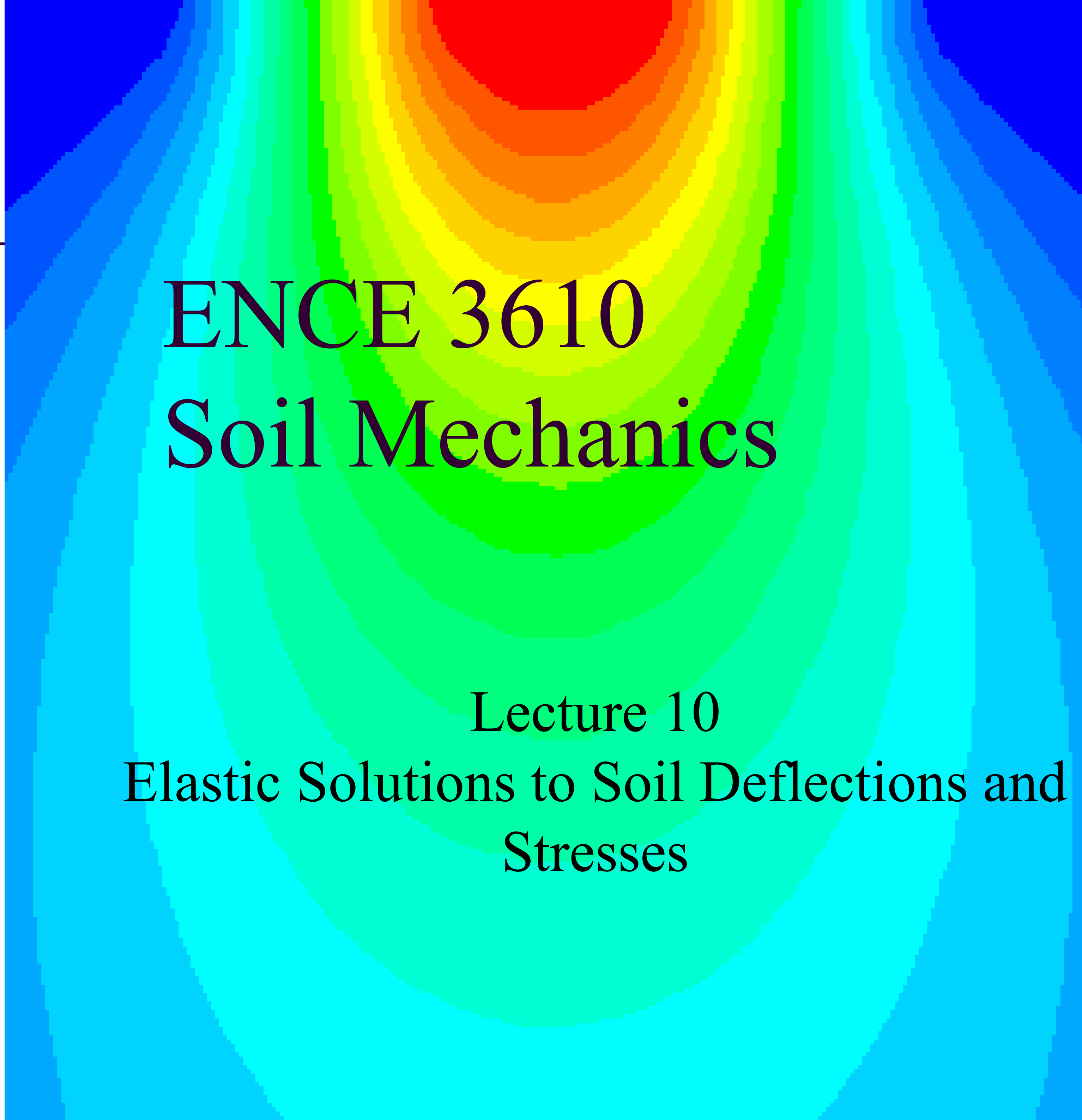


ENCE 3610
Soil Mechanics

Lecture 10
Elastic Solutions to Soil Deflections and Stresses

Using Theory of Elasticity for Estimating Stresses and Deflections

- Since it concerns both stress and strain/deflection, theory of elasticity can be used to estimate deflections in the elastic region
  - In this presentation, we will emphasize stresses, will consider deflections in more detail later
- Soils are highly nonlinear; thus, the strains must be restricted to small ones
  - Will not consider strain-softening effects, which means that the shear modulus/modulus of elasticity varies with the proximity to the load
- Theory of elasticity is applied to a semi-infinite solid (the soil) and the stresses vary (decrease) as one gets further from the load source
- For distributed loads, either a flexible or rigid foundation can be assumed, depending upon the situation at hand
  - Most solutions here—and those commonly used—assume a flexible foundation

Implementation of Theory of Elasticty

- Boussinesq Theory
  - Based on theory of elasticity
  - Homogeneous, isotropic material Semi-infinite solid
  - Original equations describe loading at a point; can be applied to various foundation shapes
  - Can be used to determine both deflections and stresses
  - Many of these solutions assume values of Poisson's Ratio for simplicity
- In all cases, our main focus will be on vertical stresses, although horizontal and shear stresses can be computed using this and other theories
- Westergaard Theory
  - Similar to Boussinesq, but no lateral deformations of the soil are assumed
  - Used with soils of alternating layers of materials
  - Used extensively in airfield pavement design
- Newmark's Method
  - Adaptation of Boussinesq Theory for structures that do not have a simple shape
- 2:1 Method
  - Empirical method commonly used to estimate structure induced stresses (FHWA)

Shapes and Solution Methods

- Shapes
  - Shapes that will be considered here
    - Point Loads
    - Line Loads (Flamant)
    - Strip Loads (Flamant)
    - Rectangular Loads
    - Circular Loads
  - Chart and analytical solutions can be combined because of superposition
  - Newmark's Method can handle foundations of any shape
- Solution Methods
  - Equations
    - For simple cases, can be very useful
    - For more complex cases, generally not practical
    - Mistakes are common in literature and on the internet
  - Charts (traditionally the most extensively used, but have accuracy issues)
  - Computer program (needs check for verification)
  - Newmark's Graphical Method

Boussinesq Point Load Stresses

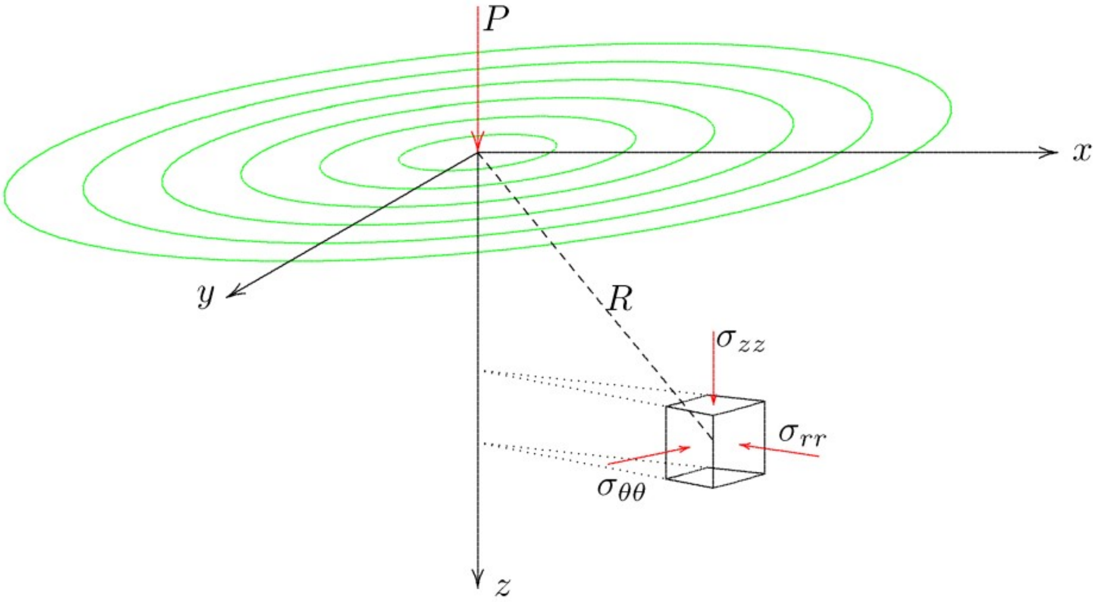


Figure 28.1: Point load on half space.

$$\sigma_{zz} = \frac{3P}{2\pi}\frac{z^3}{R^5},$$

$$\sigma_{rr} = \frac{P}{2\pi}\left[\frac{3r^2 z}{R^5} - (1-2\nu)\frac{1}{R(R+z)}\right],$$

$$\sigma_{\theta\theta} = \frac{P}{2\pi}\frac{1-2\nu}{R^2}\left(\frac{R}{R+z} - \frac{z}{R}\right),$$

$$\sigma_{rz} = \frac{3P}{2\pi}\frac{rz^2}{R^5}.$$

In these equations r is the cylindrical coordinate,

$$r = \sqrt{x^2+y^2},$$

and R is the spherical coordinate,

$$R = \sqrt{x^2+y^2+z^2}.$$

Another interesting quantity is the distribution of the stresses as a function of depth, just below the point load, i.e. for $r = 0$. This is found to be

$$r = 0: \quad \sigma_{zz} = \frac{3P}{2\pi z^2}, \tag{28.11}$$

$$r = 0: \quad \sigma_{rr} = \sigma_{\theta\theta} = -(1-2\nu)\frac{P}{4\pi z^2}. \tag{28.12}$$

These stresses decrease with depth, of course.

Boussinesq Point Load Displacements

The solution for the displacements is

$$u_r = \frac{P(1+\nu)}{2\pi ER}[\frac{r^2 z}{R^3} - (1-2\nu)(1-\frac{z}{R})], \tag{28.7}$$

$$u_\theta = 0, \tag{28.8}$$

$$u_z = \frac{P(1+\nu)}{2\pi ER}[2(1-\nu) + \frac{z^2}{R^2}]. \tag{28.9}$$

The vertical displacement of the surface is particularly interesting. This is

$$z = 0 : \quad u_z = \frac{P(1-\nu^2)}{\pi ER}. \tag{28.10}$$

For $R \to 0$ this tends to infinity, indicating that at the point of application of the point load the displacement is infinitely large. This singular behavior is a consequence of the singularity in the surface load, as in the origin the stress is infinitely large. That the displacement in that point is also infinitely large may not be so surprising.

We don't use these very often because of the singularity issue.

Boussinesq Point Load Illustration

- Given
  - Point Load, 45 kN
  - Point 3 m directly below the point load
- Find
  - Additional Vertical and Shear Stress Created by Point Load (overburden/ effective stresses not considered)

- Solution
  - $z = 3$ m
  - $R = (x^2 + y^2 + z^2)^{1/2} = z = 3$ m
  - $r = 0$
  - $\sigma_{zz} = ((3)(45)(3)^3)/((2)(\pi)(3)^5) = 2.387$ kPa
  - $\sigma_{rz} = ((3)(45)(0)(3)^2)/((2)(\pi)(3)^5) = 0$ kPa

Flamant Line Load Stresses

In 1892 the French scientist M. Flamant obtained the solution for a vertical line load on a homogeneous isotropic linear elastic half space, see Figure 30.1. This is the two-dimensional equivalent of Boussinesq's basic problem. It can be considered as the superposition of an infinite number of point loads, uniformly distributed along the y-axis. A derivation of this solution is given in Appendix B.

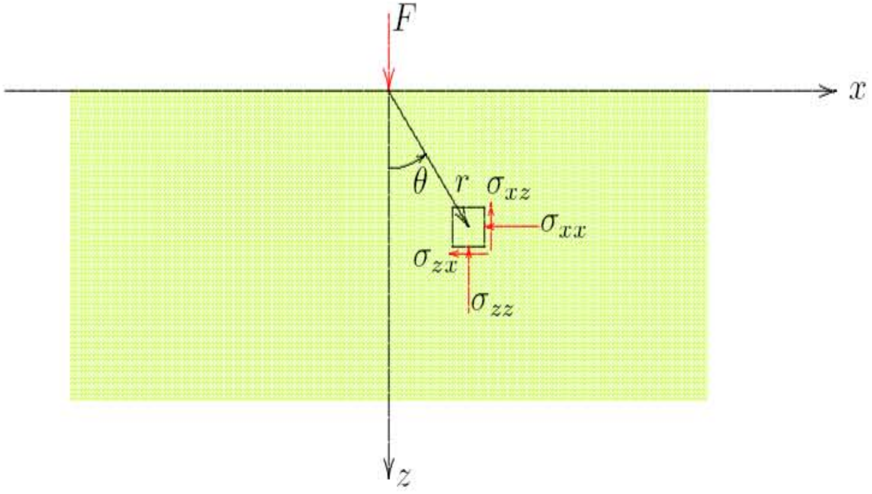


Figure 30.1: Flamant's Problem.

In this case the stresses in the x, z-plane are

$$\sigma_{zz} = \frac{2F}{\pi}\frac{z^3}{r^4} = \frac{2F}{\pi r}\cos^3\theta, \tag{30.1}$$

$$\sigma_{xx} = \frac{2F}{\pi}\frac{x^2 z}{r^4} = \frac{2F}{\pi r}\sin^2\theta\cos\theta, \tag{30.2}$$

$$\sigma_{xz} = \frac{2F}{\pi}\frac{xz^2}{r^4} = \frac{2F}{\pi r}\sin\theta\cos^2\theta. \tag{30.3}$$

In these equations $r = \sqrt{x^2 + z^2}$. The quantity F has the dimension of a force per unit length, so that F/r has the dimension of a stress.

Expressions for the displacements are also known, but these contain singular terms, with a factor $\ln r$. This factor is infinitely large in the origin and at infinity. Therefore these expressions are not so useful.

Flamant Strip Load Stresses

On the basis of Flamant's solution several other solutions may be obtained using the principle of superposition. An example is the case of a uniform load of magnitude p on a strip of width $2a$, see Figure 30.2. In this case the stresses are

$$\sigma_{zz} = \frac{p}{\pi}[(\theta_1 - \theta_2) + \sin\theta_1\cos\theta_1 - \sin\theta_2\cos\theta_2], \quad (30.4)$$

$$\sigma_{xx} = \frac{p}{\pi}[(\theta_1 - \theta_2) - \sin\theta_1\cos\theta_1 + \sin\theta_2\cos\theta_2], \quad (30.5)$$

$$\sigma_{xz} = \frac{p}{\pi}[\cos^2\theta_2 - \cos^2\theta_1]. \quad (30.6)$$

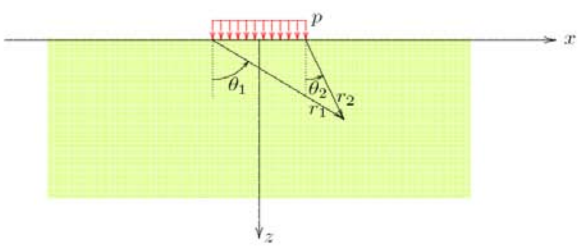


Figure 30.2: Strip load.

In the center of the plane, for $x = 0$, $\theta_2 = -\theta_1$. Then the stresses are

$$x = 0 \ : \ \sigma_{zz} = \frac{2p}{\pi}[(\theta_1 + \sin\theta_1\cos\theta_1], \quad (30.7)$$

$$x = 0 \ : \ \sigma_{xx} = \frac{2p}{\pi}[(\theta_1 - \sin\theta_1\cos\theta_1], \quad (30.8)$$

$$x = 0 \ : \ \sigma_{xz} = 0. \quad (30.9)$$

That the shear stress $\sigma_{xz} = 0$ for $x = 0$ is a consequence of the symmetry of this case. The stresses σ_{xx} and σ_{zz} are shown in Figure 30.3, as functions of the depth z. Both stresses tend towards zero for $z \to \infty$, of course, but the horizontal normal stress appears to tend towards zero much faster than the vertical normal stress. It also appears that at the surface the horizontal stress is equal to the vertical stress. At the surface this vertical stress is equal to the load p, of course, because that is a boundary condition of the problem. Actually, in every point of the surface below the strip load the normal stresses are $\sigma_{xx} = \sigma_{zz} = p$.

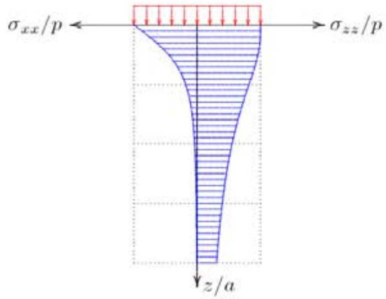


Figure 30.3: Stresses for $x = 0$.

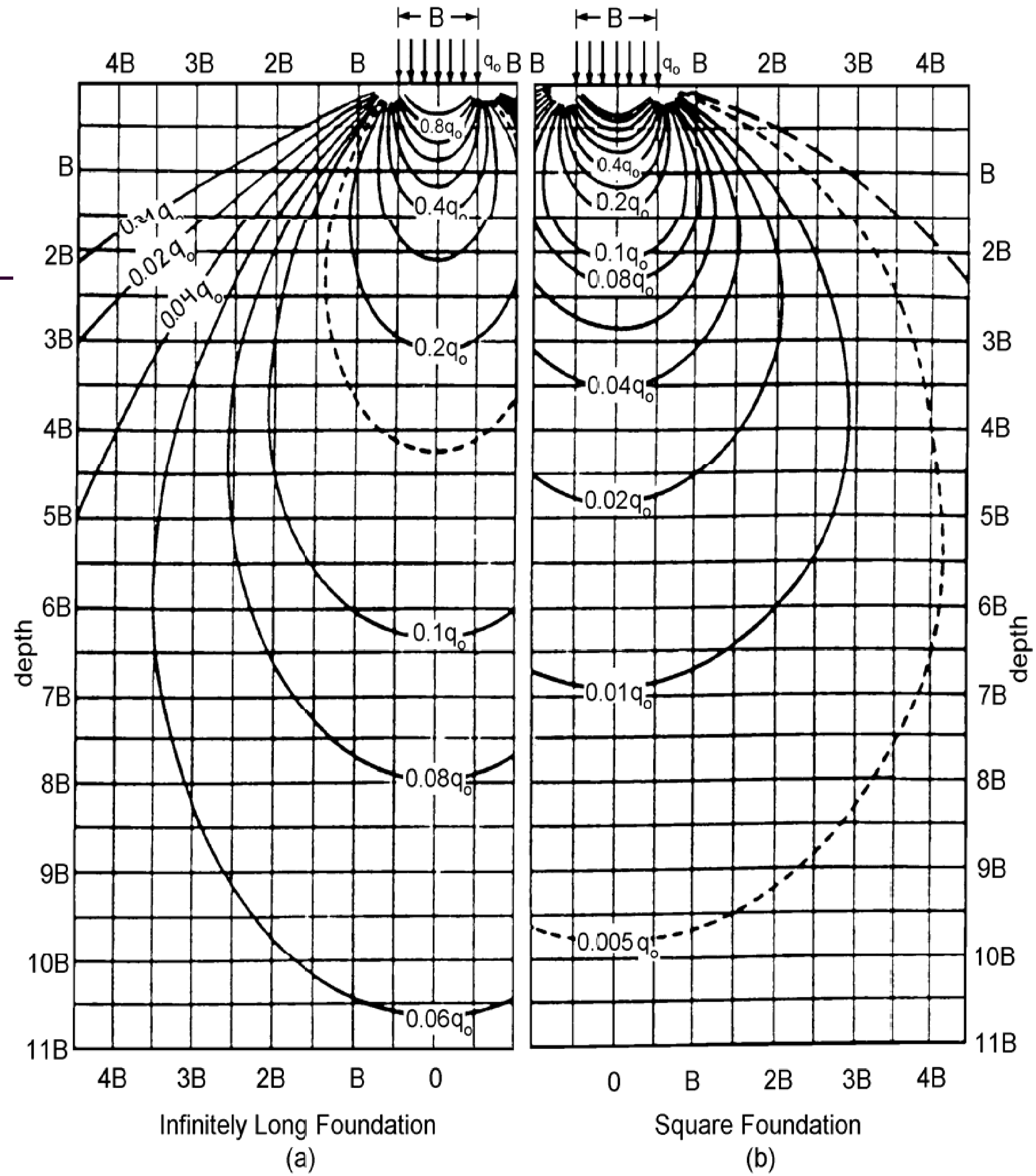
It may be interesting to further explore the result that the shear stress $\sigma_{xz} = 0$ along the axis of symmetry $x = 0$ in the case of a strip load, see Figure 30.2. It can be expected that this symmetry also holds for the horizontal displacement, so that $u_x = 0$ along the axis $x = 0$. This means that this solution can also be used as the solution of the problem that is obtained by considering the right half of the strip problem only, see Figure 30.4. In this problem the quarter plane $x > 0, z > 0$ is supposed to be loaded by a strip load of width a on the surface $z = 0$, and the boundary conditions on the boundary $x = 0$ are that the displacement $u_x = 0$ and the shear stress $\sigma_{xz} = 0$, representing a perfectly smooth and rigid vertical wall. The wall is supposed to extend to an infinite depth, which is impractical. For a smooth rigid wall of finite depth the solution may be considered as a first approximation.

The formulas (30.7) and (30.8) can also be written as

$$x = 0 \ : \ \sigma_{zz} = \frac{2p}{\pi}[\arctan(\frac{a}{z}) + \frac{az}{a^2 + z^2}], \quad (30.10)$$

$$x = 0 \ : \ \sigma_{xx} = \frac{2p}{\pi}[\arctan(\frac{a}{z}) - \frac{az}{a^2 + z^2}]. \quad (30.11)$$

Chart for Strip and Square Loads



Boussinesq Rectangle and Square Stresses

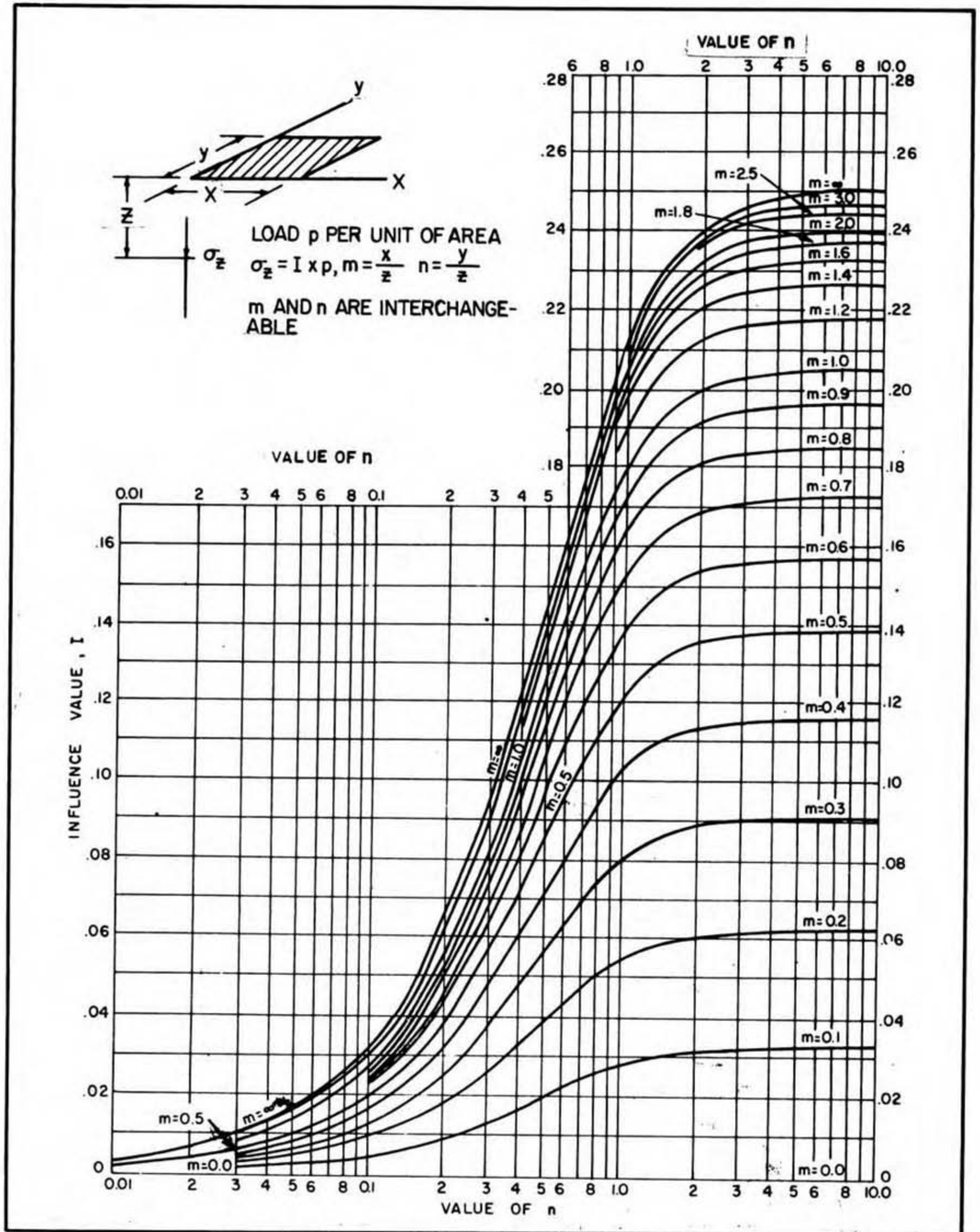


FIGURE 4
Influence Value for Vertical Stress Beneath a Corner of a Uniformly Loaded Rectangular Area (Boussinesq Case)

Rectangular Influence Coefficient I Using Equations

Equations for Boussinesq Influence Coefficients for Rectangles at Corner (using DM 7 notation):

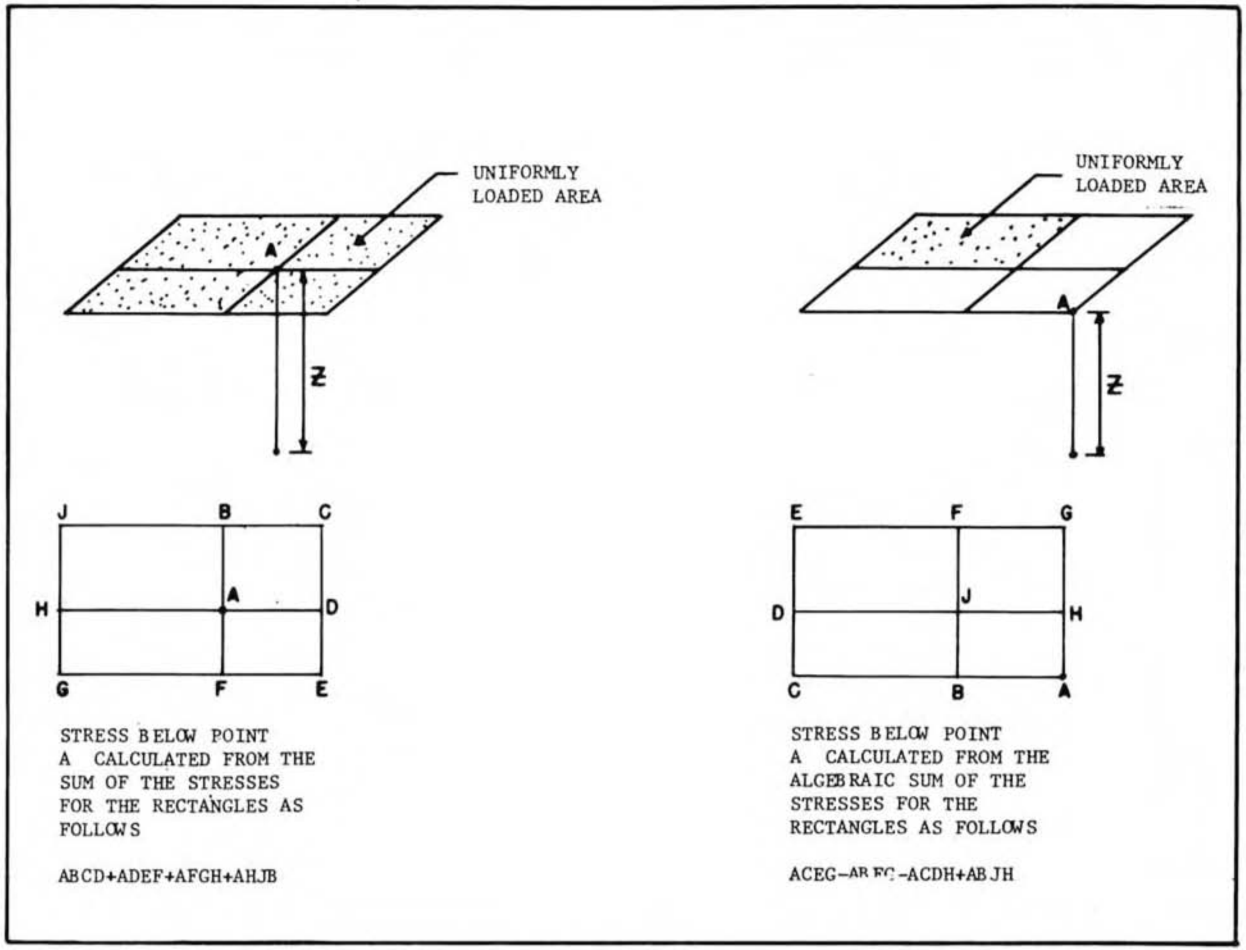
If $m^2n^2 > m^2 + n^2 + 1$:

$$1/4\left(2\,\frac{mn\,(m^2+n^2+2)}{\sqrt{m^2+n^2+1}\,(m^2+n^2+1+m^2n^2)} + \arctan(2\,\frac{mn\sqrt{m^2+n^2+1}}{m^2+n^2+1-m^2n^2}) + \pi\right)\pi^{-1}$$

Otherwise:

$$1/4\left(2\,\frac{mn\,(m^2+n^2+2)}{\sqrt{m^2+n^2+1}\,(m^2+n^2+1+m^2n^2)} + \arctan(2\,\frac{mn\sqrt{m^2+n^2+1}}{m^2+n^2+1-m^2n^2})\right)\pi^{-1}$$

Using Superposition with Boussinesq Charts



Verrujt Newmark Example

- Given
  - Buildings as shown to the right
  - Yellow building has uniform load of 5 kPa
  - Brown building has a uniform load of 15 kPa
- Find
  - Vertical stress 8m below point A

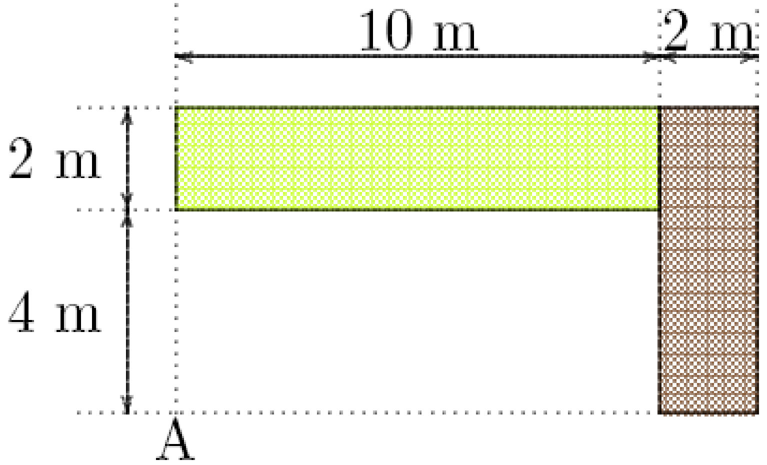


Figure 29.4: Example.

Verruijt Newmark Example

- Solution
  - Since there are two different loads, best way is to analyze two loads separately and add them together using superposition
  - Notation is per previous chart
  - In both cases, it was simpler to compute a "large" area and then subtract a void from that area
  - Influence coefficients were from the equations, can also be obtained from the charts

Yellow Foundation									
Rectangle	B, m	L, m	Z, m	m	n	Iz	Pressure, kPa	Stress, kPa	
ABFG (+)	6	10	8	0.75	1.25	0.16	5.00	0.823	
ABJH (-)	4	10	8	0.5	1.25	0.13	-5.00	-0.637	
							Total	0.186	kPa
Brown Foundation									
Rectangle	B, m	L, m	Z, m	m	n	Iz	Pressure, kPa	Stress, kPa	
ACEG (+)	6	12	8	0.75	1.5	0.17	15.00	2.551	
ABFG (-)	6	10	8	0.75	1.25	0.16	-15.00	-2.468	
							Total	0.083	kPa
							Complete Total	0.269	kPa

Boussinesq Circular Stresses (Analytic Solution)

As an example consider the case of a uniform load of magnitude p over a circular area, of radius a. The solution for this case can be found by integration over a circular area (S.P. Timoshenko & J.N. Goodier, *Theory of Elasticity*, paragraph 124), see Figure 28.3.

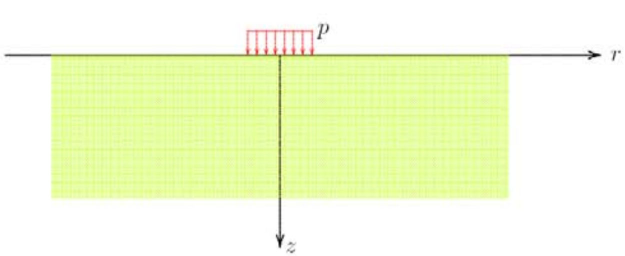


Figure 28.3: Uniform load over circular area.

The stresses along the axis $r = 0$, i.e. below the center of the load, are found to be

$$r = 0 : \quad \sigma_{zz} = p(1 - \frac{z^3}{b^3}), \tag{28.13}$$

$$r = 0 : \quad \sigma_{rr} = p[(1+\nu)\frac{z}{b} - \tfrac{1}{2}(1 - \frac{z^3}{b^3})], \tag{28.14}$$

in which $b = \sqrt{z^2 + a^2}$.

The displacement of the origin is

$$r = 0,\ z = 0 : \quad u_z = 2(1-\nu^2)\frac{pa}{E}. \tag{28.15}$$

is solution will be used as the basis of a more general case in the next chapter.

Another important problem, which was already solved by Boussinesq (see S.P. Timoshenko & J.N. Goodier, *Theory of Elasticity*, paragraph) is the problem of a half space loaded by a vertical force on a rigid plate. The force is represented by $P = \pi a^2 \overline{p}$, see Figure 28.4. The distribution of the normal stresses below the plate is found to be

$$z = 0,\ 0 < r < a : \quad \sigma_{zz} = \frac{\frac{1}{2}\overline{p}}{\sqrt{1 - r^2/a^2}}. \tag{28.16}$$

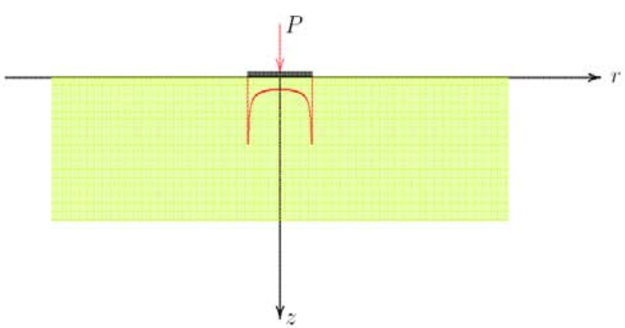


Figure 28.4: Rigid plate on half space.

This stress distribution is shown in Figure 28.4. At the edge of the plate the stresses are infinitely large, as a consequence of the constant displacement of the rigid plate. In reality the material near the edge of the plate will probably deform plastically. It can be expected, however, that the real distribution of the stresses below the plate will be of the form shown in the figure, with the largest stresses near the edge. The center of the plate will subside without much load.

The displacement of the plate is

$$z = 0,\ 0 < r < a : \quad u_z = \frac{\pi}{2}(1-\nu^2)\frac{\overline{p}a}{E}. \tag{28.17}$$

When this is compared with the displacement below a uniform load, see (28.15), it appears that the displacement of the rigid plate is somewhat smaller, as could be expected.

Circular Tank Example

- Given
  - Circular Tank, 25 metres diameter
  - Soil, 18 kN/m^3 unit weight, water table very deep
  - Weight of tank 6100 metric tons = 59,800 kN
- Find
  - Vertical stress induced by tank 10 metres below the centre of the tank
  - Effective stress induced by the overburden
  - Combined stress
- Solution
  - Bearing Pressure = 59,800 kN/ 491 m^2 = 122 kPa
  - Effective stress due to overburden = (10 m)(18 kN/m3) = 180 kPa
  - Stress induced at 10 m = 92.3 kPa (see below)
  - Combined stress with effective stress = 180+92.3 = 272.3 kPa

$$\sigma_z = p\left(1-\left(\frac{z}{b}\right)^3\right) = p\left(1-\left(\frac{z}{\sqrt{z^2+a^2}}\right)^3\right) = 122\left(1-\left(\frac{10}{\sqrt{10^2+12.5^2}}\right)^3\right) = 92.3kPa$$

Elastic Settlement

- Based on theory of elasticity
- Load applied at a point or over an area on a semi-infinite half space
- Can estimate both deflections and stresses
- Theory of Boussinesq most commonly used; will discuss stresses later
- Especially useful in computing the settlement of structures on firmer material, such as intermediate geomaterials and rock

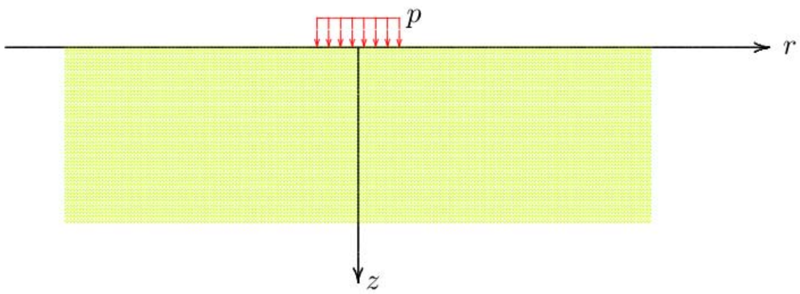


Figure 28.3: Uniform load over circular area.

As an example consider the case of a uniform load of magnitude p over a circular area, of radius a.

The displacement of the origin is

$$r = 0,\ z = 0\ :\quad u_z = 2(1-\nu^2)\frac{pa}{E}$$

Elastic Settlement; Definition of Intermediate Geomaterials and Rock

$$\delta_v = \frac{C_d \, \Delta p \, B_f (1-\nu^2)}{E_m} \quad \text{8-19}$$

where: δ_v = vertical settlement at surface
C_d = shape and rigidity factors (Table 8-13)
Δp = change in stress at top of rock surface due to applied footing load
B_f = footing width or diameter
ν = Poisson's ratio (refer to Table 5-22 in Chapter 5)
E_m = Young's modulus of rock mass (see Section 5.12.1 in Chapter 5)

The geotechnical specialist is usually concerned with the design and construction of some type of geotechnical feature constructed on or out of a geomaterial. For engineering purposes, in the context of this manual, the geomaterial is considered to be primarily rock and soil. A geomaterial intermediate between soil and rock is labeled as an intermediate geomaterial (IGM). These three classes of geomaterials are described as follows:

- **Rock** is a relatively hard, naturally formed solid mass consisting of various minerals and whose formation is due to any number of physical and chemical processes. The rock mass is generally so large and so hard that relatively great effort (e.g., blasting or heavy crushing forces) is required to break it down into smaller particles.
- **Soil** is defined as a conglomeration consisting of a wide range of relatively smaller particles derived from a parent rock through mechanical weathering processes that include air and/or water abrasion, freeze-thaw cycles, temperature changes, plant and animal activity and by chemical weathering processes that include oxidation and carbonation. The soil mass may contain air, water, and/or organic materials derived from decay of vegetation, etc. The density or consistency of the soil mass can range from very dense or hard to loose or very soft.
- **Intermediate geomaterials** (IGMs) are transition materials between soils and rocks. The distinction of IGMs from soils or rocks for geotechnical engineering purposes is made purely on the basis of strength of the geomaterials. Discussions and special design considerations of IGMs are beyond the scope of this document.

Table 8-13
Shape and rigidity factors, C_d, for calculating settlements of points on loaded areas at the surface of a semi-infinite elastic half space (after Winterkorn and Fang, 1975)

Shape	Center	Corner	Middle of Short Side	Middle of Long Side	Average
Circle	1.00	0.64	0.64	0.64	0.85
Circle (rigid)	0.79	0.79	0.79	0.79	0.79
Square	1.12	0.56	0.76	0.76	0.95
Square (rigid)	0.99	0.99	0.99	0.99	0.99
Rectangle (length/width):					
1.5	1.36	0.67	0.89	0.97	1.15
2	1.52	0.76	0.98	1.12	1.30
3	1.78	0.88	1.11	1.35	1.52
5	2.10	1.05	1.27	1.68	1.83
10	2.53	1.26	1.49	2.12	2.25
100	4.00	2.00	2.20	3.60	3.70
1000	5.47	2.75	2.94	5.03	5.15
10000	6.90	3.50	3.70	6.50	6.60

Example of Elastic Settlement in Rock

- Given
  - Same circular foundation as before, only now seated on rock, RMR = 50 (see SFH Eq. 5-29)
  - E_s = 145,000 psi = 1,000,000 kPa
  - Poisson's Ratio = 0.33
- Find
  - Deflection at centre
- Solution
  - From Table 8-13, C_d = 1.0 (for flexible foundation)
  - Result is computed below
  - Using Verruijt Equation 28.15 (online) will yield the same result

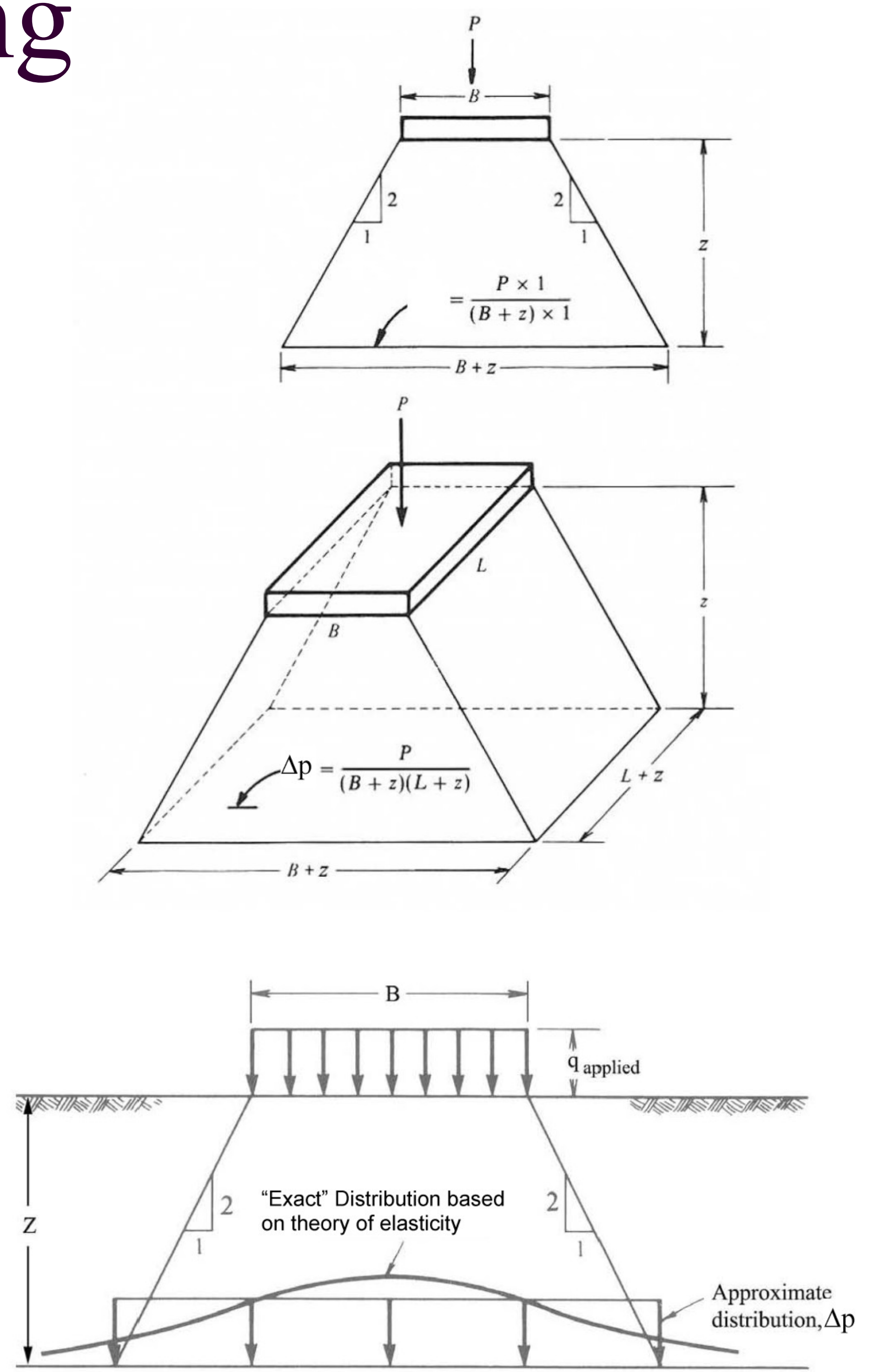
$$\delta_v = \frac{C_d \Delta p B_f \left(1 - v^2\right)}{E_m} = \frac{1 \times 122 \times 25 \times \left(1 - 0.33^2\right)}{1000000} = .0027m = 2.7mm$$

60 Degree Loading 2:1 Method

- For rectangular loads, formula in center diagram at right
  - Note the use of total load in equation
- For square and circular structures:

$$\Delta p = \frac{q}{\left(1 + \frac{z}{B}\right)^2}$$

  - Note the use of the unit load in this form, not the total load
- Should not be used anywhere other than the center of the foundation
- Example at center of foundation
  - Use same 25 metre diameter tank with 122 kPa loading
  - In this case, $z/B = 0.4$, so $\Delta\sigma_z = (122)(1/(1+0.4)^2) = 62.2$ kPa



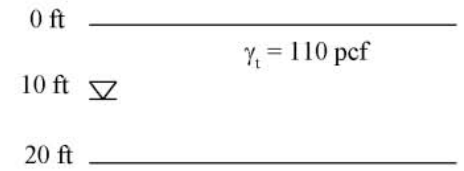
(a)

(b)

Figure 2-10. Distribution of vertical stress by the 2:1 method (after Perloff and Baron, 1976).

Combined Effective Stress and Applied Stress Example

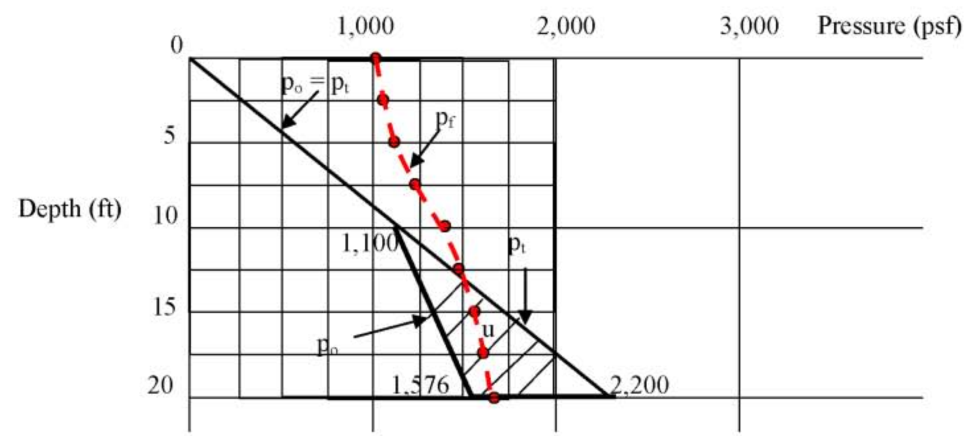
Example 2-2: For the Example 2-1 shown in Figure 2-7, assume that a 5 ft wide strip footing with a loading intensity of 1,000 psf is located on the ground surface. Compute the stress increments, Δp, under the centerline of the footing and plot them on the p_o diagram shown in Figure 2-7 down to a depth of 20 ft.



Solution:

For the strip footing, use the left chart in Figure 2-9. As per the terminology of the chart in Figure 2-9, B = 5 ft and q_o= 1,000 psf. Compile a table of stresses for various depths and plot as follows:

Depth z, ft	z/B	Isobar Value, x	Stress, Δp = x(q_o), psf	p_o, psf	p_f= p_o+Δp psf
2.5	0.5	0.80	800	(110)(2.5)=275	1,075
5.0	1.0	0.55	550	(110)(5.0)=550	1,100
7.5	1.5	0.40	400	(110)(7.5)=825	1,225
10.0	2.0	0.32	320	(110)(10.0)=1,100	1,420
12.5	2.5	0.25	250	1,100+(12.5-10.0)(110-62.4)=1,219	1,469
15.0	3.0	0.20	200	1,100+(15.0-10.0)(110-62.4)=1,338	1,538
17.5	3.5	0.18	180	1,100+(17.5-10.0)(110-62.4)=1,457	1,637
20.0	4.0	0.16	160	1,100+(20.0-10.0)(110-62.4)=1,576	1,736



Pressure Diagram

Figure 2-12. Example calculation of p_f with stress increments from strip load on p_o-diagram.

Questions?