PLASTIC SOIL MECHANICS THEORIES

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1. <u>Introduction</u>. Although civil engineers have been interested in soil mechanics for ages, only recently has it begun to be feasible to apply to it the techniques of modern plasticity theories. There is little experimental evidence as to which theory to choose. On the other hand there is still relatively little theoretical guidance as to fruitful lines of experimental attack. This report is an attempt to bring the theoretical and experimental forces somewhat nearer common ground.

To be mathematically tractable a theory of soil mechanics must drastically simplify true soil structure. Only a few important soil mechanisms may be considered. We will adopt rather naive models of these mechanisms, for simplicity is at a premium and high accuracy is not.

The basic soil structure is its <u>skeleton</u> of solid sand or silt grains, usually of quartz. They are elastic in nature. Although they are crystalline, their crystallographic axes are usually randomly oriented so that the skeleton as a whole may be regarded as isotropic.*

The interstices between the particles are filled with water, air, or both. The water contains flocculated colloidal particles much smaller than the sand or silt grains, but appearing more prominently in clays than in sands. They are subject to electro-chemical forces tending to bond together the grains of the skeleton.

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* In some alluvial deposits stream action has caused the deposit of non-randomly oriented highly flattened particles. The present theory does not apply to such cases.

A similar bonding is caused by surface tension in the water when air is present. We will account for this bonding by introducing a _cohesive pressure_ p_c, constant or at most weakly dependent on the plastic work W. The energy loss resulting from the breaking of these bonds and the rubbing together of the skeletal particles as the soil deforms will be accounted for by a mechanism of Coulomb friction.

The soil skeleton will transmit a stress τ_{ij}. In addition the water will transmit a _pore pressure_ u, dependent primarily on the strain ϵ. In many cases the cohesive pressure and the pore pressure can be lumped together into the _intrinsic pressure_

$p = u - p_c$.

A soil may be considered incompressible only if it is saturated with water and contains no air and if the water is not free to flow out.

In the first 13 sections several different plastic soil mechanics theories are developed. In section 14, the general question of testing these theories is discussed, and specialized in section 15 to the ordinary triaxial type of testing situation. In section 16 the theory of section 15 is extended for the perfect plasticity case, and a numerical example is given in section 17.

Rather accurate experimental work is required in testing the theories, as one might expect. If the experiments suggested are feasible they should definitely throw light upon the internal mechanism of soil plasticity.

NOTATION

The symbols are defined in or near the formulas given

A	(50)	t	time	σ_1	(23)
B	(50)	U	(22)	$\sigma_r,\ \sigma_\theta,\ \sigma_z$	(40)
C	(50)	u	(1)	σ_{ij}	(1)
c	(10)	u_i	(3)	$\tau_m,\ \tau_\mu,\ \tau_M$	(7)
D	(50)	$V(u)$	(22)	τ_{ij}	(1)
E	(6)	W	(18)	ϕ	(7)
E_o, E_1, E_2	(25)	x	(15)		
F	(50)	Y	(49)		
F_i	(2)	y	(15)		
F_1, F_2, F_3	(26)	α	(6)		
$f(x,y,z)$	(27)	β	(6)		
G	(50)	γ	(48)		
H	(50)	δ	(43)		
h	(21)	δ_{ij}	(1)		
I	(50)	ϵ	(20)		
J	(50)	$\epsilon_1, \epsilon_2, \epsilon_3$	(29)		
J_1, J_2, J_3	(12)	$\epsilon_r, \epsilon_\theta, \epsilon_z$	(40)		
K	(50)	ϵ_{ij}	(3)		
k	(42)	$\overset{\bullet}{\epsilon}_{ij}, \overset{\phi}{\epsilon}_{ij}$	(4)		
p	(1)	η	(14)		
p_c	(1)	κ	(13)		
$q(\epsilon)$	(39)	λ	(6)		
s	(10)	μ	(6)		
s_1	(23)	ν	(6)		
s_{ij}	(13)	σ	(37)		

2. The total stress tensor σ_{ij} and the intrinsic pressure p are defined in terms of the skeletal stress tensor τ_{ij}, the cohesive pressure p_c, and the pore pressure u by

$$\sigma_{ij} = \tau_{ij} - p\,\delta_{ij}, \qquad p = u - p_c \qquad (1)$$

where δ_{ij} is the Kronecker delta: $\delta_{ii} = 1$, $\delta_{ij} = 0$ for $i \neq j$.

The usual mechanical equations of equilibrium of plasticity and elasticity are

$$\frac{\partial \sigma_{i\alpha}}{\partial x_\alpha} + F_i = 0 \qquad (2)$$

where the summation convention is used, and where F_i is the rectangular x_i-component of external force per unit volume.

The total strain tensor ϵ_{ij} is defined by

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \qquad (3)$$

where u_i is the component of displacement in x_i-direction. This may be resolved into an elastic strain tensor ϵ_{ij}^e and a plastic strain tensor ϵ_{ij} such that

$$\epsilon_{ij} = \epsilon_{ij}^e + \epsilon_{ij}^p \qquad (4)$$

By [1; §§ 67, 69]* the elastic part is given by

$$d\epsilon_{ij}^e = \alpha\, d\sigma_{ij} - \beta\,\delta_{ij} d\sigma_{kk}, \qquad (5)$$

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* Numbers in brackets refer to the references at the end of the report.

where

$$\alpha = 1/(2\mu) = (1 + \nu)/E, \quad \beta = \lambda/[2\mu(3\lambda + 2\mu)] = \nu/E, \quad (6)$$

where λ, μ are Lame constants, E is Young's modulus, and ν is Poisson's ratio.

3. <u>Yield conditions</u>. Yield conditions for soils have usually been based on the notion of Coulomb friction, as mentioned in § 2, beginning with the work of Coulomb himself [2], and have been very extensively tested. At any given point in a soil the skeletal stresses acting on an arbitrary plane through that point may be obtained by transformation from the stress components referred to fixed axes. The ratio of shearing stress to normal stress is maximized by varying the direction of this plane. According to the Coulomb friction idea, this maximum must remain below a certain constant, $\tan\phi$, ϕ being the <u>angle of shearing resistance</u>. It is a straightforward calculus problem to prove that

$$\frac{\tau_M}{\tau_m} \geq \frac{1-\sin\phi}{1+\sin\phi} \;, \qquad (7)$$

where τ_m, τ_μ, τ_M are the three principal skeletal stresses, so arranged that

$$\tau_m \leq \tau_\mu \leq \tau_M \leq 0 \qquad (8)$$

the convention being that compressive stresses are negative.

(7) may be written in the form

$$s \leq c + \sigma' \tan\phi \;, \qquad (9)$$

where

$$s = \tfrac{1}{2}(\tau_M - \tau_m), \quad \sigma' = -p_c - \tfrac{1}{2}(\tau_M + \tau_m), \quad c = p_c \tan\phi. \qquad (10)$$

(9) is the standard form of the yield condition as used in engineering soil mechanics. [3; p. 22, formula [6]]. c is called the <u>cohesion</u> of the material.

The yield surface corresponding to (7) is a six-sided pyramid.

(7) is simple and easy to apply when the principal directions are known from symmetry or otherwise. In this respect it is similar to the Tresca condition. Like the Tresca condition it is rather complicated when expressed in invariant form. Then the yield condition is

$$\varkappa [(2\varkappa + 3) J_3 - \tfrac{2}{3}(\varkappa + 1) J_1 J_2 + \tfrac{2}{27} \varkappa J_1^3]^2 \tag{11}$$

$$+ (3\varkappa + 4)(J_3^2 - \tfrac{4}{27} J_2^3) = 0,$$

where

$$J_1 = \tau_{kk} = \tau_M + \tau_\mu + \tau_m,$$

$$J_2 = \tfrac{1}{2} \tau_{ij} \tau_{ij} - \tfrac{1}{6} J_1^2 = \tfrac{1}{2} s_{ij} s_{ij}$$

$$= \tfrac{1}{6}[(\tau_M - \tau_m)^2 + (\tau_M - \tau_\mu)^2 + (\tau_\mu - \tau_m)^2], \tag{12}$$

$$J_3 = |\tau_{ij}| + \tfrac{1}{3} J_1 J_2 - \tfrac{1}{27} J_1^3 = |s_{ij}|$$

$$= (\tau_M - \tfrac{1}{3} J_1)(\tau_\mu - \tfrac{1}{3} J_1)(\tau_m - \tfrac{1}{3} J_1),$$

$$s_{ij} = \tau_{ij} - \tfrac{1}{3} J_1 \delta_{ij}, \qquad \varkappa = \tfrac{4}{3} \tan^2 \phi. \tag{13}$$

Drucker and Prager [4; p. 153] (see also [5; p. 447]) have suggested a simpler yield condition equivalent to

$$J_2^{1/2} + 3^{-1/2} \eta J_1 = 0 \tag{14}$$

for η constant. This has a conical yield surface.

The homogeneity of conditions (11) and (14) makes a two-dimensional graphical representation possible. Define

$$x = - 3^{1/2} J_2^{1/2}/J_1, \quad y = (3^{3/2}/2) J_2^{-3/2} J_3. \tag{15}$$

Then (11) becomes

$$\varkappa[(2\varkappa + 3)y + 3(\varkappa + 1)/x - \varkappa/x^3]^2 + (3\varkappa + 4)(y^2 - 1) = 0,$$

which may be solved in the form

$$y = - \frac{3\varkappa(2\varkappa + 3)}{4(\varkappa + 1)^2} \left[x^{-1} - \frac{\varkappa x^{-3}}{3(\varkappa + 1)} \right.$$

$$\left. \pm \frac{\sqrt{(3\varkappa + 4)}}{2(\varkappa + 1)^{3/2}} \left[1 - \frac{\varkappa x^{-2}}{\varkappa + 1} \right] \sqrt{\left[x - \frac{\varkappa}{4(\varkappa + 1)} \right]}. \right. \tag{16}$$

(14) becomes

$$x = \eta \tag{17}$$

The point (x,y) as defined by (15) can cover only a finite region of the plane. By a straightforward maximum-minimum procedure, which will be omitted here, it may be shown that

$$x \geq 0, \quad -1 \leq y \leq 1, \quad y \leq \frac{1}{2}x^{-3} - \frac{3}{2}x^{-1}.$$

The region to the left of the curves (16) or (17) is elastic, and the region to the right is forbidden. During plastic deformation the point (x,y) lies on one or the other of these curves. In Fig. 1, equation (16) is plotted for the values of $\varkappa$ indicated beside each curve near the bottom. These curves are quite similar to those for (17) for small $\varkappa$, corresponding to slight soil compaction, but the difference becomes progressively more

marked as $\varkappa$ and the compaction increase.

4. <u>Strain hardening</u>. The angle of shearing resistance is fairly constant over moderate ranges of stresses provided the soil is somewhat compacted to begin with. However if the soil is initially very loose, or if the stress becomes high enough to drive water out of the soil permanently, this angle may change. This phenomenon is referred to as strain hardening. Following the custom in metal plasticity we will regard this as a function of plastic work W, where

$$dW = \sigma_{ij} de^p_{ij}. \tag{18}$$

When strain hardening is ignored so that ϕ is considered constant, the resulting theory is referred to as <u>perfect plasticity</u>.

5. <u>The pore pressure</u>. If possible the water in the soil flows in such a way as to relieve the pore pressure at the expense of the skeleton. However the inertia and viscosity of the water oppose this. Similar forces act on the air, but their magnitude is much smaller so that may usually be neglected.

By the law of Darcy,

$$\underset{\sim}{v} \propto \nabla u,$$

$\underset{\sim}{v}$ being the velocity of flow of water in the soil. Now $\nabla \cdot \underset{\sim}{v}$ represents the outflow of water per unit time from a unit volume of soil, so the increase per unit time in the space occupied by the fluid per unit volume is proportional to $\nabla^2 u$. Two factors contribute to this increase in the region occupied by the water. One is an increase dV in the volume of the trapped air, thus

forcing the water outward. The second is a decrease equal to
$-d\epsilon = -d\epsilon_{kk}$ in the pore space within the skeleton. This decrease
forces the water outward into a larger volume. Therefore we have

$$\nabla^2 u \propto \partial V/\partial t - \partial \epsilon/\partial t \qquad (19)$$

where

$$\epsilon = \epsilon_{kk} \qquad (20)$$

In saturated soil $V = 0$. In unsaturated soil $V \geq 0$,
but may be taken to depend on the pore pressure u. Therefore

$$h^2 \nabla^2 u - \partial U/\partial t = 0, \qquad (21)$$

where

$$U = V(u) - \epsilon, \quad V'(u) < 0 \qquad (22)$$

and where h is essentially constant, but may tend to drop off
toward zero for very high frequency oscillations. In this case
it might be considered a function of $\underline{V}$ or ∇u.

6. The plastic strain increment. Drucker [6] has shown
that there exists a plastic potential which is, in fact, the yield
function, if the plastic overwork due to an external agency acting
on a prestressed material is always positive. While this is
usually the case, especially in metal plasticity, in the case of
the yield condition (7) it may not be, for an external agency may
have a "trigger" action in which it releases plastic deformation
while gaining energy at the expense of the initial stressing agency.

We will assume the medium to be isotropic, which seems justi-
fied if the soil deformations are not too large. We will also
assume the plastic strain increments de_{ij}^p to be proportional

to the plastic work increment dW defined in (18), and to depend
on the stress components σ_{ij}, and possibly also on u and p_c.

Let σ_1, σ_2, σ_3 denote the principal stress components
of the stress tensor, and let

$$s_i = \sigma_i - \sigma_{\alpha\alpha} \tag{23}$$

denote the <u>principal stress deviations</u>. Then, letting $T_{ij} = \sigma_{ij}$
in the Appendix, the most general expression for the plastic
strain increments is

$$de_{ij}^p = (E_0 \delta_{ij} + E_1 s_{ij} + E_2 s_{ik} s_{kj}) \, dW, \tag{24}$$

where

$$E_0 = s_2 s_3 F_1 + s_3 s_1 F_2 + s_1 s_2 F_3,$$

$$E_1 = s_1 F_1 + s_2 F_2 + s_3 F_3, \tag{25}$$

$$E_2 = F_1 + F_2 + F_3,$$

where

$$F_1 = f(\sigma_1, \sigma_2, \sigma_3)/[(s_1 - s_2)(s_1 - s_3)],$$

$$F_2 = f(\sigma_2, \sigma_3, \sigma_1)/[(s_2 - s_3)(s_2 - s_1)], \tag{26}$$

$$F_3 = f(\sigma_3, \sigma_1, \sigma_2)/[(s_3 - s_1)(s_3 - s_2)],$$

and where $f(x,y,z)$ is arbitrary except that

$$f(x,y,z) = f(x,z,y), \tag{27}$$

and except for equation (28) below.

If we multiply (24) by σ_{ij} and sum, by (18)

$$(J_1 - 3p)E_0 + 2J_2 E_1 + (3J_3 + \tfrac{2}{3}J_1 J_2 - 2p J_2)E_2 = 1$$

Inserting (25), after some reduction,

$$xf(x,y,z) + yf(y,z,x) + zf(z,x,y) = 1. \qquad (28)$$

The principal axes for the stress are also principal axes for the strain increments. Denote principal strain increments by de_1^p, de_2^p, de_3^p. Then by (24)-(26),

$$de_1^p = f(\sigma_1, \sigma_2, \sigma_3)dW, \quad de_2^p = f(\sigma_2, \sigma_3, \sigma_1)dW,$$

$$(29)$$

$$de_3^p = f(\sigma_3, \sigma_1, \sigma_2)dW.$$

7. **The Reuss equations.** The Reuss equations in the plastic theory of metals are derived using the invariant $_2$ as a plastic potential. For these equations

$$f(\sigma_1, \sigma_2, \sigma_3) = s_1/(2J_2) \qquad (30)$$

8. **The Tresca theory.** When the Tresca yield condition is used in conjunction with a plastic potential,

$$f(\sigma_1, \sigma_2, \sigma_3) = [\text{sgn}(\sigma_1 - \sigma_2) + \text{sgn}(\sigma_1 - \sigma_3)]/[(\sigma_1 - \sigma_2)\text{sgn}(\sigma_1 - \sigma_2)$$

$$(31)$$

$$+ (\sigma_2 - \sigma_3)\text{sgn}(\sigma_2 - \sigma_3) + (\sigma_3 - \sigma_1)\text{sgn}(\sigma_3 - \sigma_1)].$$

9. **The Drucker-Prager soil theory.** In this theory [4], the yield condition (14) is used in conjunction with a plastic potential. For it,

$$f(\sigma_1, \sigma_2, \sigma_3) = [3^{-1/2}\eta + s_1/(2J_2^{1/2})]/[3^{-1/2}\eta(J_1 - 3p) + J_2^{1/2}] \qquad (32)$$

10. The first soil theory. In this paper we will consider several soil theories. In the first of these, the yield condition (7) is used in conjunction with a plastic potential. This leads to

$$f(\sigma_1, \sigma_2, \sigma_3) = [\mathrm{sgn}(\sigma_1 - \sigma_2) + \mathrm{sgn}(\sigma_1 - \sigma_3) + 2 \sin\phi$$

$$- \sin\phi \,\big|\, \mathrm{sgn}(\sigma_1 - \sigma_2) - \mathrm{sgn}(\sigma_1 - \sigma_3)\big|\,]/D,$$

$$D = (\sigma_1 - \sigma_2)\mathrm{sgn}(\sigma_1 - \sigma_2) + (\sigma_2 - \sigma_3)\mathrm{sgn}(\sigma_2 - \sigma_3) +$$

$$+ (\sigma_3 - \sigma_1)\,\mathrm{sgn}(\sigma_3 - \sigma_1) + 2\sin\phi\,(J_1 - 3p) \tag{33}$$

$$- \sin\phi\,[\,\sigma_1|\mathrm{sgn}(\sigma_1 - \sigma_2) - \mathrm{sgn}(\sigma_1 - \sigma_3)| + \sigma_2|\mathrm{sgn}(\sigma_2 - \sigma_3)$$

$$- \mathrm{sgn}(\sigma_2 - \sigma_1)| + \sigma_3|\,\mathrm{sgn}(\sigma_3 - \sigma_1) - \mathrm{sgn}(\sigma_3 - \sigma_2)|\,].$$

11. The second soil theory. If one assumes that the soil shears along the planes of maximum ratio of shearing stress to normal stress when the yield condition (7) holds, one is led to

$$f(\sigma_1, \sigma_2, \sigma_3) = \sqrt{(-\sigma_1 - p)}\,[\mathrm{sgn}(\sigma_1 - \sigma_2) + \mathrm{sgn}(\sigma_1 - \sigma_3)]/D,$$

$$D = \sigma_1\sqrt{(-\sigma_1 - p)}\,[\mathrm{sgn}(\sigma_1 - \sigma_2) + \mathrm{sgn}(\sigma_1 - \sigma_3)]$$

$$\tag{34}$$

$$+ \sigma_2\sqrt{(-\sigma_2 - p)}\,[\mathrm{sgn}(\sigma_2 - \sigma_3) + \mathrm{sgn}(\sigma_2 - \sigma_1)]$$

$$+ \sigma_3\sqrt{(-\sigma_3 - p)}\,[\mathrm{sgn}(\sigma_3 - \sigma_1) + \mathrm{sgn}(\sigma_3 - \sigma_2)].$$

12. The third soil theory. In another possible soil theory the yield condition (7) is used together with $f(\sigma_1, \sigma_2, \sigma_3)$ as defined in (30).

13. **The fourth soil theory.** In this theory, in analogy with the third soil theory, one uses the yield condition (7) together with $f(\sigma_1, \sigma_2, \sigma_3)$ as defined in (31).

14. **Testing the theories.** Beyond this point experimental evidence is required for further progress. It is essential to test the various theories suggested or others. It is necessary to determine the form of V as a function of u in (22), and the form of ϕ as a function of the plastic work W. In addition, the elastic constants must be determined for any given sample. It is desirable to isolate the unknowns in different testing programs if possible.

The problem of determining the elastic constants can be isolated. Since

$$\sigma_{ij}d\epsilon_{ij}^e = \sigma_{ij}d\epsilon_{ij} - dW,$$

by (5),

$$\alpha \Delta(\tfrac{1}{2}\sigma_{ij}\sigma_{ij}) - \beta \Delta(\tfrac{1}{2}\sigma_{ii}\sigma_{jj}) = \int \sigma_{ij}d\epsilon_{ij} - \Delta W \qquad (35)$$

The constants α and β can be determined if the stresses and strains and the generation of plastic energy can be measured in the process of a deformation. The plastic energy is converted into heat and could presumably be measured by measuring the temperature of the sample. The expressions in (35) represent lines in the α, β-plane. Experimental errors will probably prevent a common intersection of these lines, but it will probably prove easy to select by eye a point in the α, β-plane suitably near all the lines. A more sophisticated least square or linear regression procedure could be employed if desired.

The experimental quantities in equation (35) are integrated expressions, so exceptional accuracy will probably not be required in their measurement. However the determination of the functional natures of V and ϕ must use point-to-point data, so higher accuracy will be required for them. It may be desirable to devise tests not requiring the measurement of W, since this will certainly be difficult to do with high accuracy.

To avoid complications the test set-up should have symmetries of such a nature that the principal directions are known. Then by (4), (5), (29),

$$d(\epsilon_1 - \alpha\sigma_1 - \beta\sigma)/f(\sigma_1, \sigma_2, \sigma_3) =$$

$$= d(\epsilon_2 - \alpha\sigma_2 - \beta\sigma)/f(\sigma_2, \sigma_3, \sigma_1) = \tag{36}$$

$$= d(\epsilon_3 - \alpha\sigma_3 - \beta\sigma)/f(\sigma_3, \sigma_1, \sigma_2) = dW$$

where $\sigma_1, \sigma_2, \sigma_3$ are principal stresses and

$$\sigma = \sigma_1 + \sigma_2 + \sigma_3. \tag{37}$$

For the theories considered here in §§ 9-13, when common factors are cancelled, the two equations on the left of (36) have denominators that are functions of τ_M / τ_m and τ_μ / τ_m only. Eliminating the latter ratio between the two equations we may solve for τ_M / τ_m . By (7) this gives a point-by point expression for ϕ .

τ_M / τ_m being known, p can be determined from (1). In fact

$$p = (\sigma_m \tau_M / \tau_m - \sigma_M)/(1 - \tau_M / \tau_m) \tag{38}$$

gives a point-by-point determination of p.

15. Static compression of a cylindrical soil sample. We
will specialize the material in the last section to the case of
cylindrical soil samples commonly used in soil testing. The sample
will be compressed axially. The sides may be unconfined, rigidly
confined, or confined in some other manner imposing a known
relationship among the stress and strain components. We will
assume the stress and strain components and the intrinsic pressure
to be constant throughout the sample. In practice this will be
approximately true before the sample begins to exude water except
near the ends of the column. Then by (21), U = const. By (22)

$$u = q(\epsilon), \quad q'(\epsilon) < 0 \tag{39}$$

Establish a cylindrical r, θ, z-coordinate system with
z-axis parallel to the elements of the column and let σ_r, σ_θ, σ_z,
ϵ_r, ϵ_θ, ϵ_z denote principal stress and strain components. Then
$\sigma_\theta = \sigma_r = \sigma_\mu = \sigma_M$, $\sigma_z = \sigma_m$, $\epsilon_\theta = \epsilon_r$. The yield conditions
(7) and (14) take on the same form. In the elastic zones, by (1),

$$-\sigma_r - p > (-\sigma_z - p)k, \tag{40}$$

and in the plastic zones

$$-\sigma_r - p = (-\sigma_z - p)k, \tag{41}$$

where in either case

$$k = (1-\sin\phi)/(1 + \sin\phi) = (1 - \eta)/(1 + 2\eta) \tag{42}$$

By (30)-(34), (41),

$$f(\sigma_z, \sigma_r, \sigma_r) = -k^\delta/(\sigma_r - k^\delta \sigma_z), \tag{43}$$

where $\delta = 1$ for the Drucker-Prager theory and the first soil theory (which are identical in this problem), $\delta = -1/2$ for the second soil theory, and $\delta = 0$ for the third and fourth soil theories (which are identical in this problem).

By (35),

$$\alpha \Delta [\sigma_r^2 + \tfrac{1}{2}\sigma_z^2] - \beta \Delta [\tfrac{1}{2}(2\sigma_r + \sigma_z)] =$$

$$\tag{44}$$

$$= \int(2\sigma_r \, d\epsilon_r + \sigma_z \, d\epsilon_z) - \Delta W,$$

from which α and β may be determined.

By (36), (43),

$$2k^\delta = - d(\epsilon_z - \alpha \sigma_z + \beta \sigma)/d(\epsilon_r - \alpha \sigma_r + \beta \sigma), \tag{45}$$

from which k may be determined.

By (38) or (41),

$$p = (k \sigma_z - \sigma_r)/(1 - k), \tag{46}$$

from which p may be determined.

When p_c is constant u may be determined as a function of ϵ up to an additive constant. When p_c varies with W, samples of the same skeletal structure but different water contents may be run to determine, within an additive constant, the function of W and the function of ϵ which, added together, equal p.

16. The case k = const. In this case (45) may be integrated, giving

$$\alpha(-\sigma_r) + \beta'(-\sigma) + \gamma \epsilon_r - (\epsilon) = \text{const.}, \tag{47}$$

where

$$\beta = (\alpha \gamma - \beta')/(3 + \gamma), \quad k^\delta = 1 + \tfrac{1}{2}\gamma. \tag{48}$$

The constant on the right of (47) may be determined by evaluating it at a particular stress-strain state of the sample. This may then be transposed to the left side. Therefore we may take the constant on the right of (47) as zero if instead of taking the true stresses and strains on the left of (47), we take the difference between these stresses and strains and those of one particular stress-strain state of the sample.

We may now evaluate α, β', and γ in (47) by a least squares procedure. Suppose n observations are made as the sample is compressed and denote the observed quantities by adding a subscript i to the notations used above, where $i = 1, 2, \ldots, n$. We wish to minimize

$$Y = \frac{1}{n} \sum_{i=1}^{n} [\alpha(-\sigma_{ri}) + \beta'(-\sigma_i) + \gamma \epsilon_{ri} - (-\epsilon_i)]^2,$$

$$= A \alpha^2 + 2I\alpha\beta' + 2J\alpha\gamma + B\beta'^2 + 2K\beta'\gamma \qquad (49)$$

$$+ C \gamma^2 - 2 F\alpha - 2G\beta' - 2H\gamma + D,$$

where

$$A = \frac{1}{n} \sum_{i=1}^{n} (-\sigma_{ri})^2, \qquad B = \frac{1}{n} \sum_{i=1}^{n} (-\sigma_i)^2,$$

$$C = \frac{1}{n} \sum_{i=1}^{n} (\epsilon_{ri})^2, \qquad D = \frac{1}{n} \sum_{i=1}^{n} (-\epsilon_i)^2,$$

$$I = \frac{1}{n} \sum_{i=1}^{n} (-\sigma_{ri})(-\sigma_i), \qquad J = \frac{1}{n} \sum_{i=1}^{n} (-\sigma_{ri})\epsilon_{ri}, \qquad (50)$$

$$K = \frac{1}{n} \sum_{i=1}^{n} (-\sigma_i)\epsilon_{ri}, \qquad F = \frac{1}{n} \sum_{i=1}^{n} (-\sigma_{ri})(-\epsilon_i),$$

$$G = \frac{1}{n} \sum_{i=1}^{n} (-\sigma_i)(-\epsilon_i), \qquad H = \frac{1}{n} \sum_{i=1}^{n} (\epsilon_{ri})(-\epsilon_i).$$

By (49), Y is minimized when α, β', and γ are solutions of

$$A\alpha + I\beta' + J\gamma = F,$$
$$I\alpha + B\beta' + K\gamma = G, \qquad (51)$$
$$J\alpha + K\beta' + C\gamma = H.$$

The corresponding value of Y is

$$Y = D - F\alpha - G\beta' - H\gamma \qquad (52)$$

By (49) this quantity may be judged small if it is small with respect to D.

17. __An example.__ The foregoing theory was applied to one case of a soil sample measurement made with a Hveem stabilometer. The accuracy of the test was below the requirements of the theory, but the results will be included anyway simply as an illustration.

The values of the stresses and strains over the range considered are given in columns 1-4 of Table 1. The derivatives given in columns 7-9 were calculated numerically.

After subtracting the initial values in columns 3-6 from the other entries, the quantities A-H in (50) may be calculated. They are

$A = 781.000 \text{ lb}^2/\text{in}^4,$ $B = 8750.273 \text{ lb}^2/\text{in}^4,$

$C = 2.951 \cdot 10^{-6}$ $D = 7.270 \cdot 10^{-6}$

$I = 2611.383 \text{ lb}^2/\text{in}^4,$ $J = 47.456 \cdot 10^{-3} \text{ lb/in}^2,$

$K = 158.970 \cdot 10^{-3} \text{lb/in}^2,$ $F = 75.067 \cdot 10^{-3} \text{ lb/in}^2,$

$G = 251.397 \cdot 10^{-3} \text{lb/in}^2,$ $H = 4.616 \cdot 10^{-6}.$

TABLE 1.

1	2	3	4	5	6	7	8	9
$-\sigma_z$	$-\epsilon_z$	$-\sigma_r$	ϵ_r	$-\sigma$	$-\epsilon$	$10^3 d\epsilon_z/d\sigma_z$	$10^3 d\epsilon_r/d\sigma_z$	$d\sigma_r/d\sigma_z$
lb/in^2		lb/in^2		lb/in^2		in^2/lb	in^2/lb	
47.8	.0286	21.0	.0036	89.8	.0214	.251	.074	.600
55.7	.0304	26.0	.0041	107.7	.0222	.207	.063	.660
63.7	.0319	31.5	.0046	126.7	.0227	.176	.053	.715
71.6	.0332	37.5	.0050	146.5	.0232	.156	.045	.748
79.6	.0344	43.5	.0053	166.6	.0238	.140	.036	.773
87.5	.0355	49.5	.0055	186.5	.0245	.125	.030	.795
95.5	.0364	56.0	.0058	207.5	.0248	.112	.026	.810
103.5	.0372	62.5	.0060	228.5	.0252	.100	.021	.824
111.4	.0379	69.0	.0061	249.5	.0257	.090	.018	.840

The solution of equations (51) gives $\alpha = 18.5 \cdot 10^{-6}$ in^2/lb, $\beta' = 9.35 \cdot 10^{-6}$ in^2/lb, $\gamma = .763$. By (48), $\beta = 1.26 \cdot 10^{-6}$ in^2/lb. By (6), $E = 58,100$ lb/in^2, $\nu = .073$. By (48), $k^{\epsilon} = 1.381$. Since $0 < k < 1$, this indicates the appropriateness of the second soil theory in which $\delta = -1/2$. This gives $k = .524$, corresponding, by (42), to a shearing resistance angle $\phi = 18°$. By (52) the corresponding minimum value of Y is $.036 \cdot 10^{-6}$, so $Y/D = .0050$.

Using these values of α and β, point to point values of k were computed from (45) and columns 7-9 of Table 1. The results are graphed in Fig. 2. The solid line corresponds to the value $k = .524$ calculated in the last paragraph.

The quantity p may now be computed from (46) and Table 1. The results are given in Fig. 3 using k = .524.

Appendix.

The following theorem is a slight extension of one due to C. B. Morrey, R. M. Lakness, and E. Parzen.

Theorem Let S_{ij} and T_{ij} be symmetric tensors connected by a relation

$$S_{ij} = F_{ij}(T_{pq}) \equiv F_{ij}(T_{11}, T_{22}, T_{33}, T_{12}, T_{23}, T_{31}) \qquad (A)$$

which is invariant to cartesian coordinate transformations. Then there exists a function $f(x,y,z)$ for which

$$f(x,y,z) = f(x,z,y) \qquad (B)$$

such that

$$S_{ij} = \Phi_0 \delta_{ij} + \Phi_1 T_{ij} + \Phi_2 T_{i\gamma} T_{\gamma j} \qquad (C)$$

where

$$\Phi_0 = T_2 T_3 F_1 + T_3 T_1 F_2 + T_1 T_2 F_3,$$

$$\Phi_1 = -(T_2 + T_3)F_1 - (T_3 + T_1)F_2 - (T_1 + T_2)F_3, \qquad (D)$$

$$\Phi_2 = F_1 + F_2 + F_3,$$

where

$$F_1 = f(T_1, T_2, T_3)/[(T_1 - T_2)(T_1 - T_3)],$$

$$F_2 = f(T_2, T_3, T_1)/[(T_2 - T_3)(T_2 - T_1)], \qquad (E)$$

$$F_3 = f(T_3, T_1, T_2)/[(T_3 - T_1)(T_3 - T_2)],$$

T_1, T_2, T_3 being the principal values of the tensor T_{ij}, i.e., the roots of the equation

$$T^3 - I_1 T^2 + I_2 T - I_3 = 0 \qquad (F)$$

where

$$I_1 = T_{\alpha\alpha}$$

$$I_2 = T_{11}T_{22} + T_{22}T_{33} + T_{33}T_{11} - T_{12}^2 - T_{23}^2 - T_{31}^2 \tag{G}$$

$$I_3 = |T_{ij}| \, .$$

<u>Proof</u>. A cartesian coordinate transformation may be written

$$x_i = a_{i\gamma} y_\gamma \, , \tag{H}$$

where

$$a_{\alpha i} \, a_{\alpha j} = \delta_{ij}, \qquad a_{i\beta} \, a_{j\beta} = \delta_{ij}. \tag{I}$$

The matrices $\|S_{ij}\|$ and $\|T_{ij}\|$ of tensor components are transformed by (H) to new matrices $\|s_{ij}\|$ and $\|t_{ij}\|$ such that

$$S_{ij} = s_{\alpha\beta} a_{i\alpha} a_{j\beta} \, , \qquad T_{ij} = t_{\alpha\beta} a_{i\alpha} a_{j\beta} \, . \tag{J}$$

Since the relationship (A) is invariant,

$$s_{\alpha\beta} = F_{\alpha\beta} \, (t_{mn})$$

Substituting this and (J) into (A),

$$F_{ij}(t_{\alpha\beta} a_{p\alpha} a_{q\beta}) = F_{\alpha\beta} (t_{mn}) a_{i\alpha} a_{j\beta} \tag{K}$$

In particular [8; Ch. I, § 3], there exists a transformation, with matrix $\|\bar{a}_{ij}\|$, to principal axes, i.e., such that the tensor components $\bar{t}_{ij}$ satisfy

$$\bar{t}_{ij} = t_i \, \delta_{ij} \tag{L}$$

Then by (J), (K), taking $a_{ij} = \bar{a}_{ij}$,

$$T_{ij} = T_\gamma \, \bar{a}_{i\gamma} \, \bar{a}_{j\gamma} , \tag{M}$$

$$F_{ij}(T_{pq}) = f_{\alpha\beta}(T_1,T_2,T_3) \, \bar{a}_{i\alpha} \, \bar{a}_{j\beta} , \tag{N}$$

where

$$f_{ij}(T_1,T_2,T_3) = F_{ij}(T_m \, \delta_{mn}) \tag{O}$$

Now let $t_{ij} = \bar{t}_{ij}$ in (K),

$$F_{ij}(T_\alpha \, a_{p\alpha} a_{q\alpha}) = f_{\alpha\beta}(T_1,T_2,T_3) a_{i\alpha} \, a_{j\beta} \tag{P}$$

Insert

$$\| a_{ij} \| = \begin{Vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{Vmatrix}$$

Then

$$\| T_\alpha \, a_{i\alpha} a_{j\alpha} \| = \begin{Vmatrix} T_2 & 0 & 0 \\ 0 & T_3 & 0 \\ 0 & 0 & T_1 \end{Vmatrix}$$

Therefore by (O) and (P),

$$\| f_{ij}(T_2,T_3,T_1) \| = \begin{Vmatrix} f_{22} & f_{23} & f_{12} \\ f_{23} & f_{33} & f_{13} \\ f_{12} & f_{13} & f_{11} \end{Vmatrix}$$

the arguments of the functions f_{ij} on the right being t_1,t_2,t_3. This relation gives

$$f_{22}(t_1,t_2,t_3) = f_{11}(t_2,t_3,t_1), \quad f_{33}(t_1,t_2,t_3) = f_{11}(t_3,t_1,t_2),$$
$$\tag{Q}$$
$$f_{13}(t_1,t_2,t_3) = f_{12}(t_3,t_1,t_2), \quad f_{23}(t_1,t_2,t_3) = f_{12}(t_2,t_3,t_1).$$

Now insert

$$\|a_{ij}\| = \begin{Vmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{Vmatrix}$$

in (P). This gives

$$\|t_\alpha a_{i\alpha} a_{j\alpha}\| = \begin{Vmatrix} t_1 & 0 & 0 \\ 0 & t_3 & 0 \\ 0 & 0 & t_2 \end{Vmatrix} ,$$

so by (0) and (P),

$$\|f_{ij}(t_1,t_3,t_2)\| = \begin{Vmatrix} f_{11} & f_{13} & -f_{12} \\ f_{13} & f_{33} & -f_{23} \\ -f_{12} & -f_{23} & f_{22} \end{Vmatrix} .$$

This gives

$$f_{11}(t_1,t_2,t_3) = f_{11}(t_1,t_3,t_2), \quad f_{12}(t_1,t_2,t_3) = -f_{13}(t_1,t_3,t_2)$$

$$f_{13}(t_1,t_2,t_3) = f_{12}(t_1,t_3,t_2). \tag{R}$$

Defining

$$f(x,y,z) = f_{11}(x,y,z),$$

(B) follows from (R). Also, from (R), $f_{12} = 0$. Therefore by (Q), $f_{13} = 0$, $f_{23} = 0$, and $f_{22}(x,y,z) = f(y,z,x)$, $f_{33}(x,y,z) = f(z,x,y)$.

To complete the proof we must show that the right-hand sides of (C) and (N) are equal. Inserting (M) into (C), and using (I),

$$S_{ij} = (\Phi_0 + \Phi_1 T_\gamma + \Phi_2 T_\gamma^2)\bar{a}_{ir}\,\bar{a}_{jr} \tag{S}$$

Now by (D),

$$\Phi_0 + \Phi_1 T_1 + \Phi_2 T_1^2 = (T_1 - T_2)(T_1 - T_3) F_1 = f(T_1, T_2, T_3)$$

$$\Phi_0 + \Phi_1 T_2 + \Phi_2 T_2^2 = (T_2 - T_3)(T_2 - T_1) F_2 = f(T_2, T_3, T_1)$$

$$\Phi_0 + \Phi_1 T_3 + \Phi_2 T_3^2 = (T_3 - T_1)(T_3 - T_2) F_3 = f(T_3, T_1, T_2)$$

Substituting this into (S), we obtain the right-hand side of (N), completing the proof.

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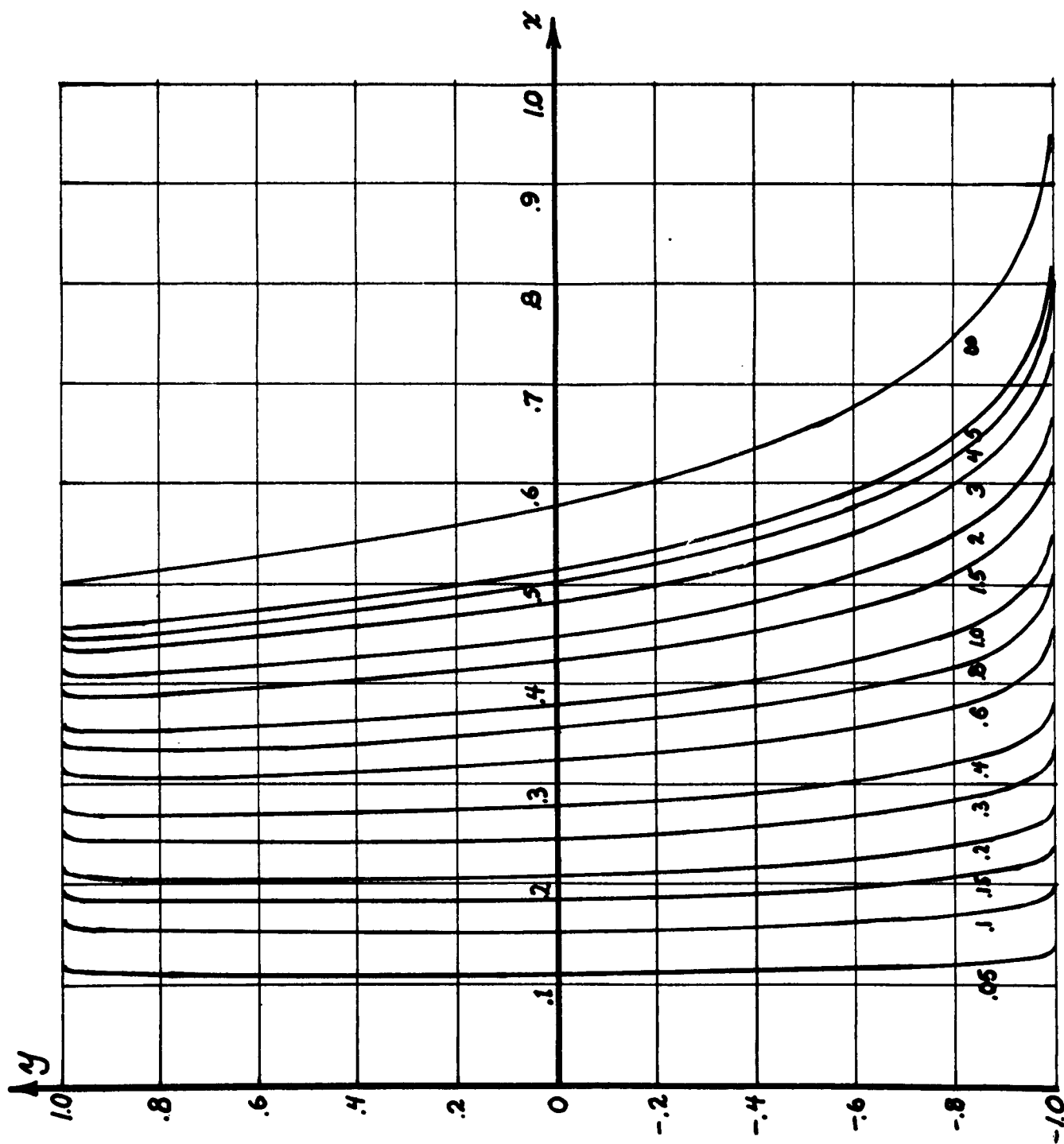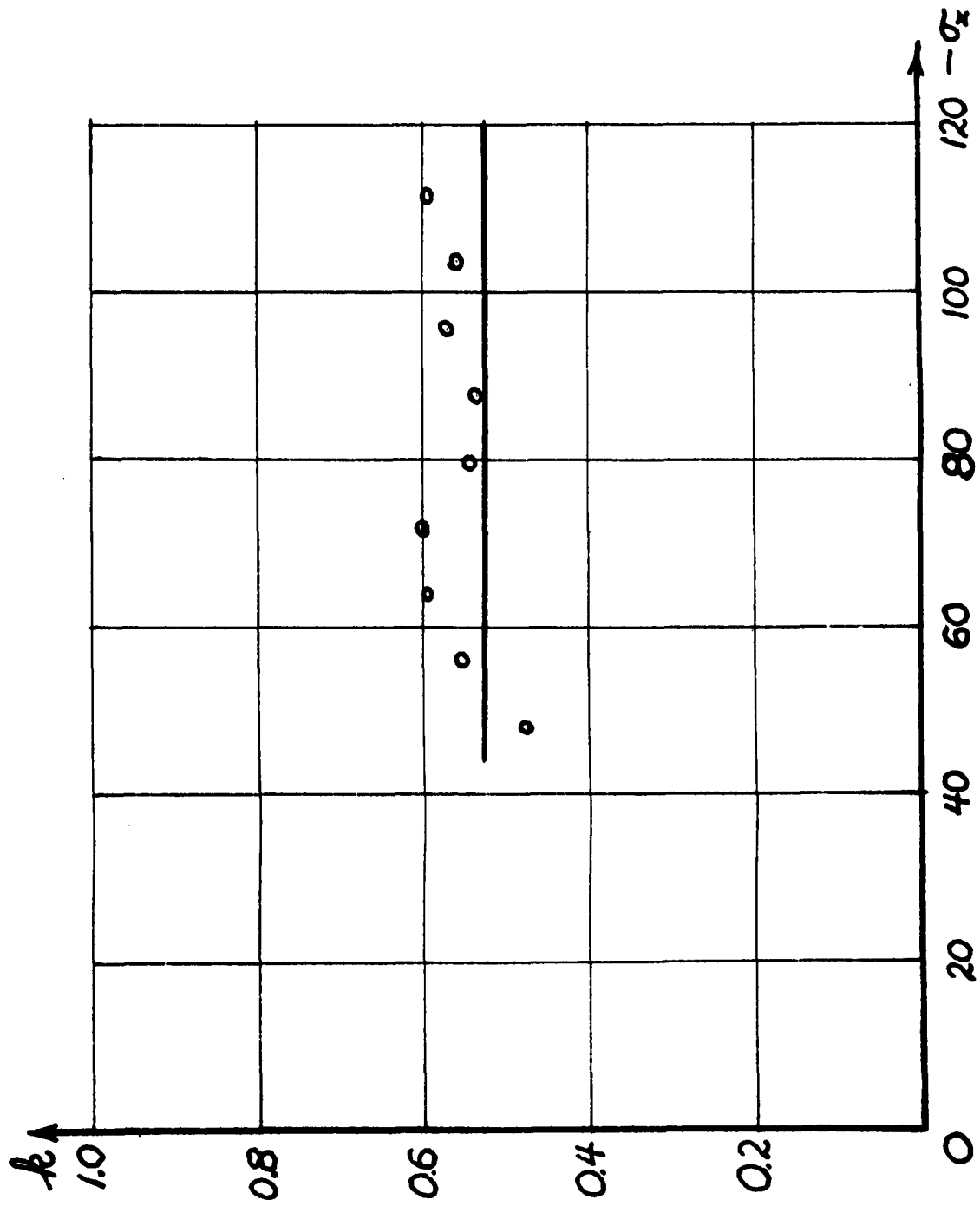
Fig. 1

Fig. 2

Fig.3