta^{-3\beta}\left(\frac{d^2\eta^{k_1}}{d\eta^2}\right)^2 \eta d\eta,$$

which is proportional to

$$\int_0^1 \eta^{2k_1-3\beta-3} d\eta.$$

Now this is finite provided

$$2k_1 - 3\beta - 2 > 0;$$

that is, provided

$$2k_1 > l. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18.239)$$

If this last inequality holds the series beginning with η^{k_1} will indicate a state of strain in which the total energy in the disk is finite. Now k_1 can have any of the four values in (18.231). If we take $(l+n)$ for k_1 it is clear that (18.239) holds. Again if we take $-n$ for k_1 it is clear that (18.239) does not hold. If we take n for k_1 (18.239) becomes

$$2n > l, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18.240)$$

whereas if we take $(l-n)$ for k_1 (18.239) gives

$$2l - 2n > l;$$

that is,

$$l > 2n. \quad . \quad . \quad . \quad . \quad (18.241)$$

Now omitting the critical case where $2n = l$ (which needs special treatment) one of the two inequalities (18.240) or (18.241) must be true. If $2n > l$ the series beginning with η^n is applicable to a complete disk; while if $2n < l$ the series beginning with η^{l-n} must replace the one beginning with η^n. The other series applicable to a complete disk is the one beginning with η^{l+n}.

Thus we may write, when $n > 0$,

$$z = Az_1 + Bz_2 \quad . \quad . \quad . \quad . \quad (18.242)$$

where z_1 is a series beginning with η^{l+n}, and z_2 is a series beginning with η^{k_1}, k_1 being the larger of the two numbers n and $(l-n)$. Moreover this value of k_1 is the index of the lowest power of η occurring in z. Thus we get the following values of k_1 for the given values of n and β.

n	β	$l-n$	k_1
2	$\frac{1}{2}$	$\frac{3}{2}$	2
2	1	3	3
3	1	2	3
3	2	5	5

The series beginning with η^{l+n} is

$$z_1 = \eta^{l+n}\left\{\begin{aligned}&1 + \frac{\mu\eta^m}{m(m+2n)(m+l)(m+l+2n)}\\ &+ \frac{\mu^2\eta^{2m}}{m.2m(m+2n)(2m+2n)(m+l)(2m+l)(m+l+2n)(2m+l+2n)}\\ &+ \text{etc.}\end{aligned}\right\} \quad (18.243)$$

If we write the last equation in the form

$$z_1 = \eta^{l+n}\,F(m, n, l) \quad . \quad . \quad . \quad . \quad (18.244)$$

then the two series beginning with η^n and η^{l-n} are respectively

$$z_2 = \eta^n F(m, n, -l), \quad . \quad . \quad . \quad . \quad (18.245)$$

and

$$z_3 = \eta^{l-n} F(m, -n, l). \quad . \quad . \quad . \quad . \quad (18.246)$$

It is useful to recall here that

$$l = 2 + 3\beta; \; m = 4 + 2\beta. \quad . \quad . \quad . \quad (18.247)$$

It is worth while to notice that, if $\beta = 0$, (in which case $l = 2$ and $m = 4$) the series in (18.243) and (18.245) become

$$z_1 = 2^{n+1}\underline{|n+1}\,\{I_u(x) - J_n(x)\},$$

and

$$z_2 = 2^{n-1}\underline{|n}\,\{I_n(x) + J_n(x)\},$$

x being $\mu^{\frac{1}{4}}\eta$.

The case where $2n = l$ is more troublesome because the value of z contains $\log \eta$. The solution of the differential equation (18.227) for this case is given in the appendix.

339. The boundary conditions.

The boundary conditions for a disk with variable thickness corresponding to (18.174) and (18.175) for the uniform disk are

$$\frac{\partial^2 w}{\partial r^2} + \sigma\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right) = 0, \quad . \quad . \quad . \quad . \quad (18.248)$$

$$\frac{\partial}{\partial r}\left\{h^3\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right)\right\} + (1-\sigma)\frac{1}{r}\frac{\partial}{\partial \theta}\left\{h^3\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w}{\partial \theta}\right)\right\} = 0 \quad (18.249)$$

both the be true when $r = a$.

These two conditions reduce to

$$\frac{d^2 z}{d\eta^2} + \sigma\left(\frac{1}{\eta}\frac{dz}{d\eta} - \frac{n^2 z}{\eta^2}\right) = 0 \quad . \quad . \quad . \quad . \quad . \quad (18.250)$$

$$\eta^{3\beta}\frac{d}{d\eta}\left\{\eta^{-3\beta}\left(\frac{d^2 z}{d\eta^2} + \frac{1}{\eta}\frac{dz}{d\eta} - \frac{n^2 z}{\eta^2}\right)\right\} - \frac{(1-\sigma)n^2}{\eta}\frac{d}{d\eta}\left(\frac{z}{\eta}\right) = 0 \quad (18.251)$$

both to be true when $\eta = 1$.

Now if $$z = \Sigma C_k \eta^k \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18.252)$$

these boundary conditions become

$$\Sigma C_k\{k(k-1) + \sigma k - \sigma n^2\} = 0 \quad . \quad . \quad . \quad . \quad (18.253)$$

$$\Sigma C_k\{(k^2 - n^2)(k - l) - (1-\sigma)(k-1)n^2\} = 0 \quad . \quad . \quad (18.254)$$

If (18.252) is the same equation as (18.242) each of the last two equations contains two infinite series of powers of μ multiplied by the constants A and B. The two equations determine, therefore, the ratio A : B and the constant μ, from which the frequencies of the modes of vibration can be calculated. Just as for a uniform disk the equation for μ has an infinite number of roots. The equation for μ is of the type

$$\frac{X}{Y} = \frac{V}{W},$$

where each of the symbols X, Y, V, W, represents an infinite series of powers of μ. Although some tedious arithmetical work is involved in the calculation of the roots it will be found, when numerical values are substituted for the constants, that the series converge fairly quickly for the smaller roots; and it is the smaller roots that are important in practical applications.

340. Vibrations due to rigidity; approximate method.

Lord Rayleigh's principle, which was used in Art. 163 for a thin rod, gives excellent results for disks. The principle consists in assuming a reasonable form for the deflection and then calculating the frequency from the energy equation. The fact that the slowest frequency under any given conditions is the minimum frequency that the energy equation can give under the same conditions for any assumed value for the deflection makes it easy to get a good approximation to this frequency. The method we shall use in applying Rayleigh's principle is to assume a form for z which is reasonably like the form suggested by theory, and to leave one parameter in z to be determined by the condition that the frequency calculated from this assumed value of z is a minimum. This seldom fails to give the frequency to within one or two per cent.*

By (14.154) the elastic energy in a plate is

$$V=\tfrac{1}{3}E'\iint h^3\left[(\nabla_1{}^2 w)^2-2(1-\sigma)\left\{\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}-\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2\right\}\right]dxdy \quad (18.255)$$

Also the kinetic energy is

$$T=\tfrac{1}{2}\iint 2\rho h\left(\frac{\partial w}{\partial t}\right)^2 dxdy \quad . \quad . \quad . \quad . \quad . \quad (18.256)$$

Now in polar coordinates

$$\nabla_1{}^2 w=\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}+\frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}, \quad . \quad . \quad . \quad . \quad (18.257)$$

$$\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}-\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2=\frac{\partial^2 w}{\partial r^2}\left\{\frac{1}{r}\frac{\partial w}{\partial r}+\frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right\}-\left\{\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w}{\partial \theta}\right)\right\}^2 . \quad . \quad (18.258)$$

If we now assume that

$$w=z\sin p_2 t\sin(n\theta+a) \quad . \quad . \quad . \quad . \quad . \quad (18.259)$$

we get

$$\nabla_1{}^2 w=\left\{\frac{d^2 z}{dr^2}+\frac{1}{r}\frac{dz}{dr}-\frac{n^2 z}{r^2}\right\}\sin p_2 t\sin(n\theta+a),$$

$$\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}-\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2=\frac{d^2 z}{dr^2}\left\{\frac{1}{r}\frac{dz}{dr}-\frac{n^2 z}{r^2}\right\}\sin^2 p_2 t\sin^2(n\theta+a)$$
$$-n^2\left\{\frac{d}{dr}\left(\frac{z}{r}\right)\right\}^2\sin^2 p_2 t\cos^2(n\theta+a).$$

Therefore, since

$$\int_0^{2\pi}\sin^2(n\theta+a)d\theta=\int_0^{2\pi}\cos^2(n\theta+a)d\theta=\pi,$$

we get

$$V=\tfrac{1}{3}\pi E'\sin^2 p_2 t\int_0^a\left[\begin{array}{l}\left\{\frac{d^2 z}{dr^2}+\frac{1}{r}\frac{dz}{dr}-\frac{n^2 z}{r^2}\right\}^2\\ -2(1-\sigma)\left\{\frac{d^2 z}{dr^2}\left(\frac{1}{r}\frac{dz}{dr}-\frac{n^2 z}{r^2}\right)-n^2\left(\frac{d(zr^{-1})}{dr}\right)^2\right\}\end{array}\right]h^3 r dr.$$

* The device of expressing p^2 in terms of a parameter and then making p^2 a minimum is propounded by Rayleigh (*Sound*, Vol I, Art. 89). The method is elaborated and its usefulness stressed in the paper by Lamb and Southwell on *Spinning Disks.*

Also

$$T = \pi p_2{}^2 \cos^2 p_2 t \int_0^a \varrho h z^2 r dr$$

$$= \pi p_2{}^2 (1 - \sin^2 p_2 t) \int_0^a \varrho h z^2 r dr.$$

Now the energy equation

$$V + T = \text{constant} \quad . \quad . \quad . \quad . \quad . \quad . \quad (18.260)$$

is satisfied if the coefficient of $\sin^2 p_2 t$ is zero in this equation.

Putting $a\eta$ for r and assuming that h is equal to $c\eta^{-\beta}$, we find, by equating to zero the coefficient of $\sin^2 p_2 t$ in (18.260),

$$\frac{3\varrho p_2{}^2 a^4 (1-\sigma^2)}{Ec^2} \int_0^1 z^2 \eta^{1-\beta} d\eta$$

$$= \int_0^1 \eta^{1-3\beta} \left[\begin{array}{l} \left\{ \dfrac{d^2 z}{d\eta^2} + \dfrac{1}{\eta}\dfrac{dz}{d\eta} - \dfrac{n^2 z}{\eta^2} \right\}^2 \\ -2(1-\sigma)\left\{ \dfrac{d^2 z}{d\eta^2}\left(\dfrac{1}{\eta}\dfrac{dz}{d\eta} - \dfrac{n^2 z}{\eta^2} \right) - n^2 \left(\dfrac{d(z\eta^{-1})}{d\eta} \right)^2 \right\} \end{array} \right] d\eta \quad (18.261)$$

Now it can be verified that

$$\left. \begin{array}{l} \dfrac{2}{\eta} z \dfrac{d^2 z}{d\eta^2} + 2\eta \left(\dfrac{d(z\eta^{-1})}{d\eta} \right)^2 = \dfrac{d^2}{d\eta^2}\left(\dfrac{z^2}{\eta} \right). \\ \text{Also} \qquad 2\dfrac{d^2 z}{d\eta^2}\dfrac{dz}{d\eta} = \dfrac{d}{d\eta}\left(\dfrac{dz}{d\eta} \right)^2 \end{array} \right\} \quad . \quad . \quad . \quad (18.262)$$

Therefore (18.261) can be written thus

$$\frac{3\varrho p_2{}^2 a^4 (1-\sigma^2)}{Ec^2} \int_0^1 z^2 \eta^{1-\beta} d\eta$$

$$= \int_0^1 \left[\begin{array}{l} \eta \left\{ \dfrac{d^2 z}{d\eta^2} + \dfrac{1}{\eta}\dfrac{dz}{d\eta} - \dfrac{n^2 z}{\eta^2} \right\}^2 \\ + (1-\sigma)\left\{ n^2 \dfrac{d^2}{d\eta^2}\left(\dfrac{z^2}{\eta} \right) - \dfrac{d}{d\eta}\left(\dfrac{dz}{d\eta} \right)^2 \right\} \end{array} \right] \eta^{-3\beta} d\eta \quad (18.263)$$

This is the equation from which an approximate value of $p_2{}^2$ can be found by using a reasonable value for z.

The disk has uniform thickness if $\beta = 0$, and in this special case the term having the coefficient $(1-\sigma)$ in the last equation can be integrated. Thus putting $\beta = 0$ we get, for a uniform disk,

$$\frac{3\varrho p_2{}^2 a^4 (1-\sigma^2)}{Ec^2} \int_0^1 z^2 \eta d\eta$$

$$= \int_0^1 \left\{ \frac{d^2 z}{d\eta^2} + \frac{1}{\eta}\frac{dz}{d\eta} - \frac{n^2 z}{\eta^2} \right\}^2 \eta d\eta$$

$$+ (1-\sigma)\left[n^2 \frac{d}{d\eta}\left(\frac{z^2}{\eta} \right) - \left(\frac{dz}{d\eta} \right)^2 \right]_0^1 . \quad . \quad . \quad (18.264)$$

In the rest of this chapter we shall use the short symbol μ with the same meaning as in (18.226). For a uniform disk c is identical with h, and μ is what we previously denoted by k^4a^4.

Excellent results can be got from (18.263) by assuming for z a short series of powers of η, the lowest power being the same as the lowest power in the theoretical value of z. The constants in the series can be chosen so as to make z satisfy the boundary conditions. If all the constants are determined from these conditions the value of μ given by (18.263) is often very good, but better results can always be got by leaving one of the constants undetermined until μ has been expressed in terms of this constant, and then choosing the constant so as to make μ have a minimum value. The following examples will make the method clear.

341. Free uniform disk making symmetrical oscillations.

The centre of a free disk making symmetrical oscillations does not remain at rest; consequently one term in z must be a constant. Let us assume that

$$w = A(1 + f\eta^2 + g\eta^4)\sin p_2 t \quad . \quad . \quad . \quad . \quad (18.265)$$

We have here three constants A, f, g, but the constant A is of no use to us because it merely introduces a factor A^2 into both sides of equation (18.263). Thus it is just as good to put

$$z = 1 + f\eta^2 + g\eta^4. \quad . \quad . \quad . \quad . \quad . \quad . \quad (18.266)$$

The boundary conditions are that the bending moment and shear force on the rim are both zero. Since the shear force is always zero at the rim it follows that the total momentum of the disk is constant, and may be assumed zero. Now we should expect, in using an energy method, that the condition that the total momentum is zero would give better results than the condition that the shear force is zero at the rim. We shall therefore use the momentum condition. This condition is

$$\int_a^a 2\pi r\varrho \frac{\partial w}{\partial t}\, dr = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (18.267)$$

which becomes, in terms of z and η,

$$\int_0^1 z\eta d\eta = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18.268)$$

This gives
$$\tfrac{1}{2} + \tfrac{1}{4}f + \tfrac{1}{6}g = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (18.269)$$

The condition that the bending moment is zero at the rim is

$$\frac{\partial^2 w}{\partial r^2} + \frac{\sigma}{r}\frac{\partial w}{\partial r} = 0 \text{ where } r = a; \quad . \quad . \quad . \quad . \quad (18.270)$$

that is,

$$\frac{d^2 z}{d\eta^2} + \frac{\sigma}{\eta}\frac{dz}{d\eta} = 0 \text{ where } \eta = 1, \quad . \quad . \quad . \quad . \quad (18.271)$$

whence

$$2f + 12g + \sigma(2f + 4g) = 0. \quad . \quad . \quad . \quad . \quad (18.272)$$

If we take $\sigma = \frac{1}{4}$ equations (18.269) and (18.272) give

$$f = -\tfrac{78}{34},\ g = \tfrac{15}{34}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (18.273)$$

Now from (18.264) and (18.266) we get

$$\mu = \frac{4\{f^2 + 4fg + \frac{20}{3}g^2\} + 4\sigma(f + 2g)^2}{\frac{1}{2} + \frac{1}{2}f + \frac{1}{3}g + \frac{1}{6}f^2 + \frac{1}{4}fg + \frac{1}{10}g^2}$$

$$= 83{\cdot}86. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18.274)$$

Kirchhoff's result for this case, given in the table on page 597, is $\mu = 79{\cdot}03$.

Let us try another expression for z; let us take

$$z = 1 + f\eta^2 + g\eta^3 \quad . \quad . \quad . \quad . \quad . \quad . \quad (18.275)$$

The first power of η must not occur in z because $\dfrac{dz}{d\eta}$ must be zero at the centre of the disk. The condition for zero momentum is now

$$\tfrac{1}{2} + \tfrac{1}{4}f + \tfrac{1}{5}g = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (18.276)$$

The condition for zero bending moment at the rim is

$$(2f + 6g) + \sigma(2f + 3g) = 0. \quad . \quad . \quad . \quad . \quad (18.277)$$

Taking $\sigma = \frac{1}{4}$ these two equations give

$$f = -\tfrac{54}{19},\ g = \tfrac{20}{19}.$$

Then

$$\mu = \frac{4f^2 + 12fg + \frac{45}{4}g^2 + \sigma(2f + 3g)^2}{\frac{1}{2} + \frac{1}{2}f + \frac{2}{5}g + \frac{1}{6}f^2 + \frac{2}{7}fg + \frac{1}{8}g^2}, \quad . \quad . \quad (18.278)$$

whence

$$\mu = 80{\cdot}54 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18.279)$$

If we had not already worked out the value of μ from rigorous theory we should still know that the last result is better than the one in (18.274) merely because it is smaller. Rayleigh's principle tells us, in fact, that the theoretical value of μ is always smaller than any we can get by this method.

Let us try still another way. Let us use the principle of zero momentum to express the constant f in terms of g, and then, without making the bending moment zero at the rim, let us find the minimum value of μ.

Thus, from (18.276),

$$f = -2 - 8x,$$

where

$$x = \tfrac{1}{10}g.$$

Then (18.278) becomes

$$\frac{\mu}{21} = \frac{40 - 280x + 940x^2}{7 - 16x + 13x^2}. \quad . \quad . \quad . \quad . \quad (18.280)$$

Writing $21y$ for μ and clearing of fractions we get

$$40-7y-(280-16y)x+(940-13y)x^2=0. \quad . \quad (18.281)$$

The extreme values of y corresponding to real values of x are therefore given by the equation

$$(280-16y)^2=4(40-7y)(940-13y), \quad . \quad . \quad (18.282)$$

or

$$27y^2-4860y+18000=0. \quad . \quad . \quad . \quad . \quad (18.283)$$

The smaller root of this equation, which is the one we are seeking, is

$$y=\frac{10217}{27}.$$

The corresponding value of μ is

$$\mu=79{\cdot}45, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18.284)$$

and this is the minimum value of μ given by (18.280). The error in this is only 0·5 per cent, and, since the frequency is proportional to p_2, the error in the frequency is less than 0·3 per cent.

To find the radius of the nodal circle we have to find r by equating z to zero. Thus if we take the expression for z that led to the result in (18.279) we get, for the radius of the nodal circle,

$$1+f\eta^2+g\eta^3=0, \quad . \quad . \quad . \quad . \quad . \quad (18.285)$$

that is,

$$19-54\eta^2+20\eta^3=0. \quad . \quad . \quad . \quad . \quad . \quad (18.286)$$

The root of this equation which lies between 0 and 1 is

$$\frac{r}{a}=\eta=0{\cdot}687, \quad . \quad . \quad . \quad . \quad . \quad . \quad (18.287)$$

which compares favourably with 0·6802 obtained in (18.136).

342. Uniform disk making symmetrical oscillations with its centre fixed.

Here z and $\frac{dz}{dr}$ are both zero at the centre of the disk. Moreover we should expect the disk to have finite curvature at the centre. Then the expression for z should start with η^2. Therefore we take, for the slowest mode,

$$z=\eta^2+f\eta^3+g\eta^4 \quad . \quad . \quad . \quad . \quad . \quad (18.288)$$

The boundary conditions are that the bending moment and shear force are both zero at the rim; that is

$$\frac{d^2z}{d\eta^2}+\frac{\sigma}{\eta}\frac{dz}{d\eta}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18.289)$$

$$\frac{d}{d\eta}\left\{\frac{d^2z}{d\eta^2}+\frac{1}{\eta}\frac{dz}{d\eta}\right\}=0 \quad . \quad . \quad . \quad . \quad . \quad (18.290)$$

both where $\eta=1$.

These give, assuming that $\sigma = \frac{1}{4}$,

$$2 + 6f + 12g + \tfrac{1}{4}(2 + 3f + 4g) = 0, \quad . \quad . \quad . \quad (18.291)$$

and

$$9f + 32g = 0; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18.292)$$

whence

$$f = -\tfrac{88}{99}, \quad g = \tfrac{5}{22} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18.293)$$

Now from (18.264)

$$\mu = \frac{5 + 15f + 20g + \frac{27}{2}f^2 + \frac{198}{5}fg + \frac{92}{3}g^2}{\frac{1}{6} + \frac{2}{7}f + \frac{1}{4}g + \frac{1}{8}f^2 + \frac{2}{9}fg + \frac{1}{10}g^2}, \quad . \quad . \quad (18.294)$$

whence

$$\mu = 14{\cdot}28 = 1{\cdot}944^4, \quad . \quad . \quad . \quad . \quad . \quad (18.295)$$

which differs by about 3 % from the value of $(ka)^4$ derived from (18.164).

A slightly better result can be got by substituting the actual value of f or g from (18.293) in the expression in (18.294) and then taking μ as the minimum value of the fraction for variations in the other parameter.

343. Free uniform disk vibrating with two nodal diameters.

Theory tells us that the lowest power of r in the expression for z for a disk vibrating with n nodal diameters is r^n. Then for two nodal diameters we may take

$$z = \eta^2 + f\eta^4. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18.296)$$

With this value of z equation (18.264) gives

$$\mu = \frac{24f^2 + (1-\sigma)(8 + 24f + 12f^2)}{\frac{1}{6} + \frac{1}{4}f + \frac{1}{10}f^2}$$

$$= 240\frac{6f^2 + (1-\sigma)(2 + 6f + 3f^2)}{10 + 15f + 6f^2}. \quad . \quad . \quad . \quad (18.297)$$

The minimum values of this fraction for variations in f are

$$\mu = 30{\cdot}64 \quad (\sigma = 0{\cdot}25) \quad . \quad . \quad . \quad . \quad (18.298)$$

$$\mu = 28{\cdot}96 \quad (\sigma = 0{\cdot}3) \quad . \quad . \quad . \quad . \quad . \quad (18.299)$$

It will be safe to extrapolate from these two to get μ when $\sigma = \frac{1}{3}$. This method gives

$$\mu = 27{\cdot}84 \quad (\sigma = \tfrac{1}{3}) \quad . \quad . \quad . \quad . \quad . \quad . \quad (18.300)$$

These values of μ differ from the values in the table on page 597 by less than one per cent, which corresponds to one half per cent in the frequency.

If we take
$$z = \eta^2 + g\eta^3 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18.301)$$
we get

$$\mu = \frac{\frac{25}{4}g^2 + (1-\sigma)(11g^2 + 20g + 8)}{\frac{1}{6} + \frac{2}{7}g + \frac{1}{8}g^2}$$

$$= 42\frac{25g^2 + 4(1-\sigma)(11g^2 + 20g + 8)}{21g^2 + 48g + 28}. \quad . \quad . \quad . \quad (18.302)$$

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The minimum values of this fraction are

$$\mu = 30{\cdot}43 \quad (\sigma = \tfrac{1}{4}) \qquad (18.303)$$

$$\mu = 27{\cdot}64 \quad (\sigma = \tfrac{1}{3}). \qquad (18.304)$$

These differ by only 0·2 per cent from Kirchhoff's results, the consequent error in the frequency being only 0·1 per cent.

If the disk were attached to a shaft of radius b it would be more accurate to regard the disk as clamped at $r=b$. In that case the integrals expressing the energy should be taken from $r=b$ to $r=a$; that is, from $\eta = \frac{b}{a}$ to $\eta = 1$. Also we could assume as the expression for z,

$$z = \left(\eta - \frac{b}{a}\right)^2 + g\left(\eta - \frac{b}{a}\right)^3. \qquad (18.305)$$

This satisfies the conditions at the inner boundary $r=b$. When a numerical quantitiy is substituted for $\frac{b}{a}$ the procedure is the same as in the example we have just worked out.

344. Uniform disk vibrating with one nodal diameter.

If a disk of radius a is clamped at all points of the circumference of a concentric circle of radius b, so that w and $\frac{\partial w}{\partial r}$ are both zero where $r=b$, it is possible for the disk to vibrate with its edge free in a mode with one nodal diameter and no nodal circle. Mr. R. V. Southwell* has shown that, when $\frac{b}{a}$ is infinitely small, the period of this vibration becomes infinite. When $\frac{b}{a}$ is less than about 0·04 he has shown that

$$k^4a^4 = \frac{16}{\log_e \frac{a}{b}} \text{ approximately} \qquad (18.306)$$

From this we calculate

$$k^4a^4 = 4{\cdot}969 \quad (a = 25b), \qquad (18.307)$$

$$k^4a^4 = 3{\cdot}474 \quad (a = 100b). \qquad (18.308)$$

The case of a free disk with one nodal diameter and one nodal circle yields fairly well to the approximate method. Thus let us assume

$$w = z \sin p_2 t \sin\theta. \qquad (18.309)$$

* *On the Free Transverse Vibrations of a Uniform Circular Disk Clamped at its Centre; and on the Effects of Rotation,* by R. V. Southwell, M.A.; Proc. Roy. Soc., A, Vol 101, 1922.

The nodal diameter is the one where $\theta = 0$. Now it is clear that the angular momentum of the whole disk about this nodal diameter is zero during the vibration. The condition for this is

$$\int_0^{2\pi}\int_0^a \frac{\partial w}{\partial t} r \sin\theta \varrho r \, d\theta \, dr = 0, \quad . \quad . \quad . \quad . \quad (18.310)$$

which is equivalent to

$$\varrho p_2 \cos p_2 t \int_0^{2\pi}\int_0^a z r^2 \sin^2\theta \, d\theta \, dr = 0$$

Since z is not a function of θ, and since the last equation must be true for all values of t, the equation becomes

$$\int_0^{2\pi} \sin^2\theta \, d\theta \int_0^a z r^2 dr = 0,$$

whence

$$\int_0^1 z\eta^2 d\eta = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18.311)$$

Now when $n = 1$ the lowest power of η in the expression for z is η itself. Then let us assume

$$z = \eta + f\eta^3 + g\eta^5 \quad . \quad . \quad . \quad . \quad . \quad . \quad (18.312)$$

The condition (18.311) gives the following relation between f and g

$$\tfrac{1}{4} + \tfrac{1}{6}f + \tfrac{1}{8}g = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (18.313)$$

Equation (18.264) gives

$$\mu = \frac{8(2f^2 + 8fg + 9g^2) - 4(1-\sigma)(f + 2g)^2}{\tfrac{1}{4} + \tfrac{1}{3}f + \tfrac{1}{4}g + \tfrac{1}{8}f^2 + \tfrac{1}{5}fg + \tfrac{1}{12}g^2} \quad . \quad . \quad (18.314)$$

Using (18.313), and taking $\sigma = \tfrac{1}{4}$, we get

$$\mu = 360\frac{604x^2 - 260x + 39}{28x^2 - 36x + 15}, \quad . \quad . \quad . \quad . \quad (18.315)$$

where $x = \tfrac{1}{4}g$. The minimum value of this fraction is

$$\mu = 447{\cdot}6, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18.316)$$

which differs from the value 416·53 given in the table on page 597 by 7·5 %, corresponding to an error of about 3·6 % in the frequency. This is not so good a result as we got for the lowest frequency with $n = 0$ or $n = 2$. The difficulty seems to be due to the nodal circle. A better result could be got by taking an extra term (and therewith an extra constant) in the expression for z and making the bending moment zero at the rim.

345. Uniform disk supported without clamping at the rim, The boundary conditions for this case are

$$\left.\begin{aligned} &z = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18.317)\\ &\frac{d^2z}{d\eta^2} + \sigma\left(\frac{1}{\eta}\frac{dz}{d\eta} - \frac{n^2 z}{\eta^2}\right) = 0 \quad . \quad . \quad . \quad (18.318)\end{aligned}\right\} \text{where } \eta = 1$$

For the first symmetrical mode of vibration, where $n=0$, we may take either of the forms

$$z = 1 + f\eta^2 + g\eta^3 \quad . \quad . \quad . \quad . \quad . \quad . \quad (18.319)$$

or

$$z = 1 + f\eta^2 + g\eta^4, \quad . \quad . \quad . \quad . \quad . \quad . \quad (18.320)$$

and find f and g from the boundary conditions. The resulting values of μ for the two values of z are, when $\sigma = 0{\cdot}25$,

$$\mu = 23{\cdot}71 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18.321)$$

$$\mu = 23{\cdot}73 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18.322)$$

These are both so near the value 23.60 given in (18.219) that further refinements are unnecessary.

346. Uniform disk clamped at the rim.

Here the boundary conditions are

$$z = 0, \ \frac{dz}{d\eta} = 0, \text{ where } \eta = 1 \quad . \quad . \quad . \quad . \quad (18.323)$$

For the first symmetrical mode we may take the same expressions for z as for the supported disk, that is, the expressions given in (18.319) and (18.320).

When the constants f and g are chosen so that z satisfies the boundary conditions the values we get for μ corresponding to the two expressions for z are respectively

$$\mu = 105{\cdot}00, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18.324)$$

$$\mu = 106{\cdot}67 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18.325)$$

both of which are very near the value 104·2 given in the table at the bottom of page 597.

347. Symmetrical vibrations of disk with variable section.

When the thickness of the disk is $c\eta^{-\beta}$, the lowest power of η that occurs in the expression for z for a disk vibrating in symmetrical modes was shown in (18.237) to be $\eta^{2+3\beta}$. This is complicated a little by the fact that $\eta^{2+3\beta}\log\eta$ actually occurs in z; but we know from the corresponding problem for a uniform disk that the omission of the factor $\log\eta$ in the energy method does not seriously affect the accuracy of this method. Then we may assume

$$z = \eta^{3\beta}(\eta^2 + f\eta^3 + g\eta^4). \quad . \quad . \quad . \quad . \quad . \quad (18.326)$$

Since the method is very cumbersome unless we substitute a numerical value of β we shall work out the value of μ when $\beta = 1$. In this case

$$z = \eta^5 + f\eta^6 + g\eta^7 \quad . \quad . \quad . \quad . \quad . \quad . \quad (18.327)$$

Then (18.263) gives, with $n = 0$,

$$\mu = \frac{\displaystyle\int_0^1 \left[\eta\left\{\frac{d^2z}{d\eta^2} + \frac{1}{\eta}\frac{dz}{d\eta}\right\}^2 - (1-\sigma)\frac{d}{d\eta}\left(\frac{dz}{d\eta}\right)^2\right]\eta^{-3\beta}\,d\eta}{\displaystyle\int_0^1 z^2\eta^{1-\beta}\,d\eta}.$$

With the present value of z this becomes, assuming $\sigma=\frac{1}{4}$,

$$\mu=\frac{95+232\frac{1}{2}f+146\frac{4}{7}f^2+275g+354\frac{3}{8}fg+217\frac{7}{9}g^2}{\frac{1}{11}+\frac{1}{6}f+\frac{1}{13}f^2+\frac{2}{13}g+\frac{1}{7}fg+\frac{1}{15}g^2}. \quad (18.328)$$

We got a good result for the uniform disk vibrating in the same mode by using values of the constants determined by the boundary conditions. These boundary conditions are, for the present case,

$$\left.\begin{aligned}&\frac{d^2z}{d\eta^2}+\frac{\sigma}{\eta}\frac{dz}{d\eta}=0\\ &\frac{d}{d\eta}\left[\eta^{-3\beta}\left\{\frac{d^2z}{d\eta^2}+\frac{1}{\eta}\frac{dz}{d\eta}\right\}\right]=0\end{aligned}\right\}\text{where } \eta=1 \quad \begin{aligned}&(18.329)\\ &(18.330)\end{aligned}$$

from which

$$85+126f+175g=0, \quad (18.331)$$

and

$$36f+98g=0. \quad (18.332)$$

Therefore

$$f=-\frac{1}{8}\frac{595}{54}, \quad g=\frac{1}{8}\frac{85}{21} \quad (18.333)$$

which are very nearly the same as

$$f=-\frac{11}{8}, \quad g=\frac{1}{2}. \quad (18.334)$$

Using these latter values we get

$$\frac{3\varrho p_2{}^2a^4(1-\sigma^2)}{Ec^2}=\mu=288{\cdot}46 \quad (18.335)$$

It would not require a great deal of labour to test the accuracy of this result from the exact equation (18.237), for the series for z_1 and z_2 in (18.234) and (18.235) converge very quickly for this value of μ.

348. Vibrations of a disk of variable section in modes with nodal diameters.

For a given value of β equation (18.263) will give a good value of $p_2{}^2$ when a suitable expression for z is used. The important thing to remember for the disk whose profile is given by $h=c\eta^{-\beta}$ is that the series for z should start with η^n or η^{l-n} according as n is greater or less than $(l-n)$. The reason for this is given in Art. 338. Thus if $\beta=1$ then $l=2+3\beta=5$. Therefore $l-n>n$ if $n=2$, and the series starts with η^3. Then we might take such forms as

$$z=\eta^3+f\eta^4,$$

or

$$z=\eta^3+f\eta^5$$

for the mode with two nodal diameters.

It will be wise in a practical example to use two or three different expressions for z and calculate μ from each, and then adopt the smallest value of μ obtained. If the values of μ thus calculated differ but little from each other it is probable that they are all near the correct result.

349. The vibrations of a disk rotating with uniform speed. We require first of all the energy equation for a rotating disk. Since the energy due to spin remains constant we need not bring this into our equations.

Multiplying equation (18.3) by $2h\dfrac{\partial w}{\partial t}$, and then replacing $h^3\nabla_1{}^4 w$ by $\nabla_1{}^2(h^3\nabla_1{}^2 w)$, so as to make the result applicable to a disk of variable thickness, we get

$$\frac{1}{r}\frac{\partial}{\partial r}\left(2\,hr\mathrm{P}\frac{\partial w}{\partial r}\right)\frac{\partial w}{\partial t}+\frac{2hQ}{r^2}\frac{\partial^2 w}{\partial \theta^2}\frac{\partial w}{\partial t}$$
$$=\frac{2\mathrm{E}}{3(1-\sigma^2)}\nabla_1{}^2(h^3\nabla_1{}^2 w)\frac{\partial w}{\partial t}+2\,\varrho h\frac{\partial^2 w}{\partial t^2}\frac{\partial w}{\partial t} \quad (18.336)$$

If we multiply this by $r\,dr\,d\theta\,dt$ and integrate both sides, the limits of integration for r and θ being such as to cover the whole disk and the integration with respect to t being from any lower limit up to t, the last term on the right hand side of the equation is

$$\iiint 2\varrho h\frac{\partial^2 w}{\partial t^2}\frac{\partial w}{\partial t}\,r\,dr\,d\theta\,dt, \quad . \quad . \quad . \quad . \quad . \quad (18.337)$$

which is the same as

$$\frac{1}{2}\iint 2\varrho h\left(\frac{\partial w}{\partial t}\right)^2 r\,dr\,d\theta \quad . \quad . \quad . \quad . \quad . \quad (18.338)$$

This last expression is clearly the kinetic energy of the disk. The rest of the terms in equation (18.336) must therefore represent the potential energy. We already know what form the terms having the factor E take. We shall now put the terms containing P and Q into more convenient forms.

Thus, if the limits of integration with respect to r are b and a, we get

$$\int_b^a \frac{1}{r}\frac{\partial}{\partial r}\left(2hr\mathrm{P}\frac{\partial w}{\partial r}\right)\frac{\partial w}{\partial t}\,r\,dr$$
$$=\left[2hr\mathrm{P}\frac{\partial w}{\partial r}\frac{\partial w}{\partial t}\right]_b^a-\int_b^a 2hr\mathrm{P}\frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial r\,\partial t}\,dr \quad . \quad . \quad (18.339)$$

Now at the outer rim $\mathrm{P}=0$, and for a complete disk, $r=b=0$ at the other limit. Therefore the integrated term vanishes at both limits for a complete disk. If b is not zero but the disk is clamped at $r=b$, then $\dfrac{\partial w}{\partial r}=0$ at the inner boundary; therefore in this case also the integrated term vanishes. Consequently

$$\int\frac{1}{r}\frac{\partial}{\partial r}\left(2hr\mathrm{P}\frac{\partial w}{\partial r}\right)\frac{\partial w}{\partial t}\,r\,dr=-\int 2hr\mathrm{P}\frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial r\,\partial t}\,dr.$$

Integrating both sides of this last equation with respect to t we get

$$\iint \frac{1}{r}\frac{\partial}{\partial r}\left(2hr\mathrm{P}\frac{\partial w}{\partial r}\right)\frac{\partial w}{\partial t}\,rdrdt = -\int hr\mathrm{P}\left(\frac{\partial w}{\partial r}\right)^2 dr. \quad . \ (18.340)$$

Consequently

$$\iiint \frac{1}{r}\frac{\partial}{\partial r}\left(2hr\mathrm{P}\frac{\partial w}{\partial r}\right)\frac{\partial w}{\partial t}\,rdrd\theta dt = -\iint h\mathrm{P}\left(\frac{\partial w}{\partial r}\right)^2 rdrd\theta \quad (18.341)$$

In the same way we can prove that

$$\iiint \frac{2h\mathrm{Q}}{r^2}\frac{\partial^2 w}{\partial\theta^2}\frac{\partial w}{\partial t}\,rdrd\theta dt = -\iint h\mathrm{Q}\left(\frac{\partial w}{r\partial\theta}\right)^2 rdrd\theta \ . \ . \ (18.342)$$

Now let

$$\mathrm{T} = \iint \varrho h\left(\frac{\partial w}{\partial t}\right)^2 rdrd\theta, \ . \ . \ . \ . \ . \ (18.343)$$

$$\mathrm{V}_1 = \iint\left\{h\mathrm{P}\left(\frac{\partial w}{\partial r}\right)^2 + h\mathrm{Q}\left(\frac{\partial w}{r\partial\theta}\right)^2\right\} rdrd\theta, \ . \ . \ (18.344)$$

$$\mathrm{V}_2 = \frac{1}{3}\frac{\mathrm{E}}{1-\sigma^2}\iint h^3\left[(\nabla_1{}^2 w)^2 - 2(1-\sigma)\left\{\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x\partial y}\right)^2\right\}\right] rdrd\theta. \ (18.345)$$

By means of (18.257) and (18.258) V_2 can be expressed entirely in terms of polar coordinates. When (18.336) is multtplied by $rdrd\theta dt$ and integrated, the resulting equation is

$$\mathrm{V}_1 + \mathrm{V}_2 + \mathrm{T} = \text{constant} \ . \ . \ . \ . \ . \ (18.346)$$

Now by the assumption

$$w = u\sin pt \ . \ . \ . \ . \ . \ . \ . \ . \ (18.347)$$

equation (18.346) gives

$$(\mathrm{U}_1 + \mathrm{U}_2)\sin^2 pt + p^2\mathrm{T}'\cos^2 pt = \text{const}, \ . \ . \ (18.348)$$

where U_1 and U_2 differ from V_1 and V_2 only in containing u instead of w; and T′ differs from T in having u instead of $\frac{\partial w}{\partial t}$. From equation (18.348) we find

$$p^2\mathrm{T}' = \mathrm{U}_1 + \mathrm{U}_2,$$

whence

$$p_2 = \frac{\mathrm{U}_1 + \mathrm{U}_2}{\mathrm{T}'} \ . \ . \ . \ . \ . \ . \ (18.349)$$

Now suppose u is the correct function for the particular normal mode we are dealing with; and suppose u_1 is the correct function for the same mode when the rigidity is negligible, and u_2 the correct function for the same mode when there is no rotation. Also let $\mathrm{U}_1(u_1)$ be written for the value of U_1 when u_1 is substituted for u. Then, p_1 and p_2 being the values of p due to centrifugal force and rigidity respectively,

$$p_1{}^2 = \frac{\mathrm{U}_1(u_1)}{\mathrm{T}'(u_1)} \ . \ . \ . \ . \ . \ . \ . \ (18.350)$$

and $$p_2{}^2 = \frac{U_2(u_2)}{T'(u_2)}. \quad \text{(18.351)}$$

But by Rayleigh's theorem

$$\frac{U_1(u_1)}{T'(u_1)} < \frac{U_1(u)}{T'(u)}, \quad \text{(18.352)}$$

and $$\frac{U_2(u_2)}{T'(u_2)} < \frac{U_2(u)}{T'(u)}. \quad \text{(18.353)}$$

Therefore, by addition,

$$p_1{}^2 + p_2{}^2 < p^2. \quad \text{(18.354)}$$

Moreover, if any function u_3 but the correct function u be used in (18.349), the resulting value of p^2 is above the true value. That is,

$$p^2 < \frac{U_1(u_3) + U_2(u_3)}{T'(u_3)}. \quad \text{(18.355)}$$

Thus when $p_1{}^2$ and $p_2{}^2$ can be found exactly for any given mode we can find two limits between which p^2 must lie. Lamb and Southwell, in the paper quoted earlier in this chapter, worked out the following example for a uniform disk to illustrate this point. The disk is vibrating with two nodal diameters. Also

$$E = 2 \times 10^{12}, \ \varrho = 7{\cdot}8, \ a = 60, \ h = 1, \ \omega = 100\pi, \quad \text{(18.356)}$$

in C. G. S. units. With these values of the constants they find

$$\frac{p_1{}^2}{\pi^2} = 2{\cdot}37500 \times 10^4, \ \frac{p_2{}^2}{\pi^2} = 2{\cdot}16485 \times 10^4. \quad \text{(18.357)}$$

Therefore

$$\frac{p}{\pi} > \frac{\sqrt{p_1{}^2 + p_2{}^2}}{\pi} \text{ i. e. } > 213{\cdot}07. \quad \text{(18.358)}$$

Next putting

$$u_3 = \left(1 + f\frac{r^2}{a^2}\right)\left(\frac{r}{a}\right)^2 \sin 2\theta \sin pt \quad \text{(18.359)}$$

in (18.355), and choosing f by making the right hand side a minimum, they find

$$\frac{p}{\pi} < 213{\cdot}74. \quad \text{(18.360)}$$

Thus the frequency lies between two limits which differ only by 0·3 % This shows that it is a good enough approximation in any practical case to use the equation

$$p^2 = p_1{}^2 + p_2{}^2, \quad \text{(18.361)}$$

where $p_1{}^2$ and $p_2{}^2$ are found either by approximate or exact methods, provided only that care is taken to get somewhere near the minimum values of the fractions from which the p's are calculated.

350. Vibrations of a turbine disk.

In using Rayleigh's method for a turbine disk T should include the kinetic energy of the blades as well as that of the disk itself. It will usually be accurate enough to regard the blades as perfectly rigid bodies, and thus assume that the kinetic energy of each blade is the same as that of a rod oscillating through the same angle as that through which the radial tangent to the middle surface at the point of attachment oscillates. That is, if the equation assumed for the deflection of the disk is

$$w = f(r)\sin(n\theta + a)\sin pt$$

the angular displacement of a radial line at the rim $r = a$ of the disk is

$$\varphi = \left(\frac{\partial w}{\partial r}\right)_{r=a}$$
$$= f'(a)\sin(n\theta + a)\sin pt,$$

and therefore the angular velocity of the blade attached to the rim at θ is

$$\frac{\partial\varphi}{\partial t} = pf'(a)\sin(n\theta + a)\cos pt \quad . \quad . \quad . \quad . \quad (18.362)$$

If the blade is a uniform rod of mass m and length l its kinetic energy is $\frac{1}{6}ml^2\left(\frac{\partial\varphi}{\partial t}\right)^2$. Still more accurate results can be got by treating the blades as part of the disk, and assuming that they take a curvature during the vibrations, so that they contribute to both the kinetic energy and the potential energy. It is, however, a very complicated task to take proper account of the energy in a blade if we assume that it bends, for, owing to its shape it will not bend into a curve whose plane is perpendicular to the middle surface of the disk. It is best therefore—and in all cases will probably be quite accurate enough—to treat the blades as straight during the vibrations.

The thickness of an actual turbine disk is not usually proportional to a single power of r from the centre to the rim. If it is not considered that a good enough approximation to the period can be got by assuming that $h = \eta^{-\beta}$ over the whole of the disk then the actual value of h may be used in (18.344) and (18.345). A value may then be assumed for z such as

$$z = \eta^k(1 + f\eta + g\eta^2), \quad . \quad . \quad . \quad . \quad . \quad (18.363)$$

and this expression may be used, with the same values of the constants f and g, over the whole disk. Then p^2 can be calculated by getting the minimum value of the right hand side of (18.349) for variations in f and g. It would probably be best to work out p^2 for different values of k. If there is a very violent change in the shape of the disk at some particular radius it would improve the accuracy to assume two different equations for z in the two portions of the disk. In that case we should take

$$z = \eta^k(1 + f\eta + g\eta^2)$$

in one part of the disk, and

$$z = \eta^m(1 + f_1\eta + g_1\eta^2). \quad . \quad . \quad . \quad . \quad (18.364)$$

in the other part. It would then be necessary to make z and $\frac{dz}{d\eta}$ have the same values at the junction of the two parts. Thus k and m could be chosen to suit the two parts, and f_1 and g_1 could be expressed in terms of f and g by means of the conditions for the continuity of z and $\frac{dz}{d\eta}$.

351. The possibility of stationary nodal diameters.

The arguments used in Art. 328 concerning the rotation of the nodal diameters can be applied to any of the disks with which we have dealt in this chapter. Thus if

$$w = f(r)\sin(n\theta + \alpha)\sin pt \quad . \quad . \quad . \quad . \quad (18.365)$$

represents a normal mode of a disk whether the controlling force is tension, or rigidity, or both, then

$$w = f(r)\sin(n\theta \pm pt + \alpha) \quad . \quad . \quad . \quad . \quad (18.366)$$

is equally a normal mode. In the vibration represented by the last value of w the nodal diameters rotate, relatively to the disk, with angular velocities $\pm\frac{p}{n}$. It follows that the nodal diameters may be at rest if

$$\frac{p}{n} = \omega.$$

In the case of the turbine disk the nodal diameters may be at rest, or moving slowly, if

$$p_1{}^2 + p_2{}^2 = n^2\omega^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (18.367)$$

352. Dependence of periods of vibration on amplitude.

Throughout this chapter no account has been taken of the effect on the periods of vibration of a disk due to the stretching of the middle surface. The work in Chap. XV show sthat the restoring forces due to stretching, instead of being proportional to deflexion, as the other restoring forces are, are proportional rather to the cube of the deflexion. If the terms due to stretching be introduced into (18.346) it will be found that the fourth power of the amplitude occurs in these terms and the square of the amplitude in the other terms. Thus when the square of the amplitude is divided out there are left in the equation terms containing the square of the amplitude, from which it follows that the frequency depends on the amplitude. The frequency, in fact, increases with the amplitude.

If the maximum deflexion in any vibration is less than one fifth of the thickness of the disk the theories of this chapter can be regarded as practically accurate; for a maximum deflexion equal to the thickness the frequency might be, according to the particular mode of vibration, 10 to 25 per cent greater than we have calculated. In fact, for large amplitudes, the motion cannot be resolved into normal modes of vibration; and probably there are no pure vibrations at all, but only an irregular wobbling in which amplitude and period both change considerably from one vibration to the next.

CHAPTER XIX

ELASTIC BODIES IN CONTACT.

353. Elastic body with no accelerations and no body forces.

The equations (2.28), (2.29), (2,30), together with the boundary conditions, determine the displacements of an elastic body with given body forces and given accelerations. In a region where X, Y, Z, f_1, f_2, f_3, are all zero these equations take the forms

$$\frac{\partial \Delta}{\partial x} + (1-2\sigma)\nabla^2 u = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (19.1)$$

$$\frac{\partial \Delta}{\partial y} + (1-2\sigma)\nabla^2 v = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (19.2)$$

$$\frac{\partial \Delta}{\partial z} + (1-2\sigma)\nabla^2 w = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (19.3)$$

$(1-2\sigma)$ being substituted for $\frac{n}{m}$.

Also the stresses are given in terms of the displacements by such equations as

$$P_1 = 2n\left\{\frac{\sigma}{1-2\sigma}\Delta + \frac{\partial u}{\partial x}\right\}, \quad . \quad . \quad . \quad . \quad . \quad (19.4)$$

and

$$S_1 = n\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19.5)$$

When zero is put for X, Y, Z, in (2.31) that equation becomes

$$\nabla^2 \Delta = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19.6)$$

There are many known solutions of this last equation, which is known as Laplace's equation. Among the simplest and most useful are the functions called *Spherical Harmonics* in Thomson and Tait's *Natural Philosophy.*

If $\qquad r^2 = x^2 + y^2 + z^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19.7)$

it is easy to verify that

$$\nabla^2\left(\frac{1}{r}\right) = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19.8)$$

Now let us write D_x for $\frac{\partial}{\partial x}$. Then, by differentiating throughout the last equation with respect to x, we get

$$D_x \nabla^2\left(\frac{1}{r}\right)=0,$$

that is,

$$\nabla^2\left\{D_x\left(\frac{1}{r}\right)\right\}=0. \qquad (19.9)$$

By repeating this process any number of times we get

$$\nabla^2\left\{D_x^l\left(\frac{1}{r}\right)\right\}=0, \qquad (19.10)$$

l being any positive integer.

In the same way it follows that

$$\nabla^2\left\{D_x^l D_y^m D_z^n\left(\frac{1}{r}\right)\right\}=0 \qquad (19.11)$$

Thus we find that a solution of the equation

$$\nabla^2\varphi=0 \qquad (19.12)$$

is

$$\varphi=D_x^l D_y^m D_z^n\left(\frac{1}{r}\right). \qquad (19.13)$$

l, m, n, being integers.

Again it is not necessary that l, m, or n, should be a positive integer: for

$$\nabla^2\left\{D_x^{-1}\left(\frac{1}{r}\right)\right\}=D_x^{-1}\nabla^2\left(\frac{1}{r}\right)=\int\nabla^2\left(\frac{1}{r}\right)dx,$$

and one possible value of this quantity is clearly zero.

Also

$$\nabla^2\left\{D_x^{-k}\left(\frac{1}{r}\right)\right\}=D_x^{-k}\nabla^2\left(\frac{1}{r}\right)=0. \qquad (19.14)$$

Thus the value of φ given by (19.13) is a solution of (19.12) for all positive or negative integral values of l, m, n.

Let φ be any solution of (19.12), and let

$$\triangle=2(1-2\sigma)\frac{\partial\varphi}{\partial z}. \qquad (19.15)$$

Then clearly $\triangle$ satisfies (19.6). Also (19.1) gives

$$\nabla^2 u=-2\frac{\partial^2\varphi}{\partial x\partial z}. \qquad (19.16)$$

A particular integral of this is

$$u=-z\frac{\partial\varphi}{\partial x}; \qquad (19.17)$$

for, with this value of u,

$$\nabla^2 u = -\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right) z \frac{\partial \varphi}{\partial x}$$

$$= -z\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\frac{\partial \varphi}{\partial x} - 2\frac{\partial z}{\partial z}\frac{\partial^2 \varphi}{\partial x \partial z}$$

$$= -2\frac{\partial^2 \varphi}{\partial x \partial z}, \qquad (19.18)$$

the coefficient of z being zero in the second line by (19.12) and (19.13). A more general solution of (19.16) is

$$u = -z\frac{\partial \varphi}{\partial x} + \psi_1, \qquad (19.19)$$

where ψ_1 is also a solution of Laplace's equation; that is,

$$\nabla^2 \psi_1 = 0 \qquad (19.20)$$

Likewise the values of v and w corresponding to the assumed value of $\triangle$ are

$$v = -z\frac{\partial \varphi}{\partial y} + \psi_2, \qquad (19.21)$$

and

$$w = -z\frac{\partial \varphi}{\partial z} + \psi'_3, \qquad (19.22)$$

ψ_2 and ψ'_3 being also solutions of Laplace's equation.

The three functions of ψ_1, ψ_2, ψ'_3, are not independent since $\triangle$ depends on u, v, w. Thus

$$\triangle = \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}$$

$$= -z\nabla^2\varphi - \frac{\partial \varphi}{\partial z} + \frac{\partial \psi_1}{\partial x} + \frac{\partial \psi_2}{\partial y} + \frac{\partial \psi'_3}{\partial z}$$

$$= -\frac{\partial \varphi}{\partial z} + \frac{\partial \psi_1}{\partial x} + \frac{\partial \psi_2}{\partial y} + \frac{\partial \psi'_3}{\partial z}.$$

Therefore

$$\frac{\partial \psi_1}{\partial x} + \frac{\partial \psi_2}{\partial y} + \frac{\partial \psi'_3}{\partial z} = \triangle + \frac{\partial \varphi}{\partial z} = (3-4\sigma)\frac{\partial \varphi}{\partial z}. \qquad (19.23)$$

Our results will be in a slightly more convenient forms if we put

$$\psi'_3 = \psi_3 + (3-4\sigma)\varphi. \qquad (19.24)$$

Then (19.23) becomes

$$\frac{\partial \psi_1}{\partial x} + \frac{\partial \psi_2}{\partial y} + \frac{\partial \psi_3}{\partial z} = 0. \qquad (19.25)$$

Also the equation for w is

$$w = -z\frac{\partial \varphi}{\partial z} + (3-4\sigma)\varphi + \psi_3 \qquad (19.26)$$

We get a particular solution by putting

$$\psi_1=\frac{\partial\psi}{\partial x},\ \psi_2=\frac{\partial\psi}{\partial y},\ \psi_3=\frac{\partial\psi}{\partial z}. \quad . \quad . \quad . \quad . \quad (19.27)$$

Since ψ must satisfy (19.25) we get

$$\nabla^2\psi=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19.28)$$

so that ψ is another solution of Laplace's equation. Thus a set of particular integrals of the equations of equilibrium are

$$u=\frac{\partial\psi}{\partial x}-z\frac{\partial\varphi}{\partial x}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19.29)$$

$$v=\frac{\partial\psi}{\partial y}-z\frac{\partial\varphi}{\partial y}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19.30)$$

$$w=\frac{\partial\psi}{\partial z}-z\frac{\partial\varphi}{\partial z}+(3-4\sigma)\varphi, \quad . \quad . \quad . \quad (19.31)$$

$$\triangle=(2-4\sigma)\frac{\partial\varphi}{\partial z}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19.32)$$

The stresses corresponding to these displacements are

$$P_1=2n\left\{2\sigma\frac{\partial\varphi}{\partial z}-z\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\psi}{\partial x^2}\right\}, \quad . \quad . \quad . \quad . \quad . \quad (19.33)$$

$$P_2=2n\left\{2\sigma\frac{\partial\varphi}{\partial z}-z\frac{\partial^2\varphi}{\partial y^2}+\frac{\partial^2\psi}{\partial y^2}\right\}, \quad . \quad . \quad . \quad . \quad (19.34)$$

$$P_3=2n\left\{(2-2\sigma)\frac{\partial\varphi}{\partial z}-z\frac{\partial^2\varphi}{\partial z^2}+\frac{\partial^2\psi}{\partial z^2}\right\}, \quad . \quad . \quad . \quad (19.35)$$

$$S_1=2n\left\{\frac{\partial^2\psi}{\partial y\partial z}-z\frac{\partial^2\varphi}{\partial y\partial z}+(1-2\sigma)\frac{\partial\varphi}{\partial y}\right\} \quad . \quad . \quad . \quad (19.36)$$

$$S_2=2n\left\{\frac{\partial^2\psi}{\partial x\partial z}-z\frac{\partial^2\varphi}{\partial x\partial z}+(1-2\sigma)\frac{\partial\varphi}{\partial x}\right\} \quad . \quad . \quad . \quad (19.37)$$

$$S_3=2n\left\{\frac{\partial^2\psi}{\partial x\partial y}-z\frac{\partial^2\varphi}{\partial x\partial y}\right\}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19.38)$$

354. Pressure concentrated at a point on the surface of an infinite solid.

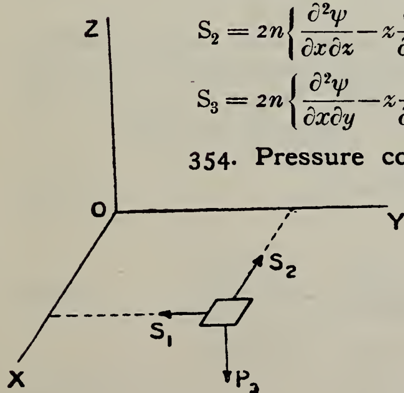

Fig. 178

Suppose a concentrated normal thrust W is applied at the origin to an infinite elastic solid which is bounded by the plane $z=0$ and extends throughout the space where z is positive. The stresses on the surface $z=0$ of this solid are P_3, S_1, S_2, as shown in fig. 178.

Our object is to make these three stresses vanish except at the origin O.

Now clearly S_1 and S_2 will be zero over the plane $z=0$ provided that

$$\frac{\partial\psi}{\partial z}=-(1-2\sigma)\varphi, \quad . \quad . \quad . \quad . \quad . \quad . \quad (19.39)$$

and provided also that $\frac{\partial\varphi}{\partial z}$ is finite over this same surface. Let us therefore assume that (19.39) is true. Then

$$S_1=-2nz\frac{\partial^2\varphi}{\partial y\partial z}, \quad S_2=-2nz\frac{\partial^2\varphi}{\partial x\partial z}, \quad . \quad . \quad . \quad (19.40)$$

$$P_3=2n\left\{\frac{\partial\varphi}{\partial z}-z\frac{\partial^2\varphi}{\partial z^2}\right\}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19.41)$$

It is only necessary that $\frac{\partial\varphi}{\partial z}$ should contain a factor z in order that P_3 may be zero over all the surface $z=0$. If we take

$$\varphi=\frac{1}{r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19.42)$$

then

$$\frac{\partial\varphi}{\partial z}=-\frac{z}{r^3}, \quad \frac{\partial^2\varphi}{\partial z^2}=-\frac{1}{r^3}+\frac{3z^2}{r^5}; \quad . \quad . \quad . \quad . \quad (19.43)$$

and consequently

$$S_1=-6n\frac{yz^2}{r^5}, \quad S_2=-6n\frac{xz^2}{r^5}, \quad . \quad . \quad . \quad . \quad (19.44)$$

$$P_3=-6n\frac{z^3}{r^5}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19.45$$

These stresses are all zero at the surface $z=0$, except possibly at the origin where r is also zero. Let S be written for the resultant of the component shear stresses S_1 and S_2. Then, since $-S_1$ and $-S_2$ act parallel to the axes OY and OX respectively, and since

$$\frac{-S_1}{y}=\frac{-S_2}{x}=6n\frac{z^2}{r^5}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (19.46)$$

it follows that

$$S^2=36n^2(x^2+y^2)\left(\frac{z^2}{r^5}\right)^2=36n^2r^2\left(\frac{z^2}{r^5}\right)^2, \quad . \quad . \quad (19.47)$$

because $r^2=x^2+y^2$ in the plane $z=0$. Therefore

$$S=6n\frac{z^2}{r^4}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19\ 48)$$

and S acts on the surface of the body along the radius vector from the origin to the point $(x, y, 0)$. Thus the stress system we have got is symmetrical about the z-axis.

To find the resultant force at the origin we may consider the equilibrium of a small cylindrical portion of the solid having the z-axis as its axis of symmetry. Suppose F is this resultant. Let the faces of the cylinder be in the planes $z=0$, $z=c$, and let the radius of the cylinder be a. We may assume also that $\frac{a}{c}$ is infinite while a itself is finite. On these assumptions the resultant of the shear stress S acting on the infinitesimal curved area of the cylinder is zero. The resultant of the stresses P_3 on the circle of radius a is therefore equal to $-F$. Thus

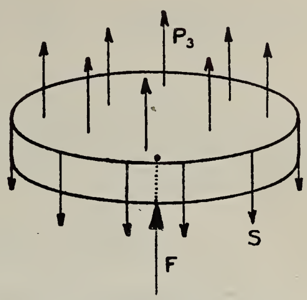

Fig. 179

$$F = -\int_0^a 2\pi\varrho d\varrho P_3 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19.49)$$

where

$$\varrho^2 = x^2 + y^2.$$

But

$$P_3 = -6n\frac{z^3}{r^5} = -6n\frac{c^3}{(\varrho^2+c^2)^{\frac{5}{2}}};$$

therefore

$$F = 12\pi n\int_0^a \frac{c^3 \varrho d\varrho}{(\varrho^2+c^2)^{\frac{5}{2}}}$$

$$= 12\pi n\left[\frac{-c^3}{3(\varrho^2+c^2)^{\frac{3}{2}}}\right]_0^a = 4\pi n\left\{1 - \frac{c^3}{(a^2+c^2)^{\frac{3}{2}}}\right\}, \quad . \quad . \quad . \quad (19.50)$$

which becomes, since $\frac{c}{a} = 0$,

$$F = 4\pi n. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19.51)$$

If we had taken

$$\varphi = \frac{W}{4\pi n}\frac{1}{r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19.52)$$

instead of (19.42) we should have got finally

$$F = W. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19.53)$$

The displacement w corresponding to φ is, by (19.31) and (19.39),

$$w = 2(1-\sigma)\varphi - z\frac{\partial\varphi}{\partial z}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (19.54)$$

whence we get, by means of (19.52),

$$w=\frac{W}{4\pi n}\left\{\frac{2(1-\sigma)}{r}+\frac{z^2}{r^3}\right\}. \quad . \quad . \quad . \quad . \quad . \quad (19.55)$$

At the surface $z=0$ this becomes

$$w=\frac{1-\sigma}{2\pi n}\frac{W}{r}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19.56)$$

The concentrated force W is the only external force acting on the body, and the stresses and displacements are due entirely to this force. The stresses due to W are

$$S_1=-\frac{3W}{2\pi}\frac{yz^2}{r^5}, \quad S_2=-\frac{3W}{2\pi}\frac{xz^2}{r^5}, \quad P_3=-\frac{3W}{2\pi}\frac{z^3}{r^5}. \quad . \quad (19.57)$$

355. Distributed pressure on the face of an infinite solid. The solid we are dealing with here, as in the last article, is supposed to be bounded by the plane $z=0$ and to extend throughout the infinite space where z is positive.

Let x_1, y_1, be the coordinates of a point in the plane $z=0$, and let the pressure per unit area applied at this point be p, which is supposed to be a function of x_1 and y_1. Thus the force on the area $dx_1 dy_1$ is $p dx_1 dy_1$. Let

$$R^2=(x-x_1)^2+(y-y_1)^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad (19.58)$$

so that R is the distance of the point $(x, y, 0)$ from the point $(x_1, y_1, 0)$ where the force $p dx_1 dy_1$ is applied. The displacement dw due to this pressure is, by (19.56),

$$dw=\frac{1-\sigma}{2\pi n}\frac{p dx_1 dy_1}{R}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19.59)$$

It follows therefore that the displacement w at $(x, y, 0,)$ due to the distributed pressure p on the plane $z=0$ is

$$w=\frac{1-\sigma}{2\pi n}\iint\frac{p dx_1 dy_1}{R}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (19.60)$$

This result can be got immediately by putting

$$\varphi=\frac{1}{4\pi n}\iint\frac{p dx_1 dy_1}{\sqrt{R^2+z^2}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (19.61)$$

in (19.54) and then putting $z=0$. It is easy to verify that this value of φ satisfies Laplace's equation.

We now require to find the displacement w of a point (x, y) in the plane $z=0$.

Let A and B be the points (x, y) and (x_1, y_1) in the plane $z=0$, (fig. 180). Also let

$$r^2=x^2+y^2; \quad r_1{}^2=x_1{}^2+y_1{}^2 \quad . \quad . \quad . \quad . \quad (19.62)$$

Then, taking θ to be the angle between R and r, we get

$$r_1^2 = r^2 + R^2 + 2Rr\cos\theta \quad . \quad (19.63)$$

If we use polar coordinates R, θ, in the integral in (19.60) the element of area which replaces $dx_1\,dy_1$ is $RdRd\theta$. Therefore

$$w = \frac{1-\sigma}{2\pi n}\iint p\,dRd\theta, \quad . \quad (19.64)$$

p being here regarded as a function of R and θ.

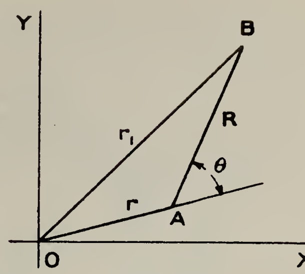

Fig. 180

356. Spherical depression produced by pressure.

Equation (19.64) gives w when p is known as a function of R and θ. It does not give p when w is known. The pressure distribution which gives rise to a spherical or ellipsoidal depression — the types of depression due to the squeezing together of two bodies — are suggested by the theory of potential. In that theory it is shown that the potential at (x, y, z) due to a distribution of mass ϱ per unit area at (x_1, y_1) on the plane $z = 0$ is

$$V = \iint \frac{\varrho\,dx_1\,dy_1}{\sqrt{R^2+z^2}} \quad . \; . \; . \; . \; . \; . \; . \quad (19.65)$$

Thus we see that w in (19.60) is the potential in the plane $z = 0$ due to a mass distribution

$$\varrho = \frac{1-\sigma}{2\pi n}p. \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (19.66)$$

per unit area. By means of this link the known results in the theory of potential suggest corresponding results in the theory of the elastic solid under pressure.

The case of a spherical depression can easily be worked out without any reference to the potential theory, and we shall confine ourselves for the present to this case.

Let us assume that a pressure p acts over a circle of radius a having its centre at the origin, the pressure at radius r_1 being

$$p = C(a^2 - r_1^2)^{\frac{1}{2}}. \quad . \; . \; . \; . \; . \; . \quad (19.67)$$

The pressure outside this circle is assumed to be zero. In this case, since p is symmetrical about the z-axis, the deflexion w will also be symmetrical about that axis; that is, the deflexion w of a point in the plane $z = 0$ is a function of r only.

In Fig. 181, in which KLH represents the circle over which p

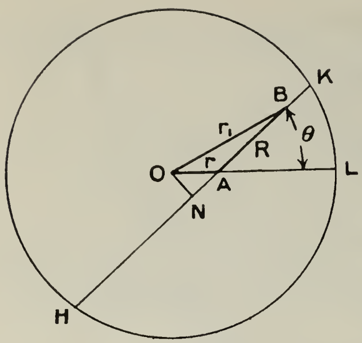

Fig. 181

acts, let ON in the perpendicular from the origin on the line of the radius vector R, which is the vector from A to B, and let the length of ON be b. Let

$$u = \mathrm{NB} = r\cos\theta + \mathrm{R}; \quad (19.68)$$

then

$$r_1{}^2 = u^2 + b^2 \quad . \quad . \quad (19.69)$$

Therefore the pressure at B is

$$p = \mathrm{C}(a^2 - b^2 - u^2)^{\frac{1}{2}}. \quad . \quad (19.70)$$

In integrating the expression in (19.64) with respect to R, both r and θ are constants. Therefore, from (19.68), $du = d\mathrm{R}$; consequently

$$\int pd\mathrm{R} = \int C(a^2 - b^2 - u^2)^{\frac{1}{2}} du \quad . \quad . \quad . \quad . \quad (19.71)$$

If we take this integral between the limits at H and K in Fig. 181 then the limits for θ are 0 and π. Now

$$\mathrm{HN} = \mathrm{NK} = \sqrt{a^2 - b^2}\,. \quad . \quad . \quad . \quad . \quad . \quad (19.72)$$

Let us write l^2 for $(a^2 - b^2)$; then the limits for u are from $-l$ to $+l$. Therefore

$$\begin{aligned}\int pd\mathrm{R} &= \int_{-l}^{l} \mathrm{C}(l^2 - u^2)^{\frac{1}{2}} du \\ &= \tfrac{1}{2}\pi \mathrm{C} l^2 = \tfrac{1}{2}\pi\mathrm{C}(a^2 - b^2) \\ &= \tfrac{1}{2}\pi\mathrm{C}(a^2 - r^2\sin^2\theta). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19.73)\end{aligned}$$

Finally

$$\begin{aligned} w &= \frac{1-\sigma}{2\pi n}\int_0^{\pi} \tfrac{1}{2}\pi\mathrm{C}(a^2 - r^2\sin^2\theta)\,d\theta \\ &= \frac{1-\sigma}{2\pi n}\cdot\tfrac{1}{2}\pi\mathrm{C}(a^2 - \tfrac{1}{2}r^2)\pi \\ &= \frac{(1-\sigma)\pi\mathrm{C}}{4n}(a^2 - \tfrac{1}{2}r^2). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19.74)\end{aligned}$$

This gives the displacement w in the direction of the pressure at a point in the circle where the pressure acts. The deflexion at the centre of the circle is

$$w_0 = \frac{(1-\sigma)\pi\mathrm{C}a^2}{4n} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19.75)$$

We may therefore write, when $r < a$,

$$w = w_0\left(1 - \frac{1}{2}\frac{r^2}{a^2}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19.76)$$

If w_0 is small in comparison with a the displacements over the circle of radius a form approximately a spherical dent in the surface. The displacement at the edge of the circle is $\frac{1}{2}w_0$.

To find the displacement at a point outside the circle over which the pressure is applied we must adjust the limits of integration with respect to θ. The limits for u are exactly the same as before, as fig. 182 shows. In this figure R denotes AB and u denotes NB. Thus the limits for u are clearly $\pm(a^2-b^2)^{\frac{1}{2}}$ as before. The upper limit for θ is, however, the value of θ at T, and the lower limit is the negative of this. Therefore, if θ_1 denotes the angle OAT, at a point outside the circle of pressure we get

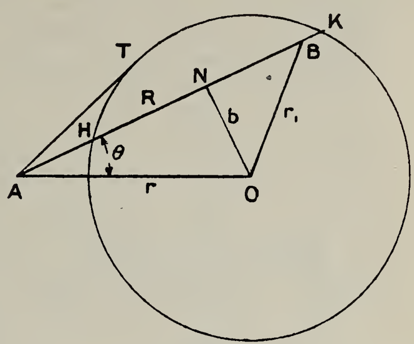

Fig. 182

$$w=\frac{1-\sigma}{2\pi n}\int_{-\theta_1}^{\theta_1}\tfrac{1}{2}\pi C(a^2-r^2\sin^2\theta)\,d\theta$$

$$=\frac{(1-\sigma)C}{4n}\int_{-\theta_1}^{\theta_1}(a^2-\tfrac{1}{2}r^2+\tfrac{1}{2}r^2\cos 2\theta)\,d\theta$$

$$=\frac{(1-\sigma)C}{4n}\{(2a^2-r^2)\theta_1+\tfrac{1}{2}r^2\sin 2\theta_1\}$$

$$=\frac{(1-\sigma)C}{4n}\left\{(2a^2-r^2)\sin^{-1}\frac{a}{r}+a(r^2-a^2)^{\frac{1}{2}}\right\}$$

$$=\frac{w_0}{\pi a^2}\left\{(2a^2-r^2)\sin^{-1}\frac{a}{r}+a(r^2-a^2)^{\frac{1}{2}}\right\} \quad . \quad . \quad . \quad (19{\cdot}77)$$

It is easy to show that this last expression for w is zero when r is infinite.

The displacements represented by (19.76) and (19.77) are shown in fig. 183:

The total thrust on the face of the solid is

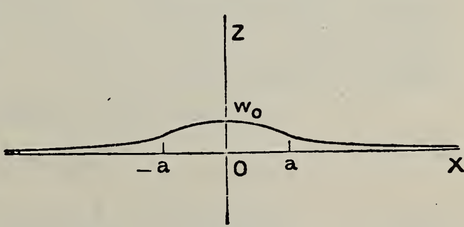

Fig. 183

$$W=\int p\,dx_1\,dy_1=\int_0^a 2\pi r_1\,p\,dr_1=\frac{2\pi}{3}Ca^3. \quad . \quad . \quad . \quad (19{\cdot}78)$$

357. Two bodies in contact.

Although the theory in the last article applies strictly only to an infinite solid bounded by a plane that same theory will nevertheless apply very accurately to a body which is neither infinite nor bounded by a plane; it is only necessary that w_0 should be small in comparison with a, and a small in comparison with the radius of curvature of the body at the place where the pressure is applied. Both these conditions will hold for most cases of bodies pressed together as long as the stresses are within the elastic limit. Thus if a sphere is squeezed between two parallel planes the displacements in the sphere and in the two bodies in contact with it in the neighbourhood of the areas of contact will be very nearly the same as those given by the theory of the last article. Making this assumption we can find approximately the change of shape of two spherical bodies when they are pressed together. As a particular case one of the bodies might have a plane boundary. Moreover the theory also applies to two bodies which are not spherical near the area of contact provided only that the area of contact under pressure is circular. Thus, for example, the theory applies to the case of two equal cylinders in contact with their axes perpendicular to each other.

Suppose two solid bodies, A and B, are in contact at a point O with no pressure between them. Let OZ_1 be the normal to the surface of the body A, the direction $\overrightarrow{OZ_1}$ being towards the inside of the body. If x, y, z_1, are the coordinates, referred to rectangular axes OX, OY, OZ_1, of a point on the surface of A, the equation to the surface in the immediate neighbourhood of O is shown in books on solid geometry to be

$$z_1 = a_1 x^2 + b_1 y^2 + 2h_1 xy, \qquad (19.79)$$

a_1, b_1, h_1, being constants. This equation is called the *indicatrix* of the surface of A in the neighbourhood of O.

Again if OZ_2 be the normal to the surface of the body B, the direction $\overrightarrow{OZ_2}$ being towards the inside of B, and therefore contrary to the direction of $\overrightarrow{OZ_1}$, the equation to the surface of B in the immediate neighbourhood of O is

$$z_2 = a_2 x^2 + b_2 y^2 + 2h_2 xy. \qquad (19.80)$$

Now let

$$\begin{aligned} z &= z_1 + z_2 \\ &= (a_1 + a_2)x^2 + (b_1 + b_2)y^2 + 2(h_1 + h_2)xy. \end{aligned} \qquad (19.81)$$

This last equation is the equation to the surface of A relative to that of B, and is called the *relative indicatrix* of the two surfaces at O. The curvatures derived from the equation are the *relative curvatures* of the two surfaces, that is, the difference of the curvatures of the two surfaces, these curvatures being both reckoned positive when their convex sides face the same way.

We intend at present to deal only with the case where the relative indicatrix is a circle. We shall therefore assume that

$$h_1 + h_2 = 0, \quad \ldots \quad (19.82)$$

$$a_1 + a_2 = b_1 + b_2 . \quad \ldots \quad (19.83)$$

Writing k for the common value of $(a_1 + a_2)$ and $(b_1 + b_2)$, the equation to the relative indicatrix is

$$z = k(x^2 + y^2) = kr^2 . \quad (19.84)$$

Let us now suppose that the two bodies are pressed together and that the surface of contact is a circle of radius a. In that case a spherical depression is made in each body, and the normal displacements in the two bodies within the area of contact (measured in each case from the tangent plane to the unstrained surface) are, by (19,74),

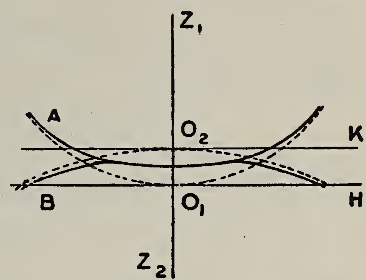

Fig. 184

$$\left.\begin{aligned} w_1 &= \frac{(1-\sigma_1)C}{4n_1}(a^2 - \tfrac{1}{2}r^2) \\ w_2 &= \frac{(1-\sigma_2)C}{4n_2}(a^2 - \tfrac{1}{2}r^2) \end{aligned}\right\} \quad \ldots \quad (19.85)$$

Here w_1 is measured from the plane O_1H in the direction O_1Z_1, and w_2 is measured from the plane O_2K in the opposite direction.

The constant C is the same for both bodies because the pressure p is the same for both. The elastic constants need not, of course, be the same for both bodies.

The distance of a point on the strained surface of A from the plane O_1H is $(z_1 + w_1)$, and the distance of a point of the strained surface of B from O_2K is $(z_2 + w_2)$. If d denote the distance between the two planes O_1H and O_2K we must have, in the area of contact,

$$(z_1 + w_1) + (z_2 + w_2) = d, \quad \ldots \quad (19.86)$$

whence

$$\begin{aligned} w_1 + w_2 &= d - (z_1 + z_2) \\ &= d - kr^2, \quad \ldots \quad (19.87) \end{aligned}$$

or, using (19.78) to express C in terms of W,

$$\frac{3W}{8\pi a^3}\left(\frac{1-\sigma_1}{n_1} + \frac{1-\sigma_2}{n_2}\right)(a^2 - \tfrac{1}{2}r^2) = d - kr^2. \quad \ldots \quad (19.88)$$

Since this is true for all values of r less than a, we get

$$\frac{3W}{8\pi a}\left(\frac{1-\sigma_1}{n_1} + \frac{1-\sigma_2}{n_2}\right) = d, \quad \ldots \quad (19.89)$$

and
$$\frac{3W}{16\pi a^3}\left(\frac{1-\sigma_1}{n_1}+\frac{1-\sigma_2}{n_2}\right)=k. \quad . \quad . \quad . \quad . \quad . \quad . \quad (19.90)$$

If the force W between the two bodies is known then equation (10.90) gives a; and this equation shows at the same time that a^3 is proportional to W for the same two bodies in contact at the same points. The constant k is known and is, in fact, the relative curvature of the surfaces of the unstrained bodies at the point of contact. Thus

$$k=a_1+a_2=\frac{1}{2}\left(\frac{\partial^2 z_1}{\partial x^2}+\frac{\partial^2 z_2}{\partial x^2}\right). \quad . \quad . \quad . \quad . \quad (19.91)$$

Also

$$k=b_1+b_2=\frac{1}{2}\left(\frac{\partial^2 z_1}{\partial y^2}+\frac{\partial^2 z_2}{\partial y^2}\right). \quad . \quad . \quad . \quad . \quad (19.92)$$

By (19.67) we see that the maximum pressure is

$$p_0=Ca=\frac{3W}{2\pi a^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19.93)$$

Since W is proportional to a^3 it follows that p_0^3 is proportional to W.

358. Particular examples of a sphere on a plane, and a sphere on a sphere.

Suppose a steel ball with a diameter of half an inch is thrust against a plane face of a much larger steel body, the total thrust between them being 500 pounds. We shall find the maximum stress and the area of contact.

Let us suppose that n and σ have the same values for both bodies. We shall take

$$\left.\begin{aligned} n_1 &= n_2 = 6000 \text{ tons per square inch,} \\ \sigma_1 &= \sigma_2 = 0{\cdot}3\,, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (19.94)$$

Now from (19.90)

$$a^3=\frac{3(1-\sigma_1)}{8\pi}\frac{W}{kn_1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19.95)$$

But k denotes half the sum of the curvatures of the sections of the two bodies by any plane containing the common normal. In this case k is merely half the curvature of the sphere; that is,

$$\frac{1}{k}=\frac{1}{2}\text{ inch}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19.96)$$

Therefore

$$a^3=\frac{3(1-\sigma_1)}{8\pi}\frac{500}{12000\times 2240}\text{ cub. inches}$$

$$=\frac{6{\cdot}36}{160^3}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19.97)$$

whence

$$a = \frac{1}{86 \cdot 5} \text{ inch.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19.98)$$

Again the maximum pressure is, by (19.93),

$$p_0 = \frac{3W}{2\pi a^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19.99)$$

which becomes, by means of (19.99),

$$p_0 = 797 \text{ tons per sq. inch} \quad . \quad . \quad . \quad . \quad (19.100)$$

This is a very big stress, but if we reduce the load to half a pound instead of 500 pounds the maximum stress is only thereby reduced to 79·7 tons per square inch, which is still a big stress.

If two steel balls, each having a diameter of half an inch, were pressed together with the same force W, the maximum pressure would be $2^{\frac{2}{3}}$ as much as for the sphere and plane; for, in this case, since the curvature of each sphere is 4,

$$k = \tfrac{1}{2}(4 + 4) = 4 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19.101)$$

which is twice as great as for the sphere and plane, and it is found, when a is eliminated from (19.95) and (19.99), that p^3 is proportional to k^2.

359. Cylindrical depression on the plane face of an infinite solid.

We shall assume that the pressure p is distributed over the rectangle in the xy plane whose sides are $x = \pm a$, $y = \pm b$ We shall also assume that the pressure p at (x_1, y_1) is an even function of x_1 and is not a function of y_1; that is,

$$p = f(x_1^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19.102)$$

By equation (19.60) the displacement at $(x, y, 0)$ is

$$w = \frac{1-\sigma}{2\pi n} \int_{-b}^{b} \int_{-a}^{a} \frac{p\, dx_1\, dy_1}{(R^2 + z^2)^{\frac{1}{2}}}, \quad . \quad . \quad . \quad . \quad (19.103)$$

wherein z must be finally made to approach zero.

Since we are going to assume that b is infinite we need only find w at a point on the x-axis. Therefore we may put $y = 0$ on the expression for w, and thus we get

$$R^2 + z^2 = (x - x_1)^2 + y_1^2 + z^2 = \varrho^2 + y_1^2, \quad . \quad . \quad (19.104)$$

where

$$\varrho^2 = (x - x_1)^2 + z^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (19.105)$$

Now
$$\int_{-b}^{b}\frac{p\,dy_1}{\sqrt{\varrho^2+y_1^2}}=2f(x_1^2)\int_{0}^{b}\frac{dy_1}{\sqrt{\varrho^2+y_1^2}}$$
$$=2f(x_1^2)\Big[\log\{y_1+\sqrt{\varrho^2+y_1^2}\}\Big]_0^b$$
$$=2f(x_1^2)\log_e\frac{b+\sqrt{\varrho^2+b^2}}{\varrho}. \quad . \quad . \quad . \quad (19.106)$$

We may now assume that b is very big in comparison with either a or z. Then ϱ is very small in comparison with b. Therefore we may take, neglecting $\frac{\varrho^2}{b^2}$,

$$\int_{-b}^{b}\frac{p\,dy_1}{\sqrt{\varrho^2+y_1^2}}=2f(x_1^2)\log\frac{2b}{\varrho}$$
$$=2f(x_1^2)\log 2b-2f(x_1^2)\log\varrho \quad . \quad . \quad (19.107)$$

Consequently

$$w=\frac{1-\sigma}{\pi n}\log 2b\int_{-a}^{a}f(x_1^2)\,dx_1-\frac{1-\sigma}{\pi n}\int_{-a}^{a}f(x_1^2)\log\varrho\,dx_1$$
$$=\frac{1-\sigma}{\pi n}\frac{W}{2b}\log 2b-\frac{1-\sigma}{\pi n}\int_{-a}^{a}f(x_1^2)\log\varrho\,dx_1, \quad . \quad . \quad (19.108)$$

where W denotes the total load on the rectangle. If b is infinite this load is also infinite, but $\frac{W}{2b}$ is finite. Thus the term involving W in (19.108) is infinite on account of the factor log $2b$. There is nothing very startling about this infinite displacement; it is due to the fact that we have assumed the body to be fixed at an infinite distance from the plane face. If a finite load were attached at the free end of an infinite elastic string this free end would have an infinite displacement due to a finite strain in the whole string.

The change of shape of the plane surface is due entirely to the finite term in the expression for w. We may therefore ignore the infinite constant term and take

$$w=-\frac{1-\sigma}{2\pi n}\int_{-a}^{a}f(x_1^2)\log\varrho^2\,dx_1. \quad . \quad . \quad . \quad (19.109)$$

Now

$$\int_{-a}^{a}f(x_1^2)\log\varrho^2dx_1=\int_{-a}^{0}f(x_1^2)\log\varrho^2\,dx_1+\int_{0}^{a}f(x_1^2)\log\varrho^2dx_1 \quad (19.110)$$

Putting $x_1=-\xi$ and consequently $dx_1=-d\xi$ in the first integral on the right hand side of the last equation, we get

$$\int_{-a}^{0}f(x_1^2)\log\{(x-x_1)^2+z^2\}\,dx_1=-\int_{a}^{0}f(\xi^2)\log\{(x+\xi)^2+z^2\}\,d\xi$$
$$=\int_{0}^{a}f(\xi^2)\log\{(x+\xi)^2+z^2\}\,d\xi, \quad . \quad . \quad . \quad (19.111)$$

which becomes, on replacing ξ by x_1,

$$\int_{-a}^{0} f(x_1^2)\log\{(x-x_1)^2+z^2\}\,dx_1=\int_0^a f(x_1^2)\log\{(x+x_1)^2+z^2\}\,dx_1 \quad (19.112)$$

Therefore

$$\int_{-a}^{a} f(x_1^2)\log\varrho^2\,dx_1$$

$$=\int_0^a f(x_1^2)\Big[\log\{(x-x_1)^2+z^2\}+\log\{(x+x_1)^2+z^2\}\Big]dx_1$$

$$=\int_0^a f(x_1^2)\log\{(x^2-x_1^2)^2+2z^2(x^2+x_1^2)+z^4\}\,dx_1 \,. \quad (19.113)$$

The normal displacement of a point in the xy plane, where $z=0$, is therefore

$$w=-\frac{1-\sigma}{2\pi n}\int_0^a f(x_1^2)\log(x^2-x_1^2)^2\,dx_1 \quad . \quad . \quad . \quad (19.114)$$

Therefore

$$-\frac{2\pi n}{1-\sigma}\frac{dw}{dx}=\int_0^a f(x_1^2)\frac{4x\,dx_1}{x^2-x_1^2};$$

that is,

$$-\frac{1}{2}\frac{\pi n}{(1-\sigma)}\frac{dw}{dx}=x\int_0^a \frac{f(x_1^2)\,dx_1}{x^2-x_1^2} \quad . \quad . \quad . \quad . \quad (19.115)$$

If x lies between 0 and a there is a singular point at $x_1=x$ in the function to be integrated in this last equation. There are a pair of infinities with opposite signs in the integral. In fact, if we write

$$\int_0^a \frac{f(x_1^2)\,dx_1}{x^2-x_1^2}=\left\{\int_0^{x-\varepsilon}+\int_{x-\varepsilon}^{x+\varepsilon}+\int_{x+\varepsilon}^{a}\right\}\frac{f(x_1^2)\,dx_1}{x^2-x_1^2}, \quad (19.116)$$

and if we assume that ε is infinitely small, the middle integral, namely

$$\int_{x-\varepsilon}^{x+\varepsilon}\frac{f(x_1^2)\,dx_1}{x^2-x_1^2},$$

is approximately equal to

$$f(x^2)\int_{x-\varepsilon}^{x+\varepsilon}\frac{dx_1}{x^2-x_1^2}=f(x^2)\int_{x-\varepsilon}^{x+\varepsilon}\frac{dx_1}{(x+x_1)(x-x_1)}$$

$$=\frac{f(x^2)}{2x}\int_{x-\varepsilon}^{x+\varepsilon}\frac{dx_1}{x-x_1}, \quad . \quad . \quad . \quad . \quad (19.117)$$

of which the positive and negative portions must be assumed to balance, because, if we are dealing with real quantities only,

$$\int_{-\varepsilon}^{\varepsilon}\frac{du}{u}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19.118)$$

Thus we may take, assuming ε infinitesimal,

$$-\frac{1}{2}\frac{\pi n}{1-\sigma}\frac{1}{x}\frac{dw}{dx}=\int_0^{x-\varepsilon}\frac{f(x_1{}^2)dx}{x^2-x_1{}^2}+\int_{x+\varepsilon}^{a}\frac{f(x_1{}^2)dx_1}{x^2-x_1{}^2}\ . \quad (19.119)$$

Now let us assume that

$$f(x_1{}^2)=\frac{p_0}{a}\sqrt{a^2-x_1{}^2}. \quad (19.120)$$

To work out the integrals we put

$$x_1=a\sin\theta, \quad (19.121)$$

whence

$$dx_1=a\cos\theta d\theta. \quad (19.122)$$

Let the limits for θ corresponding to 0 and $(x-\varepsilon)$ be 0 and θ_1; and let the limits corresponding to $(x+\varepsilon)$ and a be θ_2 and $\frac{\pi}{2}$. Then, as ε approaches zero, θ_1 and θ_2 approach a common finite limit. This common limit is α such that

$$\sin\alpha=\frac{x}{a}. \quad (19.123)$$

Now

$$\begin{aligned}\frac{\sqrt{a^2-x_1{}^2}\,dx_1}{x^2-x_1{}^2}&=\frac{a^2\cos^2\theta d\theta}{x^2-a^2\sin^2\theta}\\&=\frac{a^2(1-\sin^2\theta)d\theta}{x^2-a^2\sin^2\theta}\\&=d\theta+\frac{(a^2-x^2)d\theta}{x^2-a^2\sin^2\theta}\\&=d\theta+\frac{(a^2-x^2)d\theta}{x^2\cos^2\theta-(a^2-x^2)\sin^2\theta}\\&=d\theta+\frac{\sec^2\theta d\theta}{\dfrac{x^2}{a^2-x^2}-\tan^2\theta}. \quad (19.124)\end{aligned}$$

But

$$\frac{x^2}{a^2-x^2}=\frac{\dfrac{x^2}{a^2}}{1-\dfrac{x^2}{a^2}}=\tan^2\alpha. \quad (19.125)$$

Therefore

$$\begin{aligned}\int_0^{x-\varepsilon}\frac{\sqrt{a^2-x_1{}^2}dx_1}{x^2-x_1{}^2}&=\int_0^{\theta_1}\left\{1+\frac{\sec^2\theta d\theta}{\tan^2\alpha-\tan^2\theta}\right\}d\theta\\&=\theta_1+\int_0^{\tan\theta_1}\frac{d(\tan\theta)}{\tan^2\alpha-\tan^2\theta}\\&=\theta_1+\tfrac{1}{2}\cot\alpha\log\frac{\tan\alpha+\tan\theta_1}{\tan\alpha-\tan\theta_1}\ . \quad (19.126)\end{aligned}$$

Also

$$\int_{x+\varepsilon}^{a} \frac{\sqrt{a^2-x_1^2}\,dx_1}{x^2-x_1^2} = \int_{\theta_2}^{\frac{\pi}{2}} \left\{ d\theta + \frac{d(\tan\theta)}{\tan^2\alpha - \tan^2\theta} \right\}$$

$$= \frac{\pi}{2} - \theta_2 - \int_{\theta_2}^{\frac{\pi}{2}} \frac{d(\tan\theta)}{\tan^2\theta - \tan^2\alpha}$$

$$= \frac{\pi}{2} - \theta_2 - \tfrac{1}{2}\cot\alpha \left[\log \frac{\tan\theta - \tan\alpha}{\tan\theta + \tan\alpha} \right]_{\theta_2}^{\frac{\pi}{2}}$$

$$= \frac{\pi}{2} - \theta_2 + \tfrac{1}{2}\cot\alpha \log \frac{\tan\theta_2 - \tan\alpha}{\tan\theta_2 + \tan\alpha}, \quad . \ (19.127)$$

the integrated term vanishing at the upper limit. Thus finally

$$-\frac{1}{2}\frac{a}{p_0}\frac{\pi n}{1-\sigma}\frac{1}{x}\frac{dw}{dx} = \frac{\pi}{2} + \theta_1 - \theta_2$$

$$+ \tfrac{1}{2}\cot\alpha \log \frac{\tan\alpha + \tan\theta_1}{\tan\alpha + \tan\theta_2} \times \frac{\tan\theta_2 - \tan\alpha}{\tan\alpha - \tan\theta_1} \quad (19.128)$$

In this result we have to make $\varepsilon \rightarrow 0$, and this makes θ_1 and θ_2 both approach α. Therefore

$$-\frac{1}{2}\frac{a}{p_0}\frac{\pi n}{1-\sigma}\frac{1}{x}\frac{dw}{dx} = \tfrac{1}{2}\pi + \lim_{\varepsilon \rightarrow 0} \tfrac{1}{2}\cot\alpha \log \frac{\tan\theta_2 - \tan\alpha}{\tan\alpha - \tan\theta_1}, \quad (19.129)$$

where

$$\sin\theta_1 = \frac{x-\varepsilon}{a}, \quad \sin\theta_2 = \frac{x+\varepsilon}{a}, \quad \sin\alpha = \frac{x}{a}. \quad . \ (19.130)$$

Now let

$$\theta_1 = \alpha - \beta, \quad \theta_2 = \alpha + \gamma, \quad . \ . \ . \ . \ (19.131)$$

β and γ being small. Then

$$\sin\theta_1 = \sin(\alpha - \beta) = \sin\alpha - \beta\cos\alpha \text{ nearly}, \quad . \ . \ (19.132)$$

and

$$\sin\theta_2 = \sin(\alpha + \gamma) = \sin\alpha + \gamma\cos\alpha. \quad . \ . \ . \ (19.133)$$

Consequently

$$\beta\cos\alpha = \sin\alpha - \sin\theta_1 = \frac{\varepsilon}{a}; \quad . \ . \ . \ . \ (19.134)$$

$$\gamma\cos\alpha = \sin\theta_2 - \sin\alpha = \frac{\varepsilon}{a}. \quad . \ . \ . \ . \ (19.135)$$

Thus β and γ are equal as far as the first power of ε.

Next

$$\tan\theta_1 = \tan(\alpha - \beta) = \tan\alpha - \beta\sec^2\alpha \quad . \ . \ . \ (19.136)$$

$$\tan\theta_2 = \tan(\alpha + \gamma) = \tan\alpha + \gamma\sec^2\alpha \quad . \ . \ . \ (19.137)$$

whence
$$\lim_{\varepsilon \to 0} \log \frac{\tan\theta_2 - \tan\alpha}{\tan\alpha - \tan\theta_1} = \lim_{\varepsilon \to 0} \log \frac{\gamma \sec^2\alpha}{\beta \sec^2\alpha}$$
$$= \lim_{\varepsilon \to 0} \log \frac{\frac{\varepsilon}{a}\sec^3\alpha}{\frac{\varepsilon}{a}\sec^3\alpha}$$
$$= 0. \qquad (19.138)$$

Therefore (19.129) gives
$$\frac{1}{x}\frac{dw}{dx} = -\frac{1-\sigma}{n}\frac{p_0}{a}, \qquad (19.139)$$

from which
$$w = C - \frac{(1-\sigma)p_0}{2na}x^2. \qquad (19.140)$$

Since the coefficient of x^2 is small in any actual case this may be regarded as the displacement due to a cylindrical depression, the curvature of which depression is
$$\frac{d^2w}{dx^2} = -\frac{(1-\sigma)p_0}{na}. \qquad (19.141)$$

The total thrust per unit length in the y-direction is
$$P = \int_{-a}^{a} p\,dx_1 = \frac{p_0}{a}\int_{-a}^{a}(a^2 - x_1^2)^{\frac{1}{2}}\,dx_1$$
$$= \tfrac{1}{2}\pi p_0 a. \qquad (19.142)$$

Therefore the curvature of the depression, expressed in terms of P, is
$$-\frac{d^2w}{dx^2} = \frac{2(1-\sigma)P}{n\pi a^2}. \qquad (19.143)$$

360. Two cylinders in contact with their axes parallel.

The results proved in the last article can be applied to two cylinders pressed together with their axes parallel. The practical problem of roller bearings is an example to which the results can be applied.

It should be noticed that the theory of the last article applies strictly only to infinitely long bodies with plane faces. Nevertheless there will be very little error in applying the results to a cylinder whose radius is much greater than the width a of the rectangle of contact, but in that case (19.143) gives the change in curvature of the cylinder produced by the pressure. Moreover, although the length of the rectangle is assumed to be infinite, there will again be very little error in using the results for a case where b is much greater than a; for example, if $\frac{b}{a}$ is not less than 10, the approximation in (19.107) is quite good for points in the plane $z=0$. We finally conclude that the pressure and the displacement at a point not very near the narrow ends of the rectangle of contact of two cylinders is approximately the same as is given by the theory of the last article. It is to be understood that a point is not near the ends if its distance from the nearest narrow end is greater than $5a$

Suppose two cylinders with radii r_1 and r_2 are pressed together till the width of the area of contact is $2a$. Let P be the thrust per unit length of cylinder, p_0 the pressure at the centre of the rectangle of contact. Then the relative curvature before the pressure was exerted, assuming the two cylinders are convex to each other, is

$$\frac{1}{r_1}+\frac{1}{r_2}.$$

This relative curvature is reduced by pressure to zero. But if n_1, σ_1, n_2, σ_2, are the elastic constants for the two cylinders, equation (19.143) tells us that the changes in the curvatures of the two cylinders are

$$-\frac{d^2w_1}{dx^2}=\frac{2(1-\sigma_1)}{n_1\pi a^2}P, \quad \quad (19.144)$$

$$-\frac{d^2w_2}{dx^2}=\frac{2(1-\sigma_2)}{n_2\pi a^2}P. \quad \quad (19.145)$$

The total change must be equal to the original relative curvature; that is,

$$\frac{2P}{\pi a^2}\left\{\frac{1-\sigma_1}{n_1}+\frac{1-\sigma_2}{n_2}\right\}=\frac{1}{r_1}+\frac{1}{r_2}. \quad \quad (19.146)$$

If P is given, this equation given a. Thus we see that a^2 is proportional to P, and therefore proportional to the total thrust between the two cylinders.

Again from (19.142) and (19.146)

$$P\left(\frac{1}{r_1}+\frac{1}{r_2}\right)=\tfrac{1}{2}\pi p_0{}^2\left\{\frac{1-\sigma_1}{n_1}+\frac{1-\sigma_2}{n_2}\right\}. \quad \quad (19.147)$$

which shows that $p_0{}^2$ is proportional to P and therefore to the total thrust.

361. A cylinder on a plane.

Suppose a steel cylinder of length one inch and diameter half an inch is pressed against a plane face of a large steel body with a total thrust of 500 lbs. To find p_0 and the width of the area of contact we shall take the values of n and σ to be the same as in (19.94). Then taking $r_1=\frac{1}{4}$ inch, $r_2=\infty$, (19.146) gives

$$\frac{1-\sigma}{n}\frac{4P}{\pi a^2}=\frac{1}{r_1}=4; \quad \quad (19.148)$$

that is,

$$a^2 = \frac{(1-\sigma)P}{\pi n} = \frac{1}{120700}; \quad . \quad . \quad . \quad . \quad (19.149)$$

whence

$$a = \tfrac{1}{347} \text{ inch} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19.150)$$

Also

$$\frac{1-\sigma}{n}\pi p_0{}^2 = \frac{P}{r_1} = 4P. \quad . \quad . \quad . \quad . \quad . \quad . \quad (19.151)$$

Therefore

$$p_0{}^2 = \frac{4Pn}{(1-\sigma)\pi}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (19.152)$$

whence

$$p_0 = 48{\cdot}9 \text{ tons per sq. inch.} \quad . \quad . \quad . \quad . \quad (19.153)$$

If we had taken $P = 1125$ lbs we should have got $p_0 = 73{\cdot}3$ tons per square inch, which is nearly the same pressure as we found for a sphere with the same diameter pressed against a plane with a total thrust of only half a pound.

APPENDIX A.

BESSEL FUNCTIONS.

The equation for Bessel functions of the n^{th} order is

$$\frac{d^2z}{dx^2}+\frac{1}{x}\frac{dz}{dx}+\left(1-\frac{n^2}{x^2}\right)z=0 \quad \text{. . . . . . (A.1)}$$

If we put $x=kr$ in this equation and then multiply through by k^2 we get

$$\frac{d^2z}{dr^2}+\frac{1}{r}\frac{dz}{dr}+\left(k^2-\frac{n^2}{r^2}\right)z=0, \quad \text{. . . . . . (A.2)}$$

which is identical with equation (18.92). To solve (A.1) put

$$z=\Sigma C_m x^m;$$

then (A.1) gives

$$\Sigma C_m x^{m-2}\left\{(m^2-n^2)+x^2\right\}=0. \quad \text{. . . . . . (A.3)}$$

Equating to zero the coefficient of x^{m-2} in this last equation we get

$$(m^2-n^2)C_m+C_{m-2}=0 \quad \text{. . . . . . . (A.4)}$$

Putting $m=n$ in this we get

$$C_{n-2}=0$$

whatever value C_n has.

Thus there is a series beginning with x^n, C_n being an arbitrary constant. Since (A.4) gives a relation between the coefficients of powers of x whose indices differ by 2 it follows that the series ascends in powers of x^2. The relation between successive coefficients is, by (A.4),

$$C_m=-\frac{1}{m^2-n^2}C_{m-2}$$

$$=-\frac{1}{(m-n)(m+n)}C_{m-2},$$

or

$$C_{n+2s}=-\frac{1}{2s(2n+2s)}C_{n+2s-2} \quad \text{. . . . . . (A.5)}$$

Thus, when A is written for C_n, the series starting with x^n is,

$$Ax^n\left\{1-\frac{x^2}{2(2n+2)}+\frac{x^4}{2.4(2n+2)(2n+4)}-\ldots\right\} \quad \text{. . (A.6)}$$

If n is not an integer there is another series starting with C_{-n}, as equation (A.5) shows. This series, obtained from the series in (A.6) by putting $-n$ for n and B for A, is

$$Bx^{-n}\left\{1-\frac{x^2}{2(2-2n)}+\frac{x^4}{2.4(2-2n)(4-2n)}-\ldots\right\} \quad . \quad . \quad (A.7)$$

If n is an integer the coefficient of x^{2n} in the brackets in this last series contains an infinite factor, and all later terms contain this same factor. We can avoid this infinite factor, however, by taking C as the coefficient of x^{2n} in the brackets and making C finite. In that case B will be zero, and therefore all the terms before x^{2n} in brackets will vanish. Then the series begins with x^n and is identical with the series in (A.6), except possibly in sign.

Now let $J_n(x)$ be a function of x defined by the equation

$$J_n(x)=\frac{x^n}{2^n\Gamma(n+1)}\left\{1-\frac{x^2}{2(2n+2)}+\frac{x^4}{2.4(2n+2)(2n+4)}\ldots\right\}, \quad (A.8)$$

$\Gamma(n+1)$ being the gamma function defined by

$$\Gamma(n+1)=\lim_{k\to\infty}\frac{1.\,2.\,3.\,\ldots\ldots\,k}{(n+1)(n+2)\ldots(n+k)}k^n. \quad . \quad . \quad (A.9)$$

The important property of the gamma function is

$$\Gamma(n+1)=n\Gamma(n).$$

Also, if n is a positive integer,

$$\Gamma(n+1)=\lfloor n.$$

Then, if n is not an integer, the complete solution of (A.1) is

$$z=AJ_n(x)+BJ_{-n}(x). \quad . \quad . \quad . \quad . \quad . \quad . \quad (A.10)$$

When n is an integer it can be shown that

$$(-1)^nJ_{-n}(x)=J_n(x) \quad . \quad . \quad . \quad . \quad . \quad . \quad (A.11)$$

In that case (A.10) does not give the complete solution of (A.1) since it contains only one arbitrary constant ($A\pm B$). One solution is still

$$z=AJ_n(x),$$

but we have now to find a second solution of (A.1).

A few particular cases of $J_n(x)$ are given below.

$$\left.\begin{aligned}
J_0(x)&=1-\frac{x^2}{2^2}+\frac{x^4}{2^2.4^2}-\frac{x^6}{2^2.4^2.6^2}+\ldots\\
J_1(x)&=\frac{x}{2}\left\{1-\frac{x^2}{2.4}+\frac{x^4}{2.4^2.6}-\frac{x^6}{2.4^2.6^2.8}+\ldots\right\}\\
J_2(x)&=\frac{x^2}{2.4}\left\{1-\frac{x^2}{2.6}+\frac{x^4}{2.4.6.8}-\frac{x^6}{2.4.6^2.8.10}+\frac{x^8}{2.4.6^2.8^2.10.12}\ldots\right\}
\end{aligned}\right\} \quad (A.12)$$

We shall first prove some properties of Bessel functions using only the differential equation itself. Let z_n be any solution of (A.1), and let us put

$$z_n = u_n x^n \quad \text{(A. 13)}$$

Then (A.1) becomes

$$x^n \frac{d^2 u_n}{dx^2} + (2n+1)x^{n-1}\frac{du_n}{dx} + x^n u_n = 0,$$

whence

$$\frac{d^2 u_n}{dx^2} + \frac{2n+1}{x}\frac{du_n}{dx} + u_n = 0. \quad \text{(A. 14)}$$

Next let

$$y = \tfrac{1}{2}x^2 \quad \text{(A. 15)}$$

Then

$$\frac{1}{x}\frac{du_n}{dx} = \frac{du_n}{dy},$$

and

$$\begin{aligned}\frac{d^2 u_n}{dx^2} &= \frac{d}{dx}\left(x\frac{du_n}{dy}\right)\\ &= \frac{du_n}{dy} + x\frac{d}{dx}\left(\frac{du_n}{dy}\right)\\ &= \frac{du_n}{dy} + x^2\frac{d^2u_n}{dy^2}\\ &= \frac{du_n}{dy} + 2y\frac{d^2u_n}{dy^2}.\end{aligned}$$

Therefore (A.14) becomes

$$2y\frac{d^2u_n}{dy^2} + 2(n+1)\frac{du_n}{dy} + u_n = 0. \quad \text{(A. 16)}$$

Differentiating through this last equation with respect to y and writing u' for $\frac{du_n}{dy}$ we get

$$2y\frac{d^2u'}{dy^2} + 2(n+2)\frac{du'}{dy} + u' = 0. \quad \text{(A. 17)}$$

A comparison of (A.16) and (A.17) shows that u' is a function similar to u_n with the difference that $(n+1)$ takes the place of n. If therefore

$$z_{n+1} = u_{n+1}x^{n+1} \quad \text{(A. 18)}$$

is a solution of the equation

$$\frac{d^2 z_{n+1}}{dx^2} + \frac{1}{x}\frac{dz_{n+1}}{dx} + \left\{1 - \frac{(n+1)^2}{x^2}\right\}z_{n+1} = 0, \quad \text{(A. 19)}$$

then one possible value of u_{n+1} is $-u'$, and therefore a possible value of z_{n+1} is

$$z_{n+1} = -x^{n+1}\frac{du_n}{dy} \quad \text{. . . . . . . (A.20)}$$

Therefore

$$x^{-(n+1)}z_{n+1} = -\frac{1}{x}\frac{du_n}{dx} = -\frac{1}{x}\frac{d}{dx}(x^{-n}z_n); \quad \text{. . (A.21)}$$

and, putting $(n-1)$ for n,

$$x^{-n}z_n = -\frac{d}{xdx}(x^{-n+1}z_{n-1}). \quad \text{. . . . . (A.22)}$$

The reason for taking the negative sign will appear when we apply the method to $J_n(x)$ below.

By means of this last equation a solution of Bessel's equation of the n^{th} order can be deduced from a solution of the equation of the $(n-1)^{th}$ order. In particuiar, if

$$z_n = J_n(x) \quad \text{. . . . . (A.23)}$$

then

$$x^{-n}z^n = \frac{1}{2^n\Gamma(n+1)}\left\{1 - \frac{x^2}{2(2n+2)} + \frac{x^4}{2.4(2n+2)(2n+4)}\ldots\right\} \quad \text{(A.24)}$$

Therefore, by direct differentiation,

$$-\frac{d}{xdx}(x^{-n}z_n) = \frac{1}{2^n\Gamma(n+1)}\left\{\frac{2}{2(2n+2)} - \frac{4x^2}{2.4(2n+2)(2n+4)} + \ldots\right\}$$

$$= \frac{1}{2^{n+1}(n+1)\Gamma(n+1)}\left\{1 - \frac{x^2}{2(2n+4)} + \frac{x^4}{2.4(2n+4)(2n+6)} - \ldots\right\}$$

$$= \frac{1}{2^{n+1}\Gamma(n+2)}\left\{1 - \frac{x^2}{2(2n+4)} + \frac{x^4}{2.4(2n+4)(2n+6)} - \ldots\right\} \quad \text{(A.25)}$$

But this last line differs from $x^{-n}z_n$ only in having $(n+1)$ instead of n. Therefore we have found by direct differentiation that

$$-\frac{d}{xdx}\{x^{-n}J_n(x)\} = x^{-n-1}J_{n+1}(x). \quad \text{. . . . . (A.26)}$$

The following are particular cases:—

$$x^{-1}J_1(x) = -\frac{d}{xdx}\{J_0(x)\}; \quad \text{. . . . . . (A.27)}$$

$$x^{-2}J_2(x) = -\frac{d}{xdx}\{x^{-1}J_1(x)\}$$

$$= -\frac{d}{xdx}\left\{-\frac{dJ_0(x)}{xdx}\right\}$$

$$= \left(\frac{d}{xdx}\right)^2 J_0(x); \quad \text{. . . . . . . (A.28)}$$

and, in general, if n is a positive integer,

$$x^{-n}J_n(x) = \left(-\frac{d}{xdx}\right)^n J_0(x) \quad . \quad . \quad . \quad . \quad . \quad (A.29)$$

Equation (A.21) suggests that we ought to be able to deduce from the second solution of Bessel's equation of zero order the second solution of Bessel's equation of the n^{th} order; that is, we should expect that, if the second solution of Bessel's equation of zero order were substituted for $J_0(x)$ on the right of (A.29) then $J_n(x)$ on the left hand side would be changed to the second solution of the equation of the n^{th} order. This is, in fact, true, as we shall show after we have found a second solution of the equation of zero order.

When $n=0$ equation (A.1) becomes

$$\frac{d^2z}{dx^2} + \frac{1}{x}\frac{dz}{dx} + z = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (A30)$$

the simplest solution of which is

$$z = AJ_0(x). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (A.31)$$

To get the general solution of (A.30) we may first put it in the form

$$x\frac{d}{dx}\left(x\frac{dz}{dx}\right) = -x^2 z. \quad . \quad . \quad . \quad . \quad . \quad . \quad (A.32)$$

Now by the substitution

$$x = e^{\theta}$$

equation (A.32) becomes

$$\frac{d^2z}{d\theta^2} = -e^{2\theta}z. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (A.33)$$

Let us next put

$$z = v_1 + v_2 + v_3 + \dots \quad . \quad . \quad . \quad . \quad . \quad (A.34)$$

Then (A.33) becomes, when D^2 is written for $\frac{d^2}{d\theta^2}$,

$$D^2v_1 + D^2v_2 + D^2v_3 + \dots = -e^{2\theta}(v_1 + v_2 + v_3 + \dots) \quad . \quad (A.35)$$

Let us now take

$$\left.\begin{aligned} D^2v_1 &= 0, \\ D^2v_2 &= -v_1 e^{2\theta}, \\ D^2v_3 &= -v_2 e^{2\theta}, \\ &\dots\dots\dots \\ D^2v_m &= -v_{m-1}e^{2\theta}. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (A.36)$$

Therefore

$$\begin{aligned} v_m &= -D^{-2}(v_{m-1}e^{2\theta}) \\ &= -e^{2\theta}(D+2)^{-2}v_{m-1} \quad . \quad . \quad . \quad . \quad . \quad (A.37) \end{aligned}$$

By a repetition of this operation we get

$$\begin{aligned} v_m &= -e^{2\theta}(D+2)^{-2}\{-e^{2\theta}(D+2)^{-2}v_{m-2}\} \\ &= e^{4\theta}(D+4)^{-2}(D+2)^{-2}v_{m-2} \\ &= -e^{6\theta}(D+6)^{-2}(D+4)^{-2}(D+2)^{-2}v_{m-3}. \end{aligned}$$

The general formula giving v_m in terms of v_1 is

$$v_m = (-1)^{m-1}e^{2(m-1)\theta}(D+2)^{-2}(D+4)^{-2}\ldots(D+2m-2)^{-2}v_1. \quad (A.38)$$

Now the solution of the first of equations (A.36) is

$$v_1 = (A+B\theta).$$

Therefore (A.38) gives

$$\begin{aligned} v_m &= \frac{(-1)^{m-1}e^{2(m-1)\theta}}{2^2.4^2.6^2.....(2m-2)^2}\{(1+\tfrac{1}{2}D)^{-2}(1+\tfrac{1}{4}D)^{-2}.......\}v_1 \\ &= \frac{(-1)^{m-1}e^{2(m-1)\theta}}{2^2.4^2.6^2.....(2m-2)^2}\{1-D-\tfrac{1}{2}D-\tfrac{1}{3}D-\ldots\}v_1, \end{aligned}$$

higher powers of D than the first being neglected because $D^2(A+B\theta)=0$. Therefore

$$v_m = \frac{(-1)^{m-1}e^{2(m-1)\theta}}{2^2.4^2.6^2.....(2m-2)^2}(A+B\theta-s_{m-1}B), \quad (A.39)$$

where

$$s_{m-1} = \frac{1}{1}+\frac{1}{2}+\frac{1}{3}+....+\frac{1}{m-1}. \quad (A.40)$$

The following are particular cases:—

$$v_2 = -\frac{x^2}{2^2}(A+B\log x - B);$$

$$v_3 = +\frac{x^4}{2^2.4^2}(A+B\log x - s_2 B);$$

$$v_4 = -\frac{x^6}{2^2.4^2.6^2}(A+B\log x - s_3 B).$$

Collecting all the terms we get

$$\begin{aligned} z &= v_1+v_2+v_3+\ldots\ldots \\ &= (A+B\log_e x)J_0(x) \\ &\quad + B\left\{\frac{x^2}{2^2}-s_2\frac{x^4}{2^2.4^2}+s_3\frac{x^6}{2^2.4^2.6^2}-\ldots.\right\}. \quad (A.41) \end{aligned}$$

Now writing

$$Z_0(x) = J_0(x)\log x + \frac{x^2}{2^2} - s_2\frac{x^4}{2^2.4^2} + s_3\frac{x^6}{2^2.4^2.6^2} - .., \quad (A.42)$$

we have got, as the complete solution of (A.29),

$$z = AJ_0(x) + BZ_0(x). \quad (A.43)$$

The function $Z_0(x)$ is Neumann's form of the second solution of (A. 30). Since the complete solution is given by (A.43) all other forms of the second solution must be included in (A.43). Thus Weber's form of the second function is

$$Y_0(x) = \frac{2}{\pi} Z_0(x) - \frac{2}{\pi}(\log 2 - \gamma) J_0(x), \quad . \quad . \quad . \quad . \quad (A.44)$$

γ being Euler's constant defined by

$$\gamma = \lim_{m \to \infty} (s_m - \log_e m). \quad . \quad . \quad . \quad . \quad . \quad . \quad (A.45)$$

When $m = \infty$ both s_m and $\log_e m$ are infinite but their difference approaches a finite limit as m approaches ∞. It is found that

$$\gamma = 0{\cdot}5772156649 \ldots\ldots \quad . \quad . \quad . \quad . \quad . \quad (A.46)$$

$J_0(x)$ is convergent for all values of x; and except when $x = 0$, $Z_0(x)$ is also convergent for all values of x. Also

$$\lim_{x \to 0} \frac{Z_0(x)}{\log_e x} = 1.$$

Thus $Z_0(x)$ approaches $-\infty$ as x approaches zero.

Equation (A. 21) shows that we can derive a Bessel function of the $(n+1)^{th}$ order from one of the n^{th} order by differentiation. Moreover we showed by direct differentiation that this method, when applied to $J_n(x)$, gives us $J_{n+1}(x)$. Also $J_n(x)$ is derived from $J_0(x)$ by n successive applications of the process. Now it is clear that n successive applications of the same process to $Z_0(x)$ gives a function different from $AJ_n(x)$. Moreover this derived function is a Bessel function of the n^{th} order. Therefore it is one form of the second solution of (A. 1) when n is an integer. Let the function z_n derived in this way be denoted by $Z_n(x)$. That is,

$$x^{-n} Z_n(x) = -\frac{1}{x}\frac{d}{dx}\{x^{-n+1} Z_{n-1}(x)\} \quad . \quad . \quad . \quad . \quad (A.47)$$

By putting $n = 1$ in this we get

$$\begin{aligned} x^{-1} Z_1(x) &= -\frac{d}{xdx}\{Z_0(x)\} \\ &= -\log x \frac{d}{xdx}\{J_0(x)\} - \frac{1}{x^2} J_0(x) \\ &\quad -\frac{1}{2} + \frac{s_2 x^2}{2^2.4} - \frac{s_3 x^4}{2^2.4^2.6} + \frac{s_4 x^6}{2^2.4^2.6^2.8} - \ldots . \end{aligned}$$

By means of (A. 27) and the first of equations (A. 12) the last equation becomes

$$\begin{aligned} x^{-1} Z_1(x) &= x^{-1} J_1(x) \log x - \frac{1}{x^2} \\ &\quad -\frac{1}{4} + \frac{x^2}{2^2.4}\left(s_2 - \frac{1}{4}\right) - \frac{x^4}{2^2.4^2.6}\left(s_3 - \frac{1}{6}\right) + \frac{x^6}{2^2.4^2.6^2.8}\left(s_4 - \frac{1}{8}\right) - \end{aligned}$$

$$=x^{-1}J_1(x)\log x-\frac{1}{x^2}$$

$$-\frac{1}{2}\left\{\frac{1}{2}-\frac{x^2}{2^2.4}\left(2s_2-\frac{1}{2}\right)+\frac{x^4}{2^2.4^2.6}\left(2s_3-\frac{1}{3}\right)-\frac{x^6}{2^2.4^2.6^2.8}\left(2s_4-\frac{1}{4}\right)+..\right\}$$

$$=x^{-1}J_1(x)\log x-\frac{1}{x^2}$$

$$-\frac{1}{2}\left\{\frac{1}{2}-\frac{x^2}{2^2.4}(s_1+s_2)+\frac{x^4}{2^2.4^2.6}(s_2+s_3)-\frac{x^6}{2^2.4^2.6^2.8}(s_3+s_4)+\ldots\right\} \quad (A.48)$$

By repeating this process we get

$$x^{-2}Z_2(x)=-\frac{d}{xdx}\{x^{-1}Z_1(x)\}$$

$$=-\log x\frac{d}{xdx}\{x^{-1}J_1(x)\}-\frac{1}{x^3}J_1(x)-\frac{2}{x^4}$$

$$-\frac{1}{2}\left\{\frac{1}{2.4}(s_1+s_2)-\frac{x^2}{2^2.4.6}(s_2+s_3)+\frac{x^4}{2.^24^2.6.8}(s_3+s_4)-\ldots\right\},$$

from which we get, by using (A. 26) and the expression for $J_1(x)$,

$$x^{-2}Z_2(x)=x^{-2}J_2(x)\log x-\frac{1}{2x^2}-\frac{2}{x^4}$$

$$-\frac{1}{2}\left\{\frac{1}{2.4}(s_1+s_2-1)-\frac{x^2}{2^2.4.6}(s_2+s_3-\tfrac{1}{2})+\frac{x^4}{2^2.4^2.6.8}(s_3+s_4-\tfrac{1}{3})\ldots\right\}$$

$$=x^{-2}J_2(x)\log x-\frac{1}{2x^2}-\frac{2}{x^4}$$

$$-\frac{1}{2}\left\{\frac{s_2}{2.4}-\frac{x^2}{2^2.4.6}(s_1+s_3)+\frac{x^4}{2^2.4^2.6.8}(s_2+s_4)-\ldots\ldots\right\} \quad (A.49)$$

By successive applications of this process we get

$$x^{-n}Z_n(x)=x^{-n}J_n(x)\log x-\frac{1}{2}x^{-n}\sum_{m=0}^{m=n-1}\frac{\lfloor n-m-1}{\lfloor m}\left(\frac{x}{2}\right)^{2m-n}$$

$$-\frac{1}{2^{n+1}}\sum_{m=0}^{m=\infty}\frac{(-1)^m}{\lfloor n+m\lfloor m}(s_m+s_{n+m})\left(\frac{x}{2}\right)^{2m} \quad . \quad . \quad . \quad (A.50)$$

In the sum from $m=0$ to $m=n-1$ there are no terms when $n=0$, and one term when $n=1$. It is to be understood that

$$\left.\begin{aligned}&s_m=\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{m};\quad s_0=0;\\&\lfloor 0=1\end{aligned}\right\} \quad . \quad . \quad (A.51)$$

It is easy to see that the equations for the particular functions $Z_0(x)$, $x^{-1}Z_1(x)$, $x^{-2}Z_2(x)$, agree with the general equation (A. 50). Also the general formula can be proved by induction.

The function $Z_n(x)$ defined by (A. 50) is C. G. Neumann's form of the second Bessel function of the n^{th} order. Weber's function $Y_n(x)$, defined in the next equation, is tabulated very completely in Watson's treatise*;

$$Y_n(x) = \frac{2}{\pi}\{Z_n(x) - \lambda J_n(x)\}, \quad \ldots\ldots \quad (A.52)$$

where λ is written for $(\log_e 2 - \gamma)$.

Now when n is an integer the complete solution of (A. 1) can be written in either of the forms

$$z = AJ_n(x) + BZ_n(x); \quad \ldots\ldots \quad (A.53)$$

or

$$z = AJ_n(x) + CY_n(x). \quad \ldots\ldots \quad (A.54)$$

Consequently the complete solution of (A. 2) is

$$z = AJ_n(kr) + BZ_n(kr), \quad \ldots\ldots \quad (A.55)$$

or

$$z = AJ_n(kr) + CY_n(kr). \quad \ldots\ldots \quad (A.56)$$

It follows that the complete solution of the equation

$$\frac{d^2z}{dr^2} + \frac{1}{r}\frac{dz}{dr} - \left(k^2 + \frac{n^2}{r^2}\right)z = 0, \quad \ldots\ldots \quad (A.57)$$

which differs from (A. 2) only in having ik for k (where $i = \sqrt{-1}$), is

$$z = AJ_n(ikr) + BZ_n(ikr). \quad \ldots\ldots \quad (A.58)$$

In order to express the solution in terms of real quantities only we define two more functions thus

$$I_n(x) = i^{-n}J_n(ix)$$

$$= \frac{x^n}{2^n\Gamma(n+1)}\left\{1 + \frac{x^2}{2(2n+2)} + \frac{x^4}{2.4(2n+2)(2n+4)} + \ldots\right\}, \quad (A.59)$$

$$H_n(x) = i^{-n}Z_n(ix) - I_n(x)\log_e i$$

$$= I_n(x)\log x - \frac{1}{2}\sum_{m=0}^{m=n-1}\frac{|\underline{n-m-1}}{|\underline{m}}(-1)^{m-n}(\tfrac{1}{2}x)^{2m-n}$$

$$-\tfrac{1}{2}(\tfrac{1}{2}x)^n\sum_{m=0}^{m=8}\frac{s_m + s_{m+n}}{|\underline{m}\,|\underline{m+n}}(\tfrac{1}{2}x)^{2m}. \quad \ldots\ldots \quad (A.60)$$

Another function which is sometimes used instead of $H_n(x)$ is

$$K_n(x) = (-1)^{n+1}\{H_n(x) - (\log_e 2 - \gamma)I_n(x)\}$$

$$= (-1)^{n+1}\{H_n(x) - \lambda I_n(x)\} \quad \ldots\ldots \quad (A.61)$$

It is now clear that (A. 58) is equivalent to either of the following

$$z = A_1 I_n(x) + B_1 H_n(x); \quad \ldots\ldots \quad (A.62)$$

$$z = A_2 I_n(x) + B_2 K_n(x). \quad \ldots\ldots \quad (A.63)$$

* *The Theory of Bessel Functions* by G. N. Watson (Camb. Univ. Press. 1922).

Very extensive tables of $I_n(x)$ and $K_n(x)$ are given in Watson's *Theory of Bessel Functions.*

The following are particular cases of the functions we have just defined.

$$I_0(x) = 1 + \frac{x^2}{2^2} + \frac{x^4}{2^2.4^2} + \frac{x^6}{2^2.4^2.6^2} + \ldots \quad \text{(A.64)}$$

$$I_1(x) = \frac{x}{2} + \frac{x^3}{2^2.4} + \frac{x^5}{2^2.4^2.6} + \frac{x^7}{2^2.4^2.6^2.8} + \ldots \quad \text{(A.65)}$$

$$K_0(x) = (\lambda - \log x) I_0(x) + \frac{x^2}{2^2} + \frac{s_2 x^4}{2^2.4^2} + \frac{s_3 x^6}{2^2.4^2.6^2} + \ldots \quad \text{(A.66)}$$

$$K_1(x) = -(\lambda - \log x) I_1(x) + \frac{1}{x}$$

$$-\frac{x}{2}\left\{\frac{1}{2} + \frac{x^2}{2^2.4}(s_1 + s_2) + \frac{x^4}{2^2.4^2.6}(s_2 + s_3) + \ldots\right\} \quad \text{(A.67)}$$

Observe that

$$\frac{1}{x}\frac{d}{dx}\{x^{-n} I_n(x)\} = \frac{1}{2^n \Gamma(n+1)}\left\{\frac{1}{2n+2} + \frac{x^2}{2(2n+2)(2n+4)} + \ldots\right\}$$

$$= \frac{1}{2^{n+1}\Gamma(n+2)}\left\{1 + \frac{x^2}{2(2n+4)} + \frac{x^4}{2.4(2n+4)(2n+6)} + \ldots\right\}$$

$$= x^{-n-1} I_{n+1}(x) \quad \text{(A.68)}$$

Also

$$\frac{1}{x}\frac{d}{dx}\{x^{-n} H_n(x)\} = \frac{1}{x}\frac{d}{dx}\{(ix)^{-n} Z_n(ix)\} - \frac{1}{x}\frac{d}{dx}\{x^{-n} I_n(x) \log i\}$$

$$= i^2 \frac{1}{ix}\frac{d}{d(ix)}\{(ix)^{-n} Z_n(ix)\} - x^{-n-1} I_{n+1}(x) \log i$$

$$= -i^2 (ix)^{-n-1} Z_{n+1}(ix) - x^{-n-1} I_{n+1}(x) \log i \quad \text{(A.69)}$$

The last step follows from (A. 47).

Thus we get

$$\frac{1}{x}\frac{d}{dx}\{x^{-n} H_n(x)\} = (ix)^{-n-1} Z_{n+1}(ix) - x^{-n-1} I_{n+1}(x) \log i$$

$$= x^{-n-1} H_{n+1}(x). \quad \text{(A.70)}$$

Thus the functions H_n and H_{n+1} are connected by the same relation as the functions I_n and I_{n+1}.

The recurrence formulae.

By putting $u_n x^{-n}$ instead of $u_n x^n$ in (A.13) we can prove that, if z_n is a Bessel function of order n, then there is a Bessel function z_{n-1}, of order $(n-1)$, which is related to z_n by the equation

$$x^{n-1} z_{n-1} = \frac{1}{x}\frac{d}{dx}(x^n z_n). \quad \text{(A.71)}$$

This suggests a relation between J_{n-1} and J_n. Now by actual differentiation we find that

$$\frac{1}{x}\frac{d}{dx}\{x^n J_n(x)\} = x^{n-1}J_{n-1}(x), \quad \text{(A.72)}$$

and

$$\frac{1}{x}\frac{d}{dx}\{x^n I_n(x)\} = x^{n-1}I_{n-1}(x). \quad \text{(A.73)}$$

After performing the differentiation in (A.26) and multiplying up by x^{n+1} we get

$$-J'_n(x) + \frac{n}{x}J_n(x) = J_{n+1}(x). \quad \text{(A.74)}$$

In like manner we find, from (A.72),

$$J'_n(x) + \frac{n}{x}J_n(x) = J_{n-1}(x). \quad \text{(A.75)}$$

On eliminating $J'_n(x)$ from the last two equations we get

$$\frac{2n}{x}J_n(x) = J_{n-1}(x) + J_{n+1}(x). \quad \text{(A.76)}$$

This is the recurrence formula for Bessel functions of the first kind.

By putting ix for x in (A.76) and then using the definition of $I_n(x)$ given in (A.59) it is easy to prove that

$$\frac{2n}{x}I_n(x) = I_{n-1}(x) - I_{n+1}(x). \quad \text{(A.77)}$$

Asymptotic expansions of Bessel functions.

It is shown in treatises on Bessel functions that these functions can be expanded in asymptotic series, which are given below. The series are

$$J_n(x) = \left(\frac{2}{\pi x}\right)^{\frac{1}{2}}\{P\cos\theta - Q\sin\theta\}, \quad \text{(A.78)}$$

$$Y_n(x) = \left(\frac{2}{\pi x}\right)^{\frac{1}{2}}\{P\sin\theta + Q\cos\theta\}; \quad \text{(A.79)}$$

where

$$\theta = x - \tfrac{1}{2}n\pi - \tfrac{1}{4}\pi, \quad \text{(A.80)}$$

$$P = 1 - \frac{(4n^2-1^2)(4n^2-3^2)}{\lfloor 2\,(8x)^2} + \ldots, \quad \text{(A.81)}$$

$$Q = \frac{4n^2-1^2}{\lfloor 1\,8x} - \frac{(4n^2-1^2)(4n^2-3^2)(4n^2-5^2)}{\lfloor 3\,(8x)^3} + \ldots \quad \text{(A.82)}$$

Also

$$I_n(x) = \frac{e^x}{\sqrt{2\pi x}}(P_1 - Q_1), \quad \text{(A.83)}$$

$$\mathrm{K}_n(x)=\left(\frac{\pi}{2x}\right)^{\frac{1}{2}}e^{-x}(\mathrm{P}_1+\mathrm{Q}_1); \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{A}.84)$$

where

$$\mathrm{P}_1=1+\frac{(4n^2-1^2)(4n^2-3^2)}{\lfloor 2\,(8x)^2}+\ldots, \quad . \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{A}.85)$$

$$\mathrm{Q}_1=\frac{4n^2-1^2}{\lfloor 1\,(8x)}+\frac{(4n^2-1^2)(4n^2-3^2)(4n^2-5^2)}{\lfloor 3\,(8x)^3}+\ldots \quad (\mathrm{A}.86)$$

For only moderately large values of x each of the first few terms in the series for P, Q, P_1, Q_1, is much smaller than the preceding one, and a term is soon reached which is very small in comparison with unity. The series ultimately diverge for all values of x, but it can be shown that the actual error in any one of these quantities due to summing as far as any particular term in the series is of the same order of magnitude as the next term in the series.

It is not difficult to show that the asymptotic forms are solutions of the differential equation for Bessel functions; but it is much more difficult to show that these forms are identical with $\mathrm{J}_n(x)$, $\mathrm{Y}_n(x)$, $\mathrm{I}_n(x)$, and $\mathrm{K}_n(x)$. We shall content ourselves with showing the general character of the functions.

When (A. 1) is multiplied by $x^{\frac{1}{2}}$ that equation becomes

$$x^{\frac{1}{2}}\frac{d^2z}{dx^2}+x^{-\frac{1}{2}}\frac{dz}{dx}+\left(1-\frac{n^2}{x^2}\right)zx^{\frac{1}{2}}=0; \quad . \quad . \quad . \quad (\mathrm{A}.87)$$

and if u is written for $zx^{\frac{1}{2}}$ this equation becomes

$$\frac{d^2u}{dx^2}=-\left(1-\frac{n^2-\frac{1}{4}}{x^2}\right)u. \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{A}.88)$$

When $(n^2-\frac{1}{4})$ is greater than x^2 we see that $\frac{d^2u}{dx^2}$ has the same sign as u. Therefore the curve whose ordinate is u and abscissa x is convex to the x-axis, between $x=0$ and $x=\sqrt{n^2-\frac{1}{4}}$. Where x has the latter value the curvature changes sign, and for larger values of x the curve is concave towards the x-axis, just like a sine or cosine curve. Moreover, when x^2 is very big in comparison with $(n^2-\frac{1}{4})$ then

$$\frac{d^2u}{dx^2}=-u \text{ approximately.}$$

The solution of this is

$$u=\mathrm{A}\cos x+\mathrm{B}\sin x, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{A}.89)$$

whence

$$z=x^{-\frac{1}{2}}\{\mathrm{A}\cos x+\mathrm{B}\sin x\}. \quad . \quad . \quad . \quad . \quad (\mathrm{A}.90)$$

Thus we see that both $x^{\frac{1}{2}}\,\mathrm{J}_n(x)$ and $x^{\frac{1}{2}}\,\mathrm{Y}_n(x)$ are oscillatory functions of x somewhere beyond where $x=\sqrt{n^2-\frac{1}{4}}$; but neither function can vanish more than once from $x=0$ to $x=\sqrt{n^2-\frac{1}{4}}$. This latter result follows from the fact that the curves for the functions are convex to the x-axis in this region. $\mathrm{J}_n(x)$ vanishes when $x=0$, and consequently its next zero must be greater than $\sqrt{n^2-\frac{1}{4}}$. Actually the first zero after $x=0$ is somewhere in the neighbourhood of $(\frac{1}{2}n+\frac{3}{4})\pi$.

Instead of (A.89) we might have taken

$$u=\mathrm{A}e^{ix}, \qquad \ldots\ldots\ldots \qquad (\mathrm{A}.94)$$

and by starting with this as a first approximation to the solution of (A.88) the asymptotic series in (A.78) and (A.79) can be derived. While this method shows that the asymptotic series are solutions of the differential equation it does not, of course, show the connection between these series and the functions $\mathrm{J}_n(x)$ and $\mathrm{Y}_n(x)$ which have previously been defined. It is enough for our purpose, however, to have shown the periodic character of the functions

Roots of the equation

$J_n(x)=0$

	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$
1	2·405	3·832	5·135	6·379	7·586	8·780
2	5·520	7·016	8·147	9·760	11·064	12·339
3	8·654	10·173	11·620	13·017	14·373	15·700
4	11·792	13·323	14·796	16·224	17·616	18·982
5	14·931	16·470	17·960	19·410	20·827	22·220
6	18·071	19·616	21·117	22·583	24·018	25·431
7	21·212	22·760	24·270	25·749	27·200	28·628
8	24·353	25·903	27·421	28·909	30·371	31·813
9	27·494	29·047	30·571	32·050	33·512	34·983

APPENDIX B.

Solution of equation (18.227) when $2n = l$.

Equation (18.227) can be written in the form

$$\left(\eta^2\frac{d^2}{d\eta^2}+\eta\frac{d}{d\eta}-n^2\right)\left\{\eta^{-l}\left(\eta^2\frac{d^2z}{d\eta^2}+\eta\frac{dz}{d\eta}-n^2z\right)\right\}=\mu\eta^{2-\beta}z \quad \text{(B.1)}$$

l being written for $(2+3\beta)$.

Now let a new variable y be taken such that

$$e^y = \eta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(B.2)}$$

Then the equation for z becomes

$$(D^2 - n^2)\left\{e^{-ly}(D^2 - n^2)z\right\} = \mu z e^{(2-\beta)} \quad . \quad . \quad . \quad . \quad \text{(B.3)}$$

D^2 being written for $\dfrac{d^2}{dy^2}\cdot$

Now a property of the operator D is expressed by the equation

$$D^k(e^{-ly}u) = e^{-ly}(D-l)^k u \quad . \quad . \quad . \quad . \quad . \quad \text{(B.4)}$$

Therefore (B. 3) becomes

$$e^{-ly}\left\{(D-l)^2 - n^2\right\}(D^2 - n^2)z = \mu z e^{(2-\beta)y}\ , \quad . \quad . \quad \text{(B.5)}$$

whence we get, on multiplying by e^{ly} and writing m for $(4+2\beta)$,

$$\left\{(D-l)^2 - n^2\right\}(D^2 - n^2)z = \mu z e^{my}\ , \quad . \quad . \quad . \quad . \quad \text{(B.6)}$$

or

$$(D-l-n)(D-l+n)(D-n)(D+n)z = \mu z e^{my}\ . \quad . \quad \text{(B.7)}$$

Putting $2n$ for l in this we get

$$(D-3n)(D-n)^2(D+n)z = \mu z e^{my} \quad . \quad . \quad . \quad . \quad \text{(B.8)}$$

Now the direct method of solution in a series of powers of η used in Chapter 18 gives three series beginning with η^n, η^{-n}, and η^{3n}. The indices of these three powers of η are n, $-n$, and $(l+n)$, and are identical with the first three given in (18.231). The present case is troublesome because the last of the four indices, namely $(l-n)$, happens to be equal to one of the others. Our present problem is to get the fourth solution which the method of integration by series fails to give.

Now let

$$z = v_1 + v_2 + v_3 + \dots \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (B.9)$$

Then (B. 8) is satisfied if

$$(D-3n)(D-n)^2(D+n)v_1 = 0, \quad . \quad . \quad . \quad . \quad . \quad (B.10)$$

$$(D-3n)(D-n)^2(D+n)v_2 = \mu v_1 e^{my}, \quad . \quad . \quad . \quad (B.11)$$

and, in general,

$$(D-3n)(D-n)^2(D+n)v_q = \mu v_{q-1} e^{my}, \quad . \quad . \quad (B.12)$$

provided that the series $(v_1 + v_2 + v_3 + \dots)$ is convergent.

Now one solution of (B. 10) is

$$v_1 = Aye^{ny} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (B.13)$$

This value of v_1 leads to the solution that the method in Chapter 18 failed to give.

From (B. 12) we get

$$v_q = (D-3n)^{-1}(D+n)^{-1}(D-n)^{-2}(\mu v_{q-1} e^{my}) . \quad . \quad (B.14)$$

Thus

$$v_2 = \mu A (D-3n)^{-1}(D+n)^{-1}(D-n)^{-2} ye^{(m+n)y}$$
$$= \mu A e^{(m+n)y}(D+m-2n)^{-1}(D+m+2n)^{-1}(D+m)^{-2}y \quad (B.15)$$

By expanding the operators on the right hand side of the last equation in ascending powers of D and neglecting all powers beyond the first because $D^2y = 0$, we get

$$v_2 = \mu A e^{(m+n)y}\frac{y-c_2}{m^2(m-2n)(m+2n)}, \quad . \quad . \quad . \quad (B.16)$$

where

$$c_2 = \frac{1}{m-2n} + \frac{1}{m+2n} + \frac{2}{m} . \quad . \quad . \quad . \quad . \quad (B.17)$$

Likewise

$$v_3 = (D-3n)^{-1}(D+n)^{-1}(D-n)^{-2}\mu v_2 e^{my}$$
$$= \frac{\mu^2 A e^{(2m+n)y}}{m^2(m-2n)(m+2n)}(D+2m-2n)^{-1}(D+2m+2n)^{-1}(D+2m)^{-2}(y-c_2)$$
$$= \frac{\mu^2 A e^{(2m+n)y}(y-c_3)}{m^2(2m)^2(m-2n)(2m-2n)(m+2n)(2m+2n)}, \quad . \quad . \quad . \quad . \quad . \quad (B.18)$$

where

$$c_3 = \frac{1}{2m-2n} + \frac{1}{2m+2n} + \frac{2}{2m} + c_2 . \quad . \quad . \quad . \quad . \quad (B.19)$$

In the same way we find

$$v_4 = \frac{\mu^3 A e^{(3m+n)y}(y-c_4)}{m^2(2m)^2(3m)^2(m-2n)(2m-2n)(3m-2n)(m+2n)(2m+2n)(3m+2n)} \quad (B.20)$$

where

$$c_4 = \frac{1}{3m-2n} + \frac{1}{3m+2n} + \frac{2}{3m} + c_3. \quad . \quad . \quad . \quad (\text{B}.21)$$

Thus the solution we are seeking is

$$z = \text{A}\eta^n \log\eta \sum_{q=0}^{q=\infty} \frac{\eta^{qm}}{\text{F}_q \text{G}_q \text{H}_q} - \text{B}\eta^n \sum_{q=1}^{q=\infty} \frac{\text{C}^q \eta^{qm}}{\text{F}_q \text{G}_q \text{H}_q}, \quad . \quad . \quad (\text{B}.22)$$

where

$$\text{F}_q = \frac{1}{m^2(2m)^2 \ldots\ldots (qm)^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{B}.23)$$

$$\text{G}_q = \frac{1}{(m-2n)(2m-2n)\ldots\ldots(qm-2n)}, \quad . \quad . \quad . \quad (\text{B}.24)$$

$$\text{H}_q = \frac{1}{(m+2n)(2m+2n)\ldots\ldots(qm+2n)}, \quad . \quad . \quad . \quad (\text{B}.25)$$

$$\begin{aligned} \text{C}_q = {} & \frac{2}{m} + \frac{2}{2m} + \frac{2}{3m} + \ldots\ldots + \frac{2}{qm} \\ & + \frac{1}{m-2n} + \frac{1}{2m-2n} + \ldots\ldots + \frac{1}{qm-2n} \\ & + \frac{1}{m+2n} + \frac{1}{2m+2n} + \ldots\ldots + \frac{1}{qm+2n}. \quad . \quad (\text{B}.26) \end{aligned}$$

It should be noticed that the coefficient of $\log\eta$ in (B. 22) is itself the solution of the differential equation (B. 1) which begins with η^n. This particular solution could have been got by the present method if we had taken $v_1 = \text{A}$ as the solution of (B. 10).

APPENDIX C.

To solve the differential equation

$$\frac{d^2M}{dx^2} + n^2M = f(x), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (C.1)$$

which occurs on page 96, and again with different symbols on page 289.

When D is written for $\frac{d}{dx}$ equation (C. 1) becomes

$$(D^2 + n^2)\,M = f(x), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (C.2)$$

that is,

$$(D + in)\left\{(D - in)\,M\right\} = f(x) \quad . \quad . \quad , \quad . \quad . \quad (C.3)$$

Multiplying through by e^{inx} we get

$$e^{inx}(D + in)\left\{(D - in)\,M\right\} = e^{inx}f(x), \quad . \quad . \quad . \quad . \quad (C.4)$$

which becomes, by the rule in (B. 4),

$$D\left\{e^{inx}(D - in)\,M\right\} = e^{inx}f(x) \quad . \quad . \quad . \quad . \quad . \quad (C.5)$$

Integrating both sides with respect to x we get

$$e^{inx}(D - in)\,M = \int^x e^{inx}f(x)\,dx + H \quad . \quad . \quad . \quad (C.6)$$

The lower limit of the integral on the right hand side of the last equation can be any constant we choose. We could make it zero. Since there is already an indefinite constant H in the equation this indefiniteness of the lower limit does not make any difference.

The result will be the same if we write u for x under the integral sign provided only that we retain x as the upper limit. Thus we may write

$$e^{inx}(D - in)\,M = \int_0^x e^{inu}f(u)\,du + H, \ldots \quad . \quad . \quad . \quad (C.7)$$

whence

$$(D - in)\,M = e^{-inx}\int_0^x e^{inu}f(u)\,du + He^{-inx}$$

$$= \int_0^x e^{-in(x-u)}f(u)\,du + He^{-inx}, \quad . \quad . \quad (C.8)$$

Likewise, by reversing the factors $(\mathrm{D}+in)$ and $(\mathrm{D}-in)$ in (C. 3), we can get

$$(\mathrm{D}+in)\mathrm{M}=\int_0^x e^{in(x-u)}f(u)\,du+\mathrm{K}e^{inx}, \quad . \quad . \quad . \quad (\mathrm{C.9})$$

By subtraction we get, from (C. 8) and (C. 9),

$$2in\mathrm{M}=\int_0^x \left\{e^{in(x-u)}-e^{-in(x-u)}\right\}f(u)\,du+\mathrm{K}e^{inx}-\mathrm{H}e^{-inx}$$

$$=\int_0^x 2i\sin n(x-u)f(u)\,du+\mathrm{K}e^{inx}-\mathrm{H}e^{-inx} \quad . \quad . \quad (\mathrm{C.10})$$

Therefore

$$\mathrm{M}=\frac{1}{n}\int_0^x \sin n(x-u)f(u)\,du+\frac{1}{2in}\left\{\mathrm{K}e^{inx}-\mathrm{H}e^{-inx}\right\},$$

which can be written in the form

$$\mathrm{M}=\frac{1}{n}\int_0^x \sin n(x-u)f(u)\,du+\mathrm{A}\cos nx+\mathrm{B}\sin nx \quad . \quad (\mathrm{C.11})$$

If one of the terms in $f(u)$ is a constant G the corresponding term in the expression for M is

$$\frac{1}{n}\int_0^x \mathrm{G}\sin n(x-u)\,du=\frac{\mathrm{G}}{n^2}-\frac{\mathrm{G}}{n^2}\cos nx.$$

The only term in this that is not already in (C. 11) is $\frac{\mathrm{G}}{n^2}$, for the other term merely combines with $\mathrm{A}\cos nx$.

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