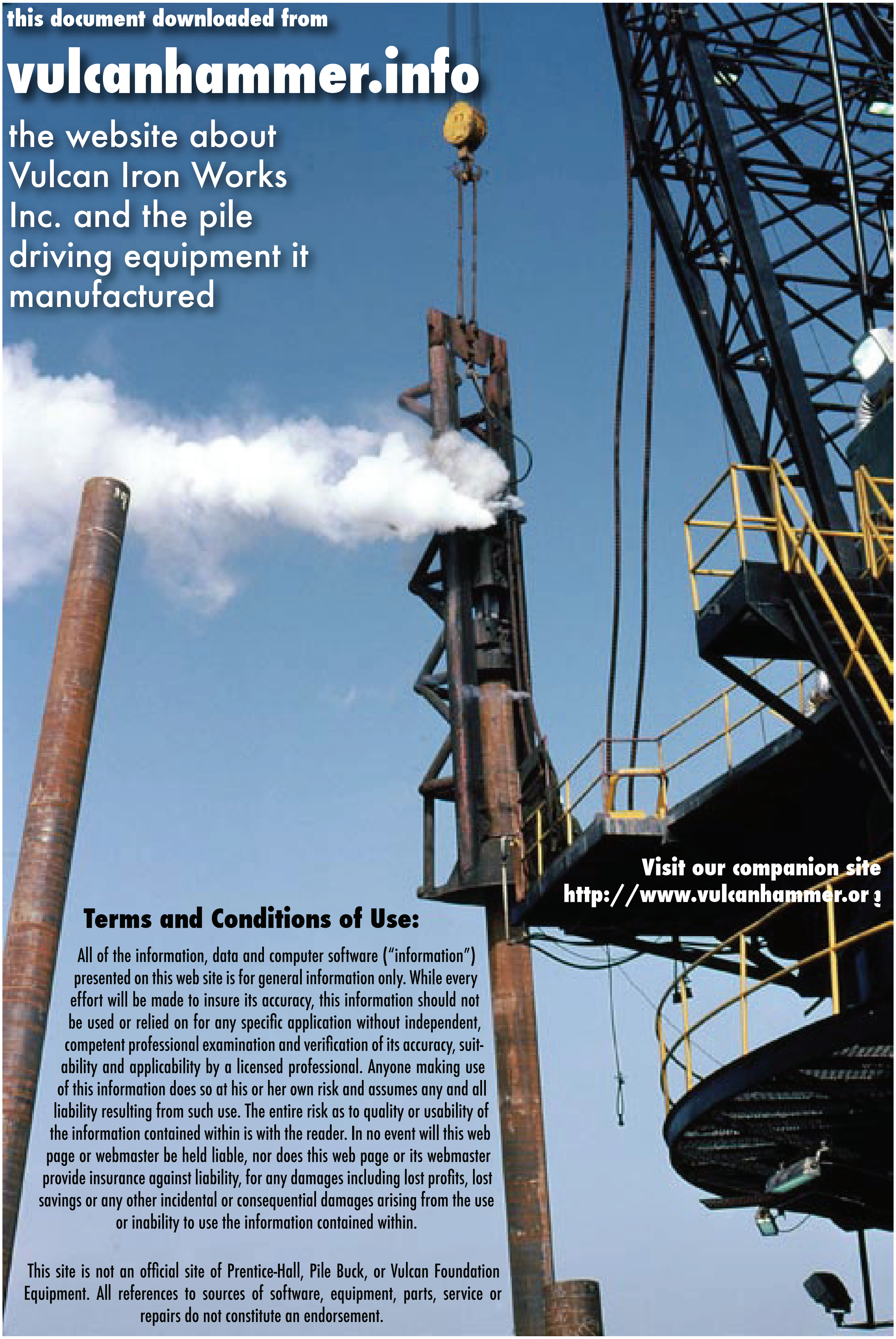

CLOSED FORM SOLUTION OF THE WAVE EQUATION FOR PILES
Internet Edition

A Thesis
Presented for the
Master of Science Degree
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Don C. Warrington
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Dedication

This work is dedicated to the memory of my father, Mr. Henry G. Warrington, and to my brother, Mr. Pembroke M. Warrington, both of whom died during the course of my pursuit of the Master's Degree and/or the writing of this thesis. Both of them, along with my mother, Mrs. Vernell S. Warrington, laboured for many years at the family business, the pile driving equipment manufacturer Vulcan Iron Works Inc.; this too passed out of our hands during the writing of this thesis. With them we look forward to the day when we can hear, "Now the dwelling of God is with men, and he will live with them. They will be his people, and God himself will be with them and be their God. He will wipe every tear from their eyes. There will be no more death or mourning or crying or pain, for the old order of things has passed away." (Rev. 21:3b–4)

Acknowledgments

Such a work as this would be impossible without the help of others. My first thanks are to the Thesis Committee: Dr. Robert H. Wynn (Chairman), Drs. Boris P. Belinskiy, Edwin P. Foster, Jr., and William Q. Gurley. Their assistance and support throughout this project were invaluable.

Unlike most research projects that pertain to public works, this one was unfunded; however, I owe a special debt of gratitude to Dr. Edward B. Perry and the U. S. Army Corps of Engineers, Waterways Experiment Station, Vicksburg, MS, for their assistance in the background research for this thesis. Without this help this thesis could not have been completed as it was done.

Others to whom I must express my appreciation to for furnishing material important to the completion of this thesis include Drs. G.A. Leonards and Richard Deschamps of Purdue University and Dr. Andrew J. Deeks of the University of Western Australia, and Mr. Alla H. Abdelhalim of the University of Tennessee at Chattanooga. Also thanks must go to the Church of God Department of Lay Ministries for their help in the printing of this thesis.

Finally I must save my special thanks for my wife Judy, whose patience and support throughout the entire pursuit of the degree and the writing of this thesis has been unfailing, and who, as a music teacher and church musician, employs the wave equation in its most beautiful form.

Abstract

This thesis details the research into the one-dimensional wave equation as applied to piles used in the support of structures for civil works and driven using impact equipment. Since the 1950's, numerical methods, both finite difference and finite element, have been used extensively for the analysis of piles during driving and are the most accepted method of analysis for the determination of driving stresses, dynamic and static resistance of piles. In this thesis the wave equation is solved in a relatively simple closed form without recourse to numerical methods. A review of past efforts to solve the wave equation in closed form is included. Problems that appear in previous related works are discussed and derived again, including the Prescott-Laura problem of the cable system stopped at one end and the solution of a hammer/cushion/cap/pile system for a semi-infinite pile. The latter is used to assist in the determination of a pile top force-time function that can be used to simulate the impact of the hammer on the pile. The basic equations, initial and boundary conditions are detailed, with the parameters adjusted to match actual soil dynamic behaviour while at the same time being a form convenient for closed form solution. To avoid difficulties due to spectral elements in the boundary conditions, a strain-based model of the radiation dampening in the pile toe was developed. The solution technique uses a Laplace transform of the semi-infinite pile problem for $0 \leq t \leq L/c$ (or for a time duration $0 \leq t \leq \delta$, where $\delta \leq L/c$) and a Fourier series solution of the Sturm-Liouville problem thereafter. This solution is applied both to undamped and damped wave equations. The work includes comparison with existing numerical methods such as WEAP87, ANSYS, and Newmark's method using Maple V.

Note on the Internet Edition

This thesis was originally submitted for examination and defence to the thesis committee in the spring of 1997. The Chairman of that committee examined the final copy of this thesis and recommended (with the concurrence of the rest of the Committee) that it be accepted in partial fulfilment of the requirements for the degree of Master of Science with a concentration in Civil Engineering. The Director of Graduate Studies subsequently accepted it.

In preparing the thesis for this Internet Edition, the text formatting was compressed (the original used double spacing) to save space. Additionally Appendices A-E (the appendices with the Maple runs) were eliminated except for a few figures, as the software has changed and importing scanned graphical copies would excessively expand the size of the file. The titles for these appendices have been retained for reference, and their salient content is included in the text itself. Also some minor corrections and harmonisations from the original manuscript were made as well.

Preface

> *Let us consider that if the ancients had kept to this deference of daring to add nothing to the knowledge transmitted to them and if their contemporaries had been as much opposed to accepting anything new, they would have deprived both themselves and their posterity of the fruit of their discoveries. Just as they used the discoveries handed down to them only as the means of making new ones, and that happy daring had opened the road for them to great achievements, so we should take the discoveries won for us by them in the same spirit, and following their example make these discoveries the means and not the end of our study, and thus by imitating the ancients try to surpass them.*

This quotation, taken from the *Preface to the Treatise on the Vacuum* by the French scientist and Christian thinker Blaise Pascal, is as fitting way of beginning such a work as this as one can find. Although the wave equation itself has been investigated since the days of Bernoulli, the application of stress-wave theory to piles is relatively recent, going back to the early 1930's. Although it is an exaggeration to refer to those who first investigated these matters as "ancients," given the acceleration of the growth of knowledge and the application of technology the time between the first investigations of this problem and the present is in reality rather long.

In any investigation such as this the ideal goal is to come up with something truly novel, and many of such works emphasize their novelty to the denigration of those who have gone on before. While in some fields of endeavour this might be appropriate, in this case such sweeping novelty cannot be claimed. This work fits the mould as outlined by Pascal above: it takes the work that has been done before, advances it a step while realizing that there are many more steps before "perfection" is achieved.

The use of the analysis of stress waves in piles to determine everything from the performance of the hammer to the capacity of the pile is widespread today. Most of these methods use numerical methods for the analysis. The use of numerical methods came rather early in this history of stress wave application to piles, earlier in fact than the computer power really needed for practical application was readily available. Closed form solutions were either abandoned entirely or applied on a limited basis or in an ancillary way to other techniques.

The acceptance of these methods without a way to really compare them with some kind of "theoretical" result have left some involved in the analysis of pile driving uneasy as to the theoretical basis of the solutions employed. A great deal of work has been done to correlate the numerical models with field data. But are these adjustments being made to actual field phenomena or to underlying deficiencies in the methods we are using? The answer to this question is critical because without a solution to this problem we may be solving the wrong problem, and thus guaranteeing surprises in the future when a breakdown in our corrections is induced by unforeseen conditions. This is especially important in a geotechnical problem because the variables in a problem are generally complex and inadequately quantified.

It is for this reason that we are "backtracking" to a closed form solution in this thesis. In doing this we are forced to take a hard look at the underlying mathematical theory of the wave equation as it can be applied to piles. Putting together sound mathematical application with the basic physics of the problem is something that is frequently lacking (generally through no fault of the investigators) in works in this field. While in this thesis we have attempted to accomplish this, we have both applied mathematics in a different way and in the process acquired a new sense of humility because the complexity of the problem stretches the mathematics applied to the limit.

With these thoughts we proceed to our subject, realizing that we are indebted to those who have gone before us and hoping to be yet another link in the chain of knowledge and understanding to those who might come after. With regard to understanding, however, we close with a quotation from the great Jewish scholar Moses Maimonedes, from his *Guide to the Perplexed:*

> *My son, so long as you are engaged in studying the Mathematical Sciences and Logic, you belong to those who go round about the palace in search of the gate...When you understand Physics, you have entered the hall; and when, after completing the study of Natural Philosophy, you master Metaphysics, you have entered the innermost court, and are with the king in the palace. You have attained the degree of the wise men, who include men of different grades of perfection. There are some who direct all their mind toward the attainment of perfection in Metaphysics, devote themselves entirely to God, exclude from their thought every other thing, and employ all their intellectual faculties in the study of the Universe, in order to derive therefrom a proof for the existence of God, and to learn in every possible way how God rules all things; they form the class of those who have entered the palace, namely the class of prophets.*

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Nomenclature

a ..... Pile Shaft Elasticity Constant, 1/sec^2

b ..... Pile Shaft Dampening constant, 1/sec

c ..... Acoustic Speed of Pile Material, m/sec

c_{ENR} ..... EN Formula Constant = 1 for Drop Hammers, 0.1-0.3 for Steam Hammers

c_s ..... Acoustic Speed of the Soil, m/sec

c' ..... Cushion Dampening/Hammer Impedance Ratio

d ..... Pile Inside Diameter, m

d' ..... Pipe Pile Diameter Ratio

$\hat{d}$ ..... Pile Shaft Damping and Elasticity Ratio

e_f ..... Correction Factor, generally 2.5

e_0 ..... Correction Factor for Pile Type

$f(t)$ ..... Displacement Function at Pile Top, m

$\hat{f}(t)$ ..... Inverse Laplace Transform of Pile Top Forcing Function

$f(x)$ ..... Initial or Momentary Displacement Distribution in Pile, m

$f(x-ct), g(x+ct)$ ..... Functions of x and t which possess continuous second derivatives, m

$g(x)$ ..... Initial or Momentary Velocity Distribution in Pile, m/sec

$\hat{g}(t)$ ..... Inverse Laplace Transform of Pile Response Function

i ..... $\sqrt{-1}$

k ..... Soil Shaft Spring or Elastic Constant per Unit Area, N/m^3

k_0 ..... Coefficient Based on the Pile, Shaft Soil Dampening, and Shaft Soil Elasticity

k_1, k_2 ..... Coefficients for Pile Top, Both Not Equal to Zero

k_t ..... Soil Toe Spring or Elastic Constant per Unit Area, N/m^3

l_1, l_2 ..... Coefficients for Pile Toe, Both Not Equal to Zero

Δm ..... Differential Mass, kg

m, n ..... Indices for Fourier and Power Series

$\hat{m}$ ..... Mass of Driving Accessory for Pile Hammer, kg

m' ..... Pile Cap/Ram Mass Ratio

r_g ..... Geometry Ratio of Pile

r_t .... Pile Toe Radius, m

$r(z)$ .... Radius of Soil Mass Below Pile Toe, m

s, s_n .... Laplace Transform Variable

s_h .... Average Penetration Per Blow under Last Few Blows, inches

t .... Time from Zero Point, seconds

t' .... Time from Transition Point t = L/c or t=δ, seconds

$u(x,t), u(x,t'), u(x,\omega)$ .... Displacement of Pile Particle, m

u_1 .... Displacement of the Pile Below the Soil Surface, m

u_2 .... Displacement of the Pile Above the Soil Surface, m

x .... Distance from Pile Top, m

x_r .... Ram Displacement, m

x_t .... Pile Top Displacement, m

Δx .... Differential Distance, m

z .... Distance Below Pile Toe, m

$\hat{z}$ .... Bessel Function Argument for Damped Case

A .... Cross-Sectional Area of Pile, m²

$A_{n1}, A_{n2}, A_{n3}, A_{n4}, A_{n5}$ .... Constants Based on Integration by Weighted Residuals

A

A_1, A_2, B_1, B_2 .... Coefficients Based on Physical Characteristics of System

C .... Cushion Dampening Coefficient, N-sec/m

$C_1, C_2, C_3 \ldots Cn$ .... Constants or Fourier Coefficients

D .... Pile Outside Diameter, m

E .... Pile Young's Modulus of Elasticity, Pa

E_s .... Soil Young's Modulus of Elasticity, Pa

$F, F(x,t), F(x,\omega)$ .... Pile Force, N

$F_0, F_0(t)$ .... Force at Pile Top (x = 0), N

F_e .... Soil Elastic Resisting Force, N

F_v .... Viscous Resisting Force of the Soil, N

$F(s)$ .... Laplace Transform of Pile Top Displacement Function

$\hat{F}(s)$ .... Laplace Transform of Pile Top Forcing Function

G_s .... Soil Shear Modulus of Elasticity, Pa

$\hat{G}(s)$ .... Laplace Transform of Pile Response Function

$H(t), H\left(t-\frac{x}{c}\right)$ .... Heaviside Step Function

H_e .... Effective Fall of Hammer, ft.

K .... Cushion Material Spring Constant, N/m

K_o .... Rebound of the Pile Top, m

K_p .... Rebound of the Pile During Driving, m

K_s .... Rebound of the Pile Toe, m

L .... Length of Pile, m

M .... Mass of Pile Hammer Ram or Cable End Mass, kg

N .... Average "N" Value for the Pile Shaft

P .... Pile Surface Perimeter, m

$P(s)$ .... Laplace Transform for Pile Top Force

R_a .... Allowable Load on Pile, Pounds

R_d .... Pile Bearing Capacity, N

S .... Maximum Displacement of the Pile During Driving, m

$T(t)$ .... Time Function in Separation of Variables

$U, U(x,s)$ .... Laplace Transform of Pile Displacement

V_0 .... Initial Velocity of Pile Hammer Ram or Cable System, m/sec

$V_t(s)$ .... Laplace Transform for Pile Top Velocity

W_h .... Ram Weight of Hammer, Pounds

$X(x)$ .... Distance Function in Separation of Variables

$X_t(s)$ .... Laplace Transform of Pile Top Displacement

Z .... Pile Impedance, N-sec/m

Z_h .... Pile Hammer Impedance, N-sec/m

Z_t .... Pile Toe Impedance, N-sec/m

Z' .... Pile-Hammer Impedance Ratio

α_0 .... Pile Top Consolidation Variable, 1/sec

$\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6$ .... Consolidation Constants for Pile Top Forces

$\hat{\alpha}$ .... Decay Constant for Damped Wave Equation

β, β_m, β_n .... Constants or Eigenvalues

β_0 .... Pile Top Consolidation Variable

$\gamma_1, \gamma_2, \gamma_3$ .... Consolidation Constants for Pile Top Forces

δ .... Time of Square Wave Simplified Impulse, sec.

$\lambda, \lambda', \lambda_m, \lambda_n$ .... Constants or Eigenvalues

λ_t .... Coefficient Based on the Pile Toe Soil Dampening and Elasticity

μ .... Shaft Soil Dampening Coefficient per Unit Area, N-sec/m^3

μ_t .... Soil Toe Dampening Constant per Unit Area, N-sec/m^3

ν .... Poisson's Ratio of Soil

ρ .... Pile Density, kg/m^3

ρ_s .... Soil Density, kg/m^3

$\sigma, \sigma(x,t)$ .... Stress in Pile, Pa

τ .... Dummy Variable for Borel's Theorem, sec.

ω .... Frequency, rad/sec

$\hat{\omega}$ .... Function of Eigenvalues of Damped Wave Equation

I. OVERVIEW OF PILE DYNAMICS

A. Piles in General

Piles driven by impact have been used to support structures on soft soils since the beginning of civilization. Because of the ease of water transportation, cities and entire nations have been located in coastal areas or along rivers. These areas are generally natural places to find weak soils and thus are sites where driven piles are used, although they are not the only places where driven piles are advantageous.

Until the end of the nineteenth century, virtually all piles were wood piles driven using a drop hammer. The drop hammer is a simple device where a weight guided by leaders is hoisted some distance about the pile top and then released to impact the pile, which is generally covered by some kind of driving accessory and cushion material. At the end of the nineteenth century, two events took place that began the serious advance of the practice of driven piles – the development of the automatic pile driver and the development of new materials for piles, specifically the steel pile and the reinforced concrete pile.

B. Pile Dynamics

The analysis of piles during their installation is a separate field altogether from the analysis of the static capacity of piles based on the soil conditions. As a result of this, the analysis of piles moving during installation is considered to be a science of its own, which is referred to as *pile dynamics*. Although pile dynamics can certainly apply to the study of piles under other types of time dependent loading (lateral and axial cyclic loading, earthquake loading, etc.), in all cases the consideration of the physics of the pile movement and the soil response are different from static soil models. This is especially true with time dependent effects and the effects of soil plasticity. In static analysis of foundations in general and piles in particular one desires an elastic response from the soil, and the soil is analysed with this objective; in pile dynamics, a plastic response is necessary for pile penetration.

C. Dynamic Formulae

Probably no branch of civil engineering depends more on the judgment and experience of its practitioners than geotechnical engineering, and this applies nowhere more than with deep foundations in general and driven piles in particular. Until the advent of a reasonable body of theory, all civil engineering was the subject of the engineer's raw experience; in some cases, structural engineers such as Thomas Telford made a virtue out of this (Billington, 1983). However, the advance in the understanding of engineering mechanics made the use of theory inevitable and indeed desirable in the advancement of the design and construction of useful structures. Because of the complexity and variability of ground conditions and soil mechanics, this process proceeded more slowly in geotechnical engineering than with any other branch of civil engineering.

It was inevitable that some kind of theory would be developed to explain the penetration of piles driven by impact and the theory first employed was that of Newtonian impact mechanics, with its assumptions of rigid body mechanics and conservation of momentum or energy. The basics of this theory are given in Jacoby and Davis (1941). The most common of these dynamic formulae is the Engineering News formula, developed by A.M. Wellington and given by the equation

$$R_a = \frac{2W_h H_e}{s_h + c_{ENR}} \quad \text{......} \quad (1)$$

where R_a = Allowable Load on Pile, Pounds
W_h = Ram Weight of Hammer, Pounds
H_e = Effective Fall of Hammer, ft.
s_h = Average Penetration Per Blow under Last Few Blows, inches
c_{ENR} = EN Formula Constant = 1 for Drop Hammers, 0.1-0.3 for Steam Hammers

The dynamic formulae were a reasonable approach to the problem as long as the assumptions of the theory were not extensively violated. To begin with, wood piling are generally short (10-15m) in length; the effects of wave propagation in them are minimal. Moreover they were a simple solution to a complex problem whose other variables (such as the interaction of the soil and the pile during loading) were not much better understood by other means. They gave all parties involved in driven piles additional information, which they did not have before, and they made it possible to make some kind of evaluation on site.

D. Shortcomings of Dynamic Formulae

Although dynamic formulae were used for many years and still are used in a large number of pile specifications, they have some serious shortcomings. These fall into two categories, namely weaknesses in the theory and changes in the application.

Weaknesses in the theory include

- the assumption that the pile is a rigid body
- inadequate modelling of the energy transfer between the hammer and the pile, or the pile and the soil, or effects of the simultaneous occurrence of both
- use of plastic soil model only without any consideration of soil elasticity or radiation dampening

The major change in the application – and the one that first generated interest in the application of wave mechanics in the first place – was the advent of concrete piles, which took place in the late 1890's. Because of the brittle nature of concrete, the possibility of damage during installation was increased; dynamic formulae were not equipped to estimate such stresses accurately. Moreover the use of materials not occurring in nature (such as concrete or steel) enabled foundation engineers to design longer piles than were possible in wood. The longer the pile, the less it could act as a rigid mass and the more its distributed mass and elasticity became an important consideration.

E. Isaacs' (1931) Research

The first observation of stress waves in piles was given by Isaacs (1931). The dynamic formulae had been developed primarily with timber piles in mind; with the growing usage of concrete piles, it became apparent that, because of the length and properties of timber piles, the dynamic formulae (with their assumption that the pile is a rigid mass) would not be sufficient for concrete piles. This became an urgent problem to solve when tension cracking took place in concrete piles. Isaacs started out by reviewing the dynamic formulae. Part of his review included a discussion of the factor of safety, where he made a statement that is still relevant:

> *It should be remembered, however, that these are not true factors of safety, but include a "factor of ignorance." The author suggests that when the ultimate resistance of any pile has been determined, in fixing the factor of safety...the most unfavourable conditions possible in the supporting strata should be judged (the range of conditions possible being narrowed with better knowledge of the subsurface conditions and of the possibility of disturbance from extraneous sources) and a proportion of the factor of safety – a "factor of ignorance" – then allowed in respect to these possible conditions, the manner of determining the ultimate load, and the type of loading to be borne. The remaining proportion of the factor of safety – or* true *margin of safety – should be approximately constant for all classes of loading and foundation conditions involving the same value of loss in case of failure; and the overall factor of safety...will then be equal to the product of the true factor of safety with the "factor of ignorance." (p. 305)*

After this, he described an experiment where rods are impacted against each other in a pendulum setup. As the rods were lengthened, the behaviour of the rods deviated more and more from Newtonian impact theory.

He then went on to do the following:

a) develop an integration technique to solve the basic equations,

b) develop a mathematical model based on the successive transmission and reflection of waves,

c) construct a drafting machine to draw out the solution and thus solve the problem graphically,

d) solve for the stresses and displacements of the pile during driving and

e) develop a set of formulae and charts to make his results accessible for analysis of piles.

In the course of the investigation, Isaacs dealt with a number of questions that would become central to stress wave analysis of piles, including tension stresses in concrete piles, the effect of ram weight (he concluded that to a point a heavier ram reduced tension stresses,) and the effect of cushion material stiffness and drive cap weight. His work also revealed the computational complexity of stress wave analysis, a complexity that ensured the dominance of dynamic formulae in pile analysis (with all of their serious limitations) for another half century.

F. Numerical Solutions

Isaacs' work demonstrated that it was one thing to show the existence of stress waves in piles and quite another to quantify them. In the next section the history of the closed form solution is discussed as background to the subject of the present thesis; however, the actual solution of the wave equation that has become the state of the art for the analysis of pile dynamics is a numerical one.

The seminal work in numerical analysis of wave mechanics of piles was that of Smith (1960). He proposed the use of a first order, finite difference scheme to solve the wave equation. He also proposed (from empirical considerations) that soils were elastic-purely plastic in nature with added viscous damping. Finally he proposed a model for the hammer system that included plasticity in the cushion material and the possibility of distributed mass and elasticity in the hammer itself.

Smith's work was an important start but it took advances in both the technique itself and the computers that ran it to make the wave equation as universal in pile dynamics as it is today. Such advances are documented in Lowery et. al. (1969), Hirsch et. al. (1976), and Goble and Rausche (1976, 1986). The wave equation has also been adapted to use in the estimation of pile capacity and the distribution of that capacity through the CAPWAP method (Rausche et. al, 1985).

Another technique that is presently coming into wide use is the finite element technique (Coutinho et. al., 1988). In contrast to the finite difference techniques, which are "one-dimensional" in their modelling of the pile and soil, finite element techniques model the soil around the pile as a continuum. Deeks (1992) analyses the numerical characteristics of many finite element techniques. In addition to impact hammers, the finite element technique is now being applied to vibratory driving of piles (Leonards et. al., 1995)

II. THEORY AND HISTORY OF THE CLOSED FORM SOLUTION OF THE WAVE EQUATION FOR PILES

A. *Definition of a Closed Form Solution*

The object of this thesis is to develop a closed form solution of the wave equation for piles. Because of the wide variety of solutions for differential equations, it is first necessary to define what kinds of solutions are "closed form." For the purposes of this thesis, a closed form solution of the wave equation for a pile is one where the solution of the governing differential equation is integrated directly, whether to an equation or system of equations or to an infinite series, without resorting to numerical methods.

B. *The Wave Equation In General*

The classical one-dimensional wave equation is given by the formula

$$u_{tt}(x,t) = c^2 u_{xx}(x,t) \quad (2)$$

where $u(x,t), u(x,t'), u(x,\omega)$ = Displacement of Pile Particle, m
x = Distance from Pile Top, m
t = Time from Zero Point, seconds
c = Acoustic Speed of Pile Material, m/sec

For longitudinal vibrations, the constant c is the acoustic speed of the material of the bar, given by the equation

$$c = \sqrt{\frac{E}{\rho}} \quad (3)$$

where E = Pile Young's Modulus of Elasticity, Pa
ρ = Pile Density, kg/m^3

Equation (2) is a hyperbolic, second order partial differential equation. Although in this form the wave equation cannot be directly applied to most real piling due to soil damping and elasticity along the side of the pile, it remains the basic equation of motion for one dimensional systems.

Following is a presentation on the various types of solutions that have been performed in the past specifically for piling. Most of this material is adapted from Warrington (1996).

C. *Types of Closed Form Solutions*

In order to attempt to make sense out of the work that has been carried out, five categories of solutions were considered:

1) solutions relating to piles of semi-infinite length,
2) solutions using the method of images,
3) Fourier series types of solutions,

4) Laplace transform solutions, and

5) solutions specific to vibratory hammers with excitation by a single frequency.

In some cases the formulas reproduced have notation changes to arrive at a consistent notation system in the thesis.

1. Semi-Infinite Pile Solutions

It may seem strange to begin a discussion of wave equation solutions for piles with a solution type that strictly speaking only exists in theory. However, the consideration of this type of solution is important in the understanding of the theory of wave mechanics in piles of a finite length as well.

a) Theory of Semi-Infinite Pile Solution

Consider Equation (2), and assume that the bar has but one boundary at x = 0. Further assume that the bar begins with no initial displacement or velocity, i.e.,

$$u(x,0) = f(x) = 0 \qquad (4)$$

and

$$u_t(x,0) = g(x) = 0 \qquad (5)$$

where $f(x)$ = Initial or Momentary Displacement Distribution in Pile, m
$g(x)$ = Initial or Momentary Velocity Distribution in Pile, m/sec

Assume also that the bar is excited at the boundary in such a way that the displacement of the end of the bar can be defined as

$$u(0,t) = f(t) \qquad (6)$$

where $f(t)$ = Displacement Function at Pile Top, m

Kreyszig (1993) shows that, if the Laplace transform of Equation (2) is taken with respect to t, the initial conditions of Equations (4) and (5) are used, and the infinite boundary condition

$$\lim_{x \to \infty} u(x,t) = 0 \qquad (7)$$

is applied, then the Laplace transform is

$$U(x,s) = F(s)e^{-\frac{sx}{c}} \qquad (8)$$

where $U, U(x,s)$ = Laplace Transform of Pile Displacement
$F(s)$ = Laplace Transform of Pile Top Displacement Function
s, s_n = Laplace Transform Variable

The inverse transform of this is

$$u(x,t) = f\left(t - \frac{x}{c}\right)H\left(t - \frac{x}{c}\right) \qquad (9)$$

where $H(t), H\left(t-\frac{x}{c}\right)$ = Heaviside Step Function

This result shows that, for completeness, it is necessary to include the Heaviside step function for a wave advancing along a bar with no previous excitation. This insures that any portion of the bar ahead of the advancing wave is at rest mathematically as it is physically. The lack of this Heaviside function is a common fault with wave equation solutions.

However, setting aside the Heaviside step functions and the Dirac delta function derivatives, the derivatives with respect to distance and time are

$$u_x = -\frac{f'\left(t-\frac{x}{c}\right)}{c} \quad \text{(10)}$$

and

$$u_t = f'\left(t-\frac{x}{c}\right) \quad \text{(11)}$$

Solving both of these equations with respect to the primed derivatives and equating,

$$u_x = -u_t \frac{1}{c} \quad \text{(12)}$$

Multiplying through by the product of Young's modulus and the cross sectional area of the pile,

$$EAu_x = -u_t \frac{EA}{c} \quad \text{(13)}$$

where A = Cross-Sectional Area of Pile, m²

Since from elasticity (assuming the sign convention of compressive stresses as positive)

$$\sigma = -Eu_x \quad \text{(14)}$$

where $\sigma, \sigma(x,t)$ = Stress in Pile, Pa

the left hand side of Equation (13) represents the stress in the pile multiplied by the area, or the pile force at any given point in the pile.

Turning to the right hand side, the pile impedance is defined by the quantity

$$Z = \frac{EA}{c} = \sqrt{\rho E}\,A \quad \text{(15)}$$

where Z = Pile Impedance, N-sec/m

Substituting Equations (14) and (15) into Equation (13),

$$\sigma A = F = Zu_t(x,t), t < \frac{x}{c} \quad \text{(16)}$$

where $F, F(x,t), F(x,\omega)$ = Pile Force, N

From this the pile impedance is obviously

$$Z = \frac{F}{u_t(x,t)} \qquad (17)$$

This result is a general type of solution; it is independent of the pile top configuration, i.e., whether the hammer is cushioned or cushionless, etc. This relationship can be used to calculate the force-time or displacement-time relationship for these various types of systems.

b) *Application of Semi-Infinite Theory to Piles*

The main result of this special case is that it is possible to model the semi-infinite pile as a velocity dependent "dashpot" with the impedance as the "dampening." This enables the analysis of the force-time characteristics of a simplified hammer-pile system by using ordinary differential equations, be they analysed in closed form or numerically. These will be developed later when the actual solution of the wave equation is presented. This is a useful result, especially for long piles where the reflections do not return sufficiently quickly to interact with the impact itself. The following are summaries of the various solutions to the problem:

Parola (1970): He first analysed the infinite pile model in a systematic way, formulating variables and computing them using an analogue computer. He also attempted to apply the results of the hammer-pile interaction at the top of the pile to the response of the soil. His work was confirmed and expanded by Warrington (1987) using numerical integration and including cushionless hammers as well as cushioned ones.

Van Koten et. al. (1980): They developed a semi-infinite pile solution which included visco-elastic shaft resistance; the resulting equation of displacement includes modified Bessel functions. The model is then converted to a finite pile model using the method of images.

Deeks (1992): This was a comprehensive solution of the equations of motion for the semi-infinite pile in true closed form with application to actual case histories. His main objective was to use these results to evaluate numerical methods of analysis for piles, an important application for closed form methods. Deeks also considered losses in the cushion material as viscous losses, which gave the possibility of analysing variations in the loading rate of the cushion material, as opposed to the static one presently used with finite difference wave equation analyses.

Parker (1996): This represents another attempt to make a relatively simple correlation between the results of semi-infinite pile theory at the top of the pile to the soil response. He only considered the impact of a rigid ram with the pile top, using empirical factors to relate this to actual observed hammer performance. Since his main interest is in offshore piling, he was able to concentrate all of the resistance at the pile toe with relatively minor loss in accuracy.

2. Method of Images

It can be shown (Wylie, 1979) that Equation (1) can be solved in the form

$$u(x,t) = f(x-ct) + g(x+ct) \qquad (18)$$

where $f(x-ct), g(x+ct)$ = Functions of x and t

This solution is in the so-called "d'Alembert Form." Using this type of solution the wave equation can be conceptualised as an odd periodic function, the period being defined by the length of the vibrating rod. The method of images is based on this concept, as it seeks to solve the wave equation by considering the effects of the periodic transmissions and reflections of the stress wave generated by the hammer along the pile. In doing this it attempts to avoid the complexities of other closed form solutions.

Glanville et. al (1938): This study was one of the first comprehensive studies on stress waves in piles in general. Equation (2) was used to develop equations to estimate the stress in the pile during driving, using the method of images. Because of the complexity of the equations, the results were reduced to a series of charts where a quantity of dimensionless stress was plotted against the ratio of hammer weight to pile weight. The charts could then be used to estimate pile stresses and resistance. The charts were applicable to concrete piles only, which was a serious limitation to such solutions, because they were applicable to a limited universe of piles.

In addition to developing a solution to the wave equation, the authors continued Isaacs' (1931) work in addressing technical issues and experimental techniques that have enduring interest in pile dynamics. These included instrumentation and data collection of stresses and forces in piles, including remote data gathering through "portable" equipment in a trailer, further research on the effect of the hammer cushion on the generation and effect of the pile stress wave (these were included in the analytical work,) drop tower testing on cushion material to determine the cushion stiffness, and further work on the relationship of ram weight to pile weight and cross section.

Hansen and Denver (1980): The authors proposed a solution of the wave equation using the method of images but also included a visco-elasto-plastic model for the shaft and toe friction. The behaviour of the pile was then calculated by successive applications of the stress wave. The authors also applied the model to pile discontinuities, both pile defects and changes in cross sectional area. The method was applied to a numerical integration technique for the analysis of actual piles.

Uto et. al. (1985): In this paper, a pile driving formula based on the solution of the wave equation was proposed. Neglecting shaft friction, toe damping, and making other assumptions concerning the displacement of the pile top and toe, the equation for the bearing capacity of the pile was given by the equation

$$R_d = \frac{AE}{2e_0}(S + K_s + 2K_0) + \frac{NPL}{e_f} \quad \text{.......... (19)}$$

or

$$R_d = \frac{AEK_p}{Le_0} + \frac{NPL}{e_f} \quad \text{.......... (20)}$$

where e_0 = Correction Factor for Pile Type
S = Maximum Displacement of the Pile During Driving, m
K_s = Rebound of the Pile Toe, m
K_o = Rebound of the Pile Top, m
N = Average "N" Value for the Pile Shaft
P = Pile Surface Perimeter, m
L = Length of Pile, m
e_f = Correction Factor, generally 2.5
K_p = Rebound of the Pile During Driving, m

Although these equations certainly used the method of images as a starting point, it is important to note that many "empirical" factors were taken into account to arrive at these formulas. The second term in each equation is not based on wave mechanics but Meyerhof's formulae for shaft friction. Also, since both maximum dynamic set and rebound are required, these equations are best applied in the field for verification of pile and hammer performance. Tada et. al. (1985) supplied additional theory to arrive at these equations and at the same time applied this equation to a hydraulic impact hammer, where they achieved good correlation in tests.

3. Fourier Series Solutions and Fourier Transformations

Fourier series solutions are those which utilize an infinite series of orthogonal eigenfunctions to describe the motion and the stress on the pile. Fourier series are described in detail in Tolstov (1962). More details on this subject are given in Petrovskii (1967), and a history of its derivation is given by McCurdy (1993). Included in these solutions are those which use Fourier integrals, Fourier transforms and inverse Fourier transforms.

a) Example of Fourier Series Solution -- the Prescott-Laura Problem

When the wave equation is solved at its most elementary level, the method of Fourier series is generally the first method to be used. However, because of the initial and boundary conditions, Fourier series have only recently been applied to piling.

The procedure for elementary solutions of the wave equation using Fourier series can be found in virtually any textbook on partial differential equations. Because of the use of this technique in the solution proposed in this thesis, a case described by Prescott (1924) and numerically analysed by Laura et. al. (1974) is analysed first. This case shows some of the difficulties with more advanced problems and their solutions. The Maple V worksheet used for this derivation is found in Appendix A.

The system considered is that shown in Figure 1. This system consists of a cable travelling at a uniform velocity V_0 and with a mass M attached at its length L. At time zero the top (x = 0) of the cable is suddenly stopped; it is necessary to analyse the motion.

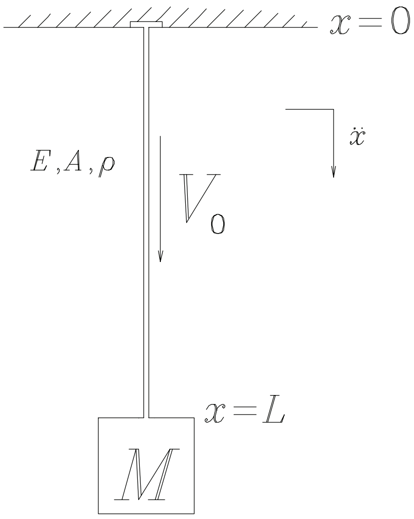


Figure 1 Diagram for Prescott-Laura Problem

The governing equation is Equation (2). The boundary conditions are

$$u(0,t)=0 \qquad (21)$$

and

$$-EAu_x(L,t)=Mu_{tt}(L,t) \qquad (22)$$

where M = Mass of Pile Hammer Ram or Cable End Mass, kg

The initial conditions are assumed to be Equation (4) and

$$u_t(x,0)=g(x)=V_0 \qquad (23)$$

where V_0 = Initial Velocity of Pile Hammer Ram or Cable System, m/sec

Using the usual method of the separation of variables, the solution is assumed to be in the form

$$u(x,t)=X(x)T(t) \qquad (24)$$

where $X(x)$ = Distance Function in Separation of Variables
$T(t)$ = Time Function in Separation of Variables

Substituting this into Equation (2), and separating the variables in the usual manner, the solution for X(x) is

$$X(x)=C_1\cos(\beta x)+C_2\sin(\beta x) \qquad (25)$$

where $C_1, C_2, C_3 \ldots Cn$ = Constants or Fourier Coefficients
β, β_m, β_n = Constants or Eigenvalues

In order to meet the requirements of the boundary condition given in Equation (21),

$$C_1=0 \qquad (26)$$

The solution for the time function is

$$T(t) = C_3\cos(\beta ct) + C_4\sin(\beta ct) \quad (27)$$

Again to satisfy the requirements of the initial condition in Equation (4),

$$C_3 = 0 \quad (28)$$

The solution for the displacement is

$$u(x,t) = C_n\sin(\beta x)\sin(\beta ct), C_n = C_2C_4 \quad (29)$$

Now the remaining initial condition and boundary condition are considered. First, differentiating Equation (29),

$$u_t(x,t) = \beta cC_n\sin(\beta x)\cos(\beta ct) \quad (30)$$

Substituting t = 0, the initial velocity profile is a sine function. Since the initial velocity is uniform, a Fourier series can be generated which when summed adds up to

$$u_t(x,0) = V_0 = c\sum_{n=1}^{\infty}\beta_nC_n\sin(\beta_nx) \quad (31)$$

where m,n = Indices for Fourier and Power Series

In view of the necessity of a Fourier series to represent the entire solution of the initial conditions, such a series is also needed for the solution.

Turning to the boundary condition given in Equation (22), substituting Equation (29) into Equation (22) and dividing out the Fourier coefficients,

$$-\beta_nEA\cos(\beta_nL)\sin(\beta_nct) = -\beta_n^2c^2M\sin(\beta_nL)\sin(\beta_nct) \quad (32)$$

and this solves to

$$\frac{\sin(\beta_nL)}{\cos(\beta_nL)} = \tan(\beta_nL) = \frac{1}{\beta_n}\frac{EA}{Mc^2} \quad (33)$$

This boundary condition introduces two complications. The first is that the values for β are transcendental and can only be solved by successive solutions of Equation (33) over each interval

$$\pi\left(n-\frac{3}{2}\right) < \beta_nL < \pi\left(n-\frac{1}{2}\right), n = 1,2,3,\ldots\infty \quad (34)$$

This produces an infinite set of eigenfunctions.

The second and potentially more serious problem is that, for the Sturm-Liouville problem to be guaranteed to be solved by a complete orthogonal set, the boundary conditions must be such that, in this case,

$$k_1u(0,t) + k_2u_x(0,t) = 0 \quad (35)$$

and

$$l_1 u(L,t) + l_2 u_x(L,t) = 0 \quad (36)$$

where k_1, k_2 = Coefficients for Pile Top, Both Not Equal to Zero
l_1, l_2 = Coefficients for Pile Toe, Both Not Equal to Zero

Because of Equation (22), there is no set of constants that satisfies Equation (36). Therefore, in the course of determining the eigenvalues and eigenfunctions, combinations of functions of β_n and β_m where n≠m as well as n=m must be considered. This can be done while determining values of C_n for all terms of the series.

First, multiplying both sides of Equation (31) as follows,

$$V_0 \sin(\beta_m x) = c \sin(\beta_m x) \sum_{n=1}^{\infty} \beta_n C_n \sin(\beta_n x) \quad (37)$$

Integrating both sides with respect to x (the right side termwise),

$$\int_0^L V_0 \sin(\beta_m x) dx = c \int_0^L \sin(\beta_m x) \sum_{n=1}^{\infty} \beta_n C_n \sin(\beta_n x) dx \quad (38)$$

Performing the integration on the right side, this results in

$$\int_0^L V_0 \sin(\beta_m x) dx = c \sum_{n=0}^{\infty} \frac{-C_n \beta_n (\beta_n \cos(\beta_n L) \sin(\beta_m L) + \beta_m \sin(\beta_n L) \cos(\beta_m L))}{(\beta_n - \beta_m)(\beta_m + \beta_n)} \quad (39)$$

Substituting Equation (33) for the resulting cosines,

$$\int_0^L V_0 \sin(\beta_m x) dx = c \sum_{n=0}^{\infty} \frac{C_n \beta_n \left(\beta_n^2 \sin(\beta_n L) \sin(\beta_m L) - \beta_m^2 \sin(\beta_n L) \sin(\beta_m L)\right) \frac{Mc^2}{EA}}{(\beta_m^2 - \beta_n^2)} \quad (40)$$

and this reduces to

$$\int_0^L V_0 \sin(\beta_m x) dx = -\frac{Mc^2}{EA} \sin(\beta_m L) c \sum_{n=0}^{\infty} C_n \beta_n \sin(\beta_n L) \quad (41)$$

Now, recognizing that the summed term times the acoustic speed of the material is in fact the initial velocity at x = L, given in Equation (23), and making the appropriate substitution,

$$\int_0^L V_0 \sin(\beta_m x) dx = -V_0 \frac{Mc^2}{EA} \sin(\beta_m L), m \neq n \quad (42)$$

Now the case where m=n is considered. Making this substitution into Equation (37) and integrating, the result is

$$\int_0^L V_0 \sin(\beta_m x) dx = \frac{cC_m}{2} \left(-\cos(\beta_m L) \sin(\beta_m L) + \beta_m L\right) \quad (43)$$

Substituting Equation (33) into Equation (43),

$$\int_0^L V_0 \sin(\beta_m x)dx = \frac{c\beta_m C_m}{2}\left(\frac{Mc^2 \sin(\beta_m L)}{EA} + L\right), m = n \quad (44)$$

Adding the two cases, the entire solution for the velocity integral is

$$\int_0^L V_0 \sin(\beta_m x)dx = \frac{c\beta_m C_m}{2}\left(\frac{Mc^2 \sin(\beta_m L)}{EA} + L\right) - V_0 \frac{Mc^2}{EA}\sin(\beta_m L) \quad (45)$$

Finally evaluating the integral on the left hand side,

$$V_0\left(\frac{-\cos(\beta_m L)+1}{\beta_m}\right) = \frac{c\beta_m C_m}{2}\left(\frac{Mc^2 \sin(\beta_m L)}{EA} + L\right) - V_0 \frac{Mc^2}{EA}\sin(\beta_m L) \quad (46)$$

Solving for C_m (now C_n) and simplifying using Equation (33),

$$C_n = \frac{2V_0}{c\beta_n(\cos(\beta_n L)\sin(\beta_n L) + \beta_n L)} \quad (47)$$

The final, complete solution for the displacement for all time after zero is

$$u(x,t) = \sum_{n=1}^{\infty} \frac{2V_0 \sin(\beta_n x)\sin(\beta_n ct)}{c\beta_n(\cos(\beta_n L)\sin(\beta_n L) + \beta_n L)} \quad (48)$$

The velocity, strain and stress can be derived accordingly.

This problem was analysed to both demonstrate the possibility of obtaining a solution using Fourier series without boundary conditions guaranteed to obtain a complete orthogonal set and to show how transcendental eigenvalues can result from this kind of problem. Laura et. al. (1974) analysed this result numerically and confirmed its convergence.

The critical point in the derivation takes place at Equation (42), where the result for n≠m is reduced to a simple expression. The simplicity is assisted by the fact that the velocity is uniform along the entire cable or rod; thus, the velocity at x = L is the uniform velocity. It is interesting to note that, if there was a velocity profile in the cable or rod such that the velocity was zero at t=0 and x = L, then this term would disappear entirely, and the solution would be in reality orthogonal according to the definition.

b) Applications of Fourier Series to Piling

Following are descriptions of this type of solution for piles:

Wang (1988): This study solved the wave equation directly using the method of weighted residuals. Using a plastic shaft resistance model, an elastic toe model, a uniform pile velocity at zero time and initial displacement, and no initial compression of the pile, the response was computed by the equation

$$u(x,t) = \sum_{n=1}^{\infty}\left(A_{n1}\cos\left(\frac{n\pi ct}{L}\right) + A_{n2}\sin\left(\frac{n\pi ct}{L}\right)\right)\sin\left(\frac{n\pi x}{L}\right) + A_{n3}x^2 + A_{n4}x + A_{n5} \quad (49)$$

where $A_{n1}, A_{n2}, A_{n3}, A_{n4}, A_{n5}$ = Constants Based on Integration by Weighted Residuals

Wang went on to use this model not directly but as part of a finite difference scheme. This enables him to overcome the greatest weakness of the model, namely the assumption of a uniform pile velocity at impact, because in fact (assuming all velocity in the pile has gone to zero from the previous blow) only the particles at the pile top have any velocity at the time of impact.

Espinoza (1991: This study attempted to bridge the gap between the analysis of driven piles and other vibrating structures (which are usually analysed spectrally rather than in real time) by first determining the displacement and force as a function of the spectral response of the system; the equations are

$$u(x,\omega) = F_0 \frac{e^{-ik_0 x} + \lambda_t e^{-ik_0(2L-x)}}{(1-\lambda_t e^{i2k_0 L}) ik_0 EA} \quad \text{.......... (50)}$$

and

$$F(x,\omega) = F_0 \frac{e^{-ik_0 x} + \lambda_t e^{-ik_0(2L-x)}}{(1-\lambda_t e^{i2k_0 L})} \quad \text{.......... (51)}$$

where $F_0, F_0(t)$ = Force at Pile Top (x = 0), N

ω = Frequency, rad/sec

i = $\sqrt{-1}$

k_0 = Coefficient Based on the Pile, Shaft Soil Dampening, and Shaft Soil Elasticity

λ_t = Coefficient Based on the Pile Toe Soil Dampening and Elasticity

A Fourier transform was applied to the top of the pile to transform the hammer impact force into a spectrum of forces, and an inverse transform is necessary to obtain the force-time and displacement-time histories of the cases studied. The model was compared with finite difference and finite element techniques. The model was found to be most useful when pile displacements were small, because the soil was modelled visco-elastically without consideration of plasticity.

4. Solutions using Laplace Transforms

Laplace transforms can be used to solve the wave equation, both damped and undamped, although if a boundary is assumed at the toe the inverse transform can be very difficult.

Zhou and Liang (1996): This solution assumed an arbitrary force-time relationship at the pile top, a fixed end at the pile toe and a Telegrapher's form of the wave equation. The ultimate result was an infinite series of convolution integrals; however, since reflections were eliminated for the purposes of analysing the pile toe for 0 < t < 2L/c, the result was the semi-infinite pile solution for the damped wave equation at the pile top.

5. Solutions for Vibratory Hammers

Although the main point of interest here is with impact hammers, vibratory hammers have been the subject of serious investigation as well. The analysis of vibratory hammers has the advantage of dealing with a forcing frequency and the steady state solution. In this case the solution generally is in terms of only one frequency, which eliminates the infinite series.

Hejazi (1963): This work is an extensive analysis of the theoretical aspects of vibratory pile driving. Part of this work consisted of the derivation of equations of an elastic rod, penetrating the soil and subject to vibrations at the top. He divided the pile into two parts; the part penetrating the soil and the part which is above the soil.

The steady state displacement for each of these parts is

$$u_1 = \left(C_1 \cos\left(A_1 x\right) + C_2 \sin\left(A_1 x\right)\right) e^{iB_1 t} \quad \text{(52)}$$

$$u_2 = \left(C_3 \cos\left(A_2 x\right) + C_4 \sin\left(A_2 x\right)\right) \cos\left(B_2 t\right) \quad \text{(53)}$$

where u_1 = Displacement of the Pile Below the Soil Surface, m
u_2 = Displacement of the Pile Above the Soil Surface, m
A_1, A_2, B_1, B_2 = Coefficients Based on Physical Characteristics of System

The complexity of the solution led Hejazi to recommend using a rigid pile type solution for vibratory pile driving.

Smart (1970): In his analysis of vibratory piles, he proposed a model with a sinusoidal force at the top and viscous toe resistance (toe impedance) at the bottom. Using a d'Alembert type of solution, the force at any time was given by the equation

$$F = F_0 \frac{Z_t \cos\left(\frac{\omega(L-x)}{c}\right) + iZ \sin\left(\frac{\omega(L-x)}{c}\right)}{Z_t \cos\left(\frac{\omega L}{c}\right) + iZ \sin\left(\frac{\omega L}{c}\right)} \quad \text{(54)}$$

where Z_t = Pile Toe Impedance, N-sec/m

6. Observations on Existing Closed Form Solutions

In preparing the formulation of a closed form solution to analyze stress waves in piles, some historical observations are in order.

- Most closed form solutions, especially those after Isaacs (1931) and Glanville et. al. (1938) are not really comprehensive, i.e., they are not intended to be used for the prediction of pile behaviour in its totality. They are designed to meet specific requirements. This is especially true for semi-infinite pile solutions, although efforts have been made to broaden these as well.

- The d'Alembert or method of images types of solutions are the most common but in most cases they do not consider the shaft friction. This is a serious omission considering the application of piles, although for offshore piles driven from the surface it may be more useful. Also, the successive reflections from the boundaries can be very complicated.

- Fourier type solutions are relatively rare because of the infinite nature of the equations but with the growth of computer capabilities they have more potential. In any case attempts to derive formulae and methods by other methods produced equations that in practical terms are little simpler than Fourier series.

- The constitutive modelling of the soils in closed form solutions has traditionally been rudimentary. It is very likely that, to accurately model the soil in these solutions, a different approach will have to be taken.

- Although the semi-infinite pile model seems to be a very special case, it is useful because it a) allows the analysis of a very important part of the system using relatively simple equations and b) and it gives a simple method of computing pile and hammer loads and stresses for a wide variety of cases. It is interesting to note that much of the work on this model after Parola (1970) has been primarily directed towards piles used in offshore platforms. Especially for piles driven from the surface of the water, the force-time profile generated at the top and the soil response at the bottom are essentially "decoupled" by the intervening length of the pile, and so the semi-infinite pile model has its best application where the piles are the longest.

III. PARAMETERS FOR A SUCCESSFUL CLOSED FORM SOLUTION

A. *Rationale for Closed Form Solutions*

Perhaps the first question that needs to be answered is simply this – why is a closed form solution of the wave equation for piles necessary?

Given the state of analysis techniques and the nature of the environment into which piles are driven, it is unlikely that closed form solutions will ever displace numerical methods *in toto*. Numerical methods are most capable in modelling the discontinuities and non-linearities (especially in the soil response) that are inherent in pile-hammer-soil systems.

These capabilities, unfortunately, have masked many of the shortcomings which are inherent in numerical methods, such as numerical instability, loss of accuracy due to the dividing up the system into finite sized units, and inadequate modelling of system elements, which is more easily "buried in the code" of a numerical technique than in a closed form solution.

In most cases, developers of numerical techniques for other applications (stress analysis, heat transfer, etc.) have had the benefit of closed form solutions to check their modelling for at least the simplest cases. However, in the case of piles the complexity of the solution encouraged the practitioners to develop numerical methods without recourse to closed form solutions, even though solutions that existed at the beginning of the development of numerical methods were no more difficult to convert to code than the numerical methods themselves.

Based on this and other considerations, there are several useful applications for these solutions; they are as follows:

1) Parametric Studies: Although finite difference programs can be used for parametric studies (Meseck, 1985,) most finite difference codes are not designed to be used parametrically, but on a "job to job" basis. Furthermore, any trends to be derived from these are either strictly qualitative in nature or reduced from numerical analysis, a technique more suitable for empirical data. Parametric studies are useful for such tasks as equipment design and general pile specifications.

2) Verification of Numerical Methods: In spite of the popularity of numerical methods, it has been shown that there are computational difficulties associated with them (van Weele and Kay, 1984; Davis and Phelan, 1988). Closed form solutions are a valuable tool in the evaluation of numerical methods for such difficulties, as illustrated by Deeks (1992). Deeks' dissertation is probably the most comprehensive comparison to date of closed form solution to numerical methods.

3) Advances in Computer Software: One of the major reasons that closed form solutions were pursued in the first place was to reduce the computations involved for pile loads and stresses using stress-wave analysis to a manageable level. The complexities of the problem, however, have made that goal unrealised, even though the programming requirements of closed form solutions are not excessive by modern standards. With the advance of mathematical software, however, the potential exists of generating a solution to this problem in closed form without recourse to specialized software.

B. Description of the Physical System

The rationale for the closed form solution having been described, the next step is to determine the requirements for a successful closed form solution.

Consider the pile/hammer/soil system described in Figure 2.

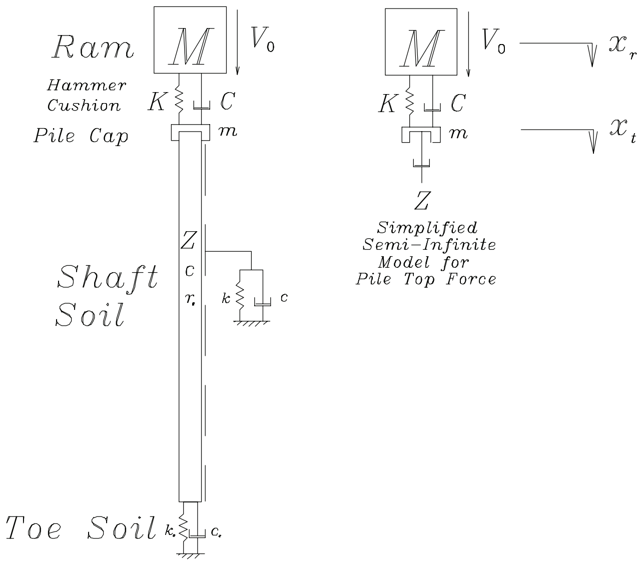


Figure 2 Hammer-Pile-Soil System

1. Hammer

The hammer is considered to be a concentrated mass which contacts the hammer cushion at t=0 with velocity V_0. The cushion is considered to be linear; the cushion can have material damping that can be modelled as a viscous damper. It is the cushion that allows the hammer to be modelled as a concentrated mass; if the cushion is removed, or there is an anvil between hammer and cushion, or the ram is long, then the ram's distributed mass and elasticity must be taken into account. The hammer cushion sits atop the driving accessory, which adapts the hammer and cushion to the pile. Because it is generally short and rigid relative to the cushion and pile it is modelled as a concentrated mass. The force generated by the contact of ram and cushion travels downward into the pile. If diesel hammers are being analysed, the explosive force of the combustion must also be taken into account.

2. Pile

The pile is assumed to be made of a homogeneous, isotropic material. Its cross section is assumed to be sufficiently small relative to its length so that radial and tangential effects are negligible and the pile can thus be modelled in one dimension. Material damping in the pile is considered negligible, especially when considered relative to the soil. (With numerical methods, material damping is added to the system for numerical stability, not to model the material itself.)

Although piling can be variable in length, cross-sectional area and material, piles analysed in this thesis were of uniform cross section and material, and also had a uniform relationship between the cross sectional area and the perimeter of the pile.

3. Soil

Modelling of soils for pile analysis can be done in one of two ways. The first is what can be called "whole soil modelling," i.e. modelling a large mass of the soil around the pile. This is generally done in conjunction with finite element analyses. In many cases the soil is assumed to be an isotropic, homogeneous mass, although plasticity can be included if the finite element code allows it. Such modelling is described by Whittle (1993), although the dynamics of pile penetration are different. Although formally this is a more satisfactory method of modelling the soil, it involves some complications that generally put it out of reach of closed form solution (Deeks, 1992). The other technique is to model the soil response – both along the pile shaft and at the top – by a visco-elastic model. The viscous part largely models radiation to the soil mass (Randolph and Simons, 1986), and the elastic portion models the elastic component of the soil resistance. Depending upon the model and the location of the soil resistance, one or both of these can be limited by the static resistance of the pile. Inertial effects can also be added to the soil model as well. This model is most commonly used with the finite difference techniques such as the Smith model and its progeny, and is used for this study.

C. Construction of the Differential Equation

1. Basic Considerations

In order to solve any ordinary or partial differential equation, three things must be defined: a) the differential equation itself, b) boundary conditions, and c) initial conditions. The various elements of the physical system can be associated with the elements of the mathematical system as follows:

Mathematical System	Physical System
Differential Equation	Pile Response
	Soil Response along Pile Shaft
Boundary Conditions	Hammer Model
	Soil Response at Pile Toe
Initial Conditions	Initial Ram Velocity
	Initial Pile Velocity
	Initial deflection of all members

2. Differential Equation

a) Formulation of the Differential Equation

The differential equation itself principally models both the pile response and the shaft soil response to the driving impulse. If Equation (2) is used as it stands, only the response of the pile is modelled without the interaction of the pile length with the soil. However, with actual piles the amount of elasticity and dampening along the pile shaft varies along the length of the pile. With finite difference codes this has been shown not to be as significant as it would seem at first glance (Meseck, 1985). Therefore, for the purposes of this analysis, the following quantities are assumed to be constant for the entire pile length:

1) Pile material, i.e., density, Young's modulus, and acoustic speed of the material..
2) Pile geometry, i.e. cross-sectional area and perimeter.
3) Soil elastic constant.
4) Soil viscous/radiation dampening constant.

To include the effects of (3) and (4), the wave equation for piling should be rewritten as

$$c^2 u_{xx}(x,t) = u_{tt}(x,t) + 2bu_t(x,t) + au(x,t) \quad (55)$$

where a = Pile Shaft Elasticity Constant, 1/sec^2
b = Pile Shaft Dampening constant, 1/sec

This is the so-called transmission line or Telegrapher's equation. Considering this equation as an equation of motion, the left hand term represents the acceleration of a differential mass at a point x which lies between the pile top (x = 0) and its toe (x = L). Since the units are those of acceleration, the differential mass has been divided out of the left side and distributed as a denominator on the right. This differential mass is assumed to be

$$\Delta m = \rho A \Delta x \quad (56)$$

where Δm = Differential Mass, kg
Δx = Differential Distance, m

The terms on the right represent the response of the various physical elements of the system. The first term $\left(c^2 u(x,t)_{xx}\right)$ represents the distributed elasticity and inertia of the system. The coefficient can be expressed as

$$c^2 = \frac{E}{\rho} = \frac{EA\Delta x}{\rho A \Delta x} = \frac{EA\Delta x}{\Delta m} \quad (57)$$

This of course represents the ratio of the elasticity of the system (represented by EA) to the inertia of the system (represented by m/Δx).

The second term represents the viscous resistance of the soil. The resisting force of the viscosity per unit mass is given by the equation

$$\frac{F_v}{\Delta m} = \frac{\mu P \Delta x}{\rho A \Delta x} u_t = 2bu_t \quad (58)$$

where F_v = Viscous Resisting Force of the Soil, N
μ = Shaft Soil Dampening Coefficient per Unit Area, N-sec/m^3

The geometry ratio is now defined as

$$r_g = \frac{A}{P^2} \quad (59)$$

where r_g = Geometry Ratio of Pile

Substituting this and solving for b,

$$b = \frac{\mu}{2\rho\sqrt{Ar_g}} \quad (60)$$

In like fashion the constant a can be determined. Again the resisting force of the soil elasticity is given by the equation

$$\frac{F_e}{m} = \frac{kP\Delta x}{\rho A\Delta x}u = au \quad (61)$$

where F_e = Soil Elastic Resisting Force, N
k = Soil Shaft Spring or Elastic Constant per Unit Area, N/m³

Solving for a,

$$a = \frac{k}{\rho\sqrt{Ar_g}} \quad (62)$$

Substituting these into the main equation,

$$c^2 u_{xx}(x,t) = u_{tt}(x,t) + \frac{\mu}{\rho\sqrt{Ar_g}}u_t(x,t) + \frac{k}{\rho\sqrt{Ar_g}}u(x,t) \quad (63)$$

This is the Telegrapher's Equation as applied to piles. This soil model is linear; it does not take into account soil plasticity.

b) Computation of Shaft Parameters from Soil Properties

Now that the basic equation is defined, it is necessary to relate these to actual soil conditions. Failure to do so may occasion the consideration of cases which, although mathematically possible, are not in reality cases which are realistic for piles.

When Smith (1960) first considered the problem, he assumed a visco-elastic soil model where the values for soil elasticity and dampening were in reality functions of a) the pile resistance and b) the elastic limit (the quake) of the pile. Generally speaking this is not the case with engineering materials; however, this model, with the numerous refinements that have been done over the years, has served the science of the analysis of piles during driving reasonably well.

More recent investigations have shown that a) it is possible to relate the properties of the soil to its basic properties, and b) most of the "dampening" is in reality radiation dampening and not viscous dampening, the effect of the soil distributed mass and elasticity.

Investigation of the nature of soil response is beyond the scope of this thesis. Based on the work of Randolph and Simons (1986) and Corté and Lepert (1986), the soil elasticity and dampening along the shaft can be computed by the equations

$$k = \pi G_s\sqrt{\frac{r_g}{A}} \quad (64)$$

and

$$\mu = \sqrt{G_s \rho_s} \quad (65)$$

where G_s = Soil Shear Modulus of Elasticity, Pa

ρ_s = Soil Density, kg/m³

The original sources cited these values per unit length of the pile; since they were derived from circular piles, it was only necessary to divide these values by the circumference of a circle to obtain values per unit surface area of the pile.

For simplicity's sake, the variables a and b are used in many of the derivations; therefore, inserting Equations (64) and (65) into (62) and (60) respectively,

$$a = \frac{\pi G_s}{A\rho} \quad (66)$$

and

$$b = \frac{\sqrt{\dfrac{G_s \rho_s}{A r_g}}}{2\rho} \quad (67)$$

It should be evident that other methods of computation can be employed as well.

3. Boundary Conditions

The shaft conditions having been dealt with in the differential equation itself, the boundaries that need to be dealt with are at the ends. From a mathematical standpoint, no single factor makes this problem more unique or difficult than its boundary conditions.

a) Pile Top (x = 0)

The pile hammer itself rests on the top of the pile and imparts both energy and impulse to the pile. There are two methods presently employed to model the hammer.

The first is to model the hammer system discretely. In simple terms, this means to input the parameters of the system, set the initial velocity of the ram to its impact velocity and the other elements, and then analyse the model. This is the method used with most of the finite difference schemes and some of the others as well.

The second is to assume that the force time history of the pile top is already solved and apply this to the pile top. This is possible when this history is taken from field data. In the case where this history is not known from such data, the best solution is to use semi-infinite pile theory, such as Deeks (1992). This eliminates the need to model both a discrete and a continuous system in the same model, which would greatly increase the complexity.

To use any force-time history derived from semi-infinite pile considerations, there are several items that need to be set forth right from the start:

- Because of reflections, semi-infinite pile theory is invalid at the pile top for times t ≥ 2L/c. Although this gives a great deal of flexibility in practical application, it is something that cannot be ignored.
- In the case of piles which respond in accordance with Equation (55), there are reflections before t = 2L/c (Van Koten et. al., 1980). Force-time histories such as those of Deeks (1992) are strictly speaking not valid. This adds greatly to the complexity of the solution.
- The interface between the hammer and cushion, and the cushion and pile cap, and the pile cap and pile, are in most impact hammer systems inextensible. Unlike vibratory hammer systems, where there is alternating tension and compression in the pile top, there is no tension at the pile top with an impact hammer, neither can there be tension in the cushion or ram-pile cap interface. Present closed form semi-infinite pile solutions do not explicitly take this into account, although they can be "stopped" when any separation takes place. In this respect numerical solutions such as Warrington (1987) are at an advantage.

In spite of these limitations, in this thesis semi-infinite pile solutions of the undamped wave equation is used extensively either as the pile top force or to assist in determining possible alternative, simplified force-time relationships. For values of time in the analysis where force-time curves cannot be applied, the pile top is assumed to be a free end. The reason for this last point is as much convenience as anything else; also, if one were to, say, model the pile with the pile cap mass at the top, then the possibility of tension at the pile top would once again take place.

There are two ways to mathematically implement this formulation of the pile top conditions. The first would be to take the velocity-time results of semi-infinite pile theory and, multiplying the velocity by the pile impedance, to equate the force to the elastic force of the pile top, i.e.,

$$-EAu_x(0,t) = Zu_t(0,t) \quad (68)$$

which is a restatement of Equation (13). In this case the $u_t(0,t)$ is a given function rather than a result.

The other solution is to essentially integrate the velocity to the displacement-time function and apply this directly as a boundary condition, i.e., to apply Equation (6).

In using either of these, it is important to keep in mind that, if tension is indicated at any time, a method must be employed to minimize or eliminate this tension in the force-time function of the pile top. Also, in the case of Equation (6), the system *must* be released from this condition no later than t ≥ 2L/c, otherwise the pile set would not exceed its elastic compression, a condition that is both theoretically possible and achieved routinely in pile driving.

When the hammer force is removed, the condition of the pile top becomes

$$u_x(0,t) = 0 \quad (69)$$

b) Pile Toe (x = L)

Generally speaking, impact pile driving produces a compressive force pulse which is transmitted down the pile. If the toe is assumed to be a free end, then the toe reflects a tension wave. If the toe is fixed, then the toe reflects a compression wave (Holloway, 1975). Use of either of these toe models would simplify the analysis of this problem.

Unfortunately, the actual response of the toe is somewhere between these two simplified models, and thus it is general practice to use a visco-elastic soil model at the pile toe, as is the case with the shaft. This is where the similarity ends; because of the difference in the interaction between the pile and the soil from shaft to toe, the implementation of this model is likewise different.

(1) Basic Model Equation and the Lysmer Analogue

For a visco-elastic pile toe without a discrete mass, the boundary equation is

$$-EAu_x(L,t) = k_t A_t u(L,t) + \mu_t A_t u_t(L,t) \quad (70)$$

where A_t = Pile Toe Area, m²

k_t = Soil Toe Spring or Elastic Constant per Unit Area, N/m³

μ_t = Soil Toe Dampening Constant per Unit Area, N-sec/m³

It is important to note that the toe area A_t is defined differently than the cross-sectional area of the pile because in some cases (such as with closed ended pipe pile) the toe area is in fact different than the shaft cross-sectional area.

As was the case with the pile shaft, Smith (1960) formulated the dampening and spring constants empirically. Although these have been used extensively, they have no relationship with the basic soil properties. Since the soil's mechanical properties would be expected to influence their response to excitation, some kind of correlation and corresponding mathematical model is natural.

Pile toe properties are difficult to quantify by themselves; however, the analogous problem of a vibrating footing on the surface of an axisymmetric semi-infinite mass of soil has received considerable attention. Since the pile toe is in fact radiating energy into a similar semi-infinite mass, it has been assumed that the two problems are similar.

One of the simplest approaches to modelling the interaction of a vibrating footing is to assume that only a certain mass of the soil actually vibrates with the footing. This approach is both assumed and experimentally verified by Perry (1963). If this were applied to the pile toe, the solution would be similar to the Prescott-Laura problem, which would be a relatively simple solution. The weakness of this approach is that the soil, being a non-reflective medium, dissipates energy, irrespective of whether soil plasticity is taken into effect or not. Experimental verification by use of a laboratory soil tank(as was done by Perry) may or may not discern this because reflections from the wall of the soil tank may retain energy within the system. Since mass (like a spring) is an energy conservative element (as opposed to the dissipative effects of viscosity) the use of mass would not model any energy dissipative effects. It is beyond the scope of this thesis to go back and analyze this in detail, even though a very simple model of pile toe soil response may result.

This entire matter of soil response to the vibration of a circular footing was taken up by Lysmer (1965). He considered two types of vibration modes; high and low frequency. What separated the two was the size of the footing relative to the wavelength propagated in the soil. If the footing was large enough, the pressure distribution of the soil under the footing was a function (expressed by the cylinder functions) of the distance from the centre of the footing.

However, since pile toes are in reality small footings (even if the pile toe diameter is 1 metre), only the low frequency case need be applied to pile toe response. In this case the pile toe is much smaller and the pressure distribution under the footing is close to the pressure distribution for the static case. This results in a much simpler distribution function. Using results from earlier analyses of an elastic semi-infinite half space, Lysmer was able to propose a simplified response analogue of the soil to the vibrating footing. His proposal was based on the following observation:

> *The practical application of the above theory has shown that only limited agreement can be achieved between observed and calculated amplitudes. This is mainly due to discrepancies between the theoretical half-space model and an actual footing-soil system, the soil phase of which is generally nonlinear, inhomogeneous and imperfectly described. The mathematical difficulties involved in the strict use of the half-space model are therefore hardly justified, and even crude approximations can be introduced in practical calculations, without reducing the reliability of the calculated response. In particular, we can attempt to replace the half-space system with a simple damped oscillator with similar dynamic properties. (Lysmer (1965), p. 42)*

His application of this idea resulted in Lysmer's Analogue, which reduced the response of the half-space (as shown in Figure 3) to a spring-dampener combination.

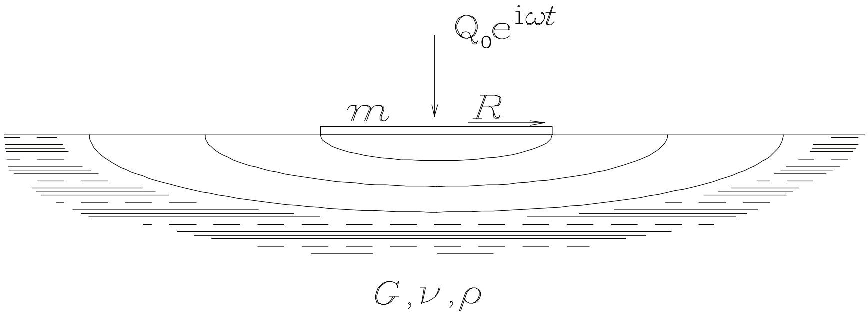


Figure 3 Oscillating Plate on Infinite Half-Space (after Holeyman (1988))

Using this, the dampening $\mu_t A_t$ and spring constant $k_t A_t$ in Equation (70) are given as (Lysmer, 1965; Holeyman, 1988)

$$\mu_t A_t = \frac{3.4 r_t^2 \sqrt{\rho_s G_s}}{1-\nu} \quad \text{(71)}$$

and

$$k_t A_t = \frac{4 G_s r_t}{1-\nu} \quad \text{(72)}$$

where r_t = Pile Toe Radius, m
ν = Poisson's Ratio of Soil

Substituting into Equation (70),

$$-EAu_x(L,t)=\frac{4G_s r_t}{1-\nu}u(L,t)+\frac{3.4r_t^2\sqrt{\rho_s G_s}}{1-\nu}u_t(L,t) \quad (73)$$

(2) Modification of Model to Eliminate Time Derivative

One of the objectives of this thesis is to produce a model that is "reasonably simple" in its formulation, although the simplicity of, the undamped string with fixed ends cannot be expected.

The existence of a first time derivative in a boundary condition is a virtual guarantee that difficulties arise in the formulation of the boundary condition, to say nothing of the orthogonality of the problem. This is because the inclusion of the first time derivative complicates the formulation of the eigenfunctions and can in some cases make them impossible. Is there a solution to this problem that can be justified from the conditions of the problem itself?

To confirm the validity of Lysmer's Analogue, Holeyman (1985, 1988) modelled an equivalent solid below the pile toe as shown in Figure 4.

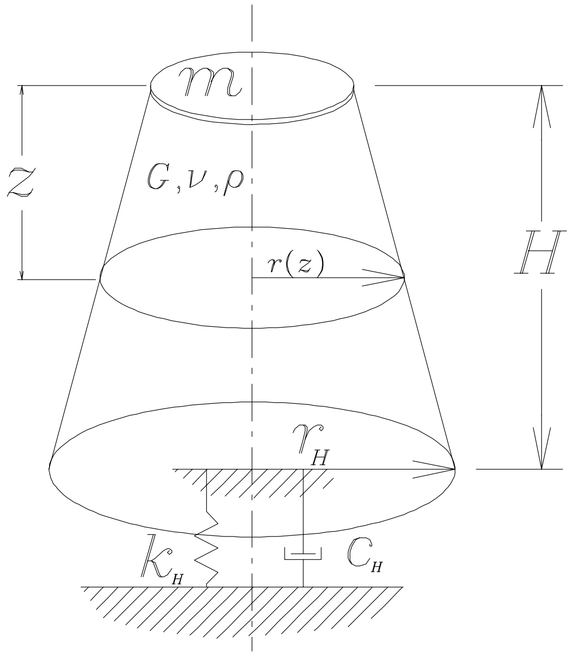


Figure 4 Schematic of Soil Model under Pile Toe (after Holeyman (1988))

The truncated cone is of an indeterminate height H and the radius of the cone is a linear function of the depth z below the pile toe. Holeyman (1988) shows that this radius is given by the equation

$$r(z)=r_t+\frac{1-\nu}{\sqrt{0.85}}z \quad (74)$$

where $r(z)$ = Radius of Soil Mass Below Pile Toe, m
z = Distance Below Pile Toe, m

The response from this equivalent solid was very close to Lysmer's Analogue.

Inside the truncated cone is assumed to be a homogeneous, isotropic, and elastic mass of soil. As the toe is excited by the arrival of the stress wave from above, stress waves are propagated in compression downward. If the height of the cone is assumed to be sufficiently large so as not to suffer reflections from the base of the cone back to the pile toe, then this system is effectively another semi-infinite bar below the pile toe.

If with the semi-infinite bar the strain at the pile top can be transformed to the velocity, the reverse transformation at the pile toe can be performed with the new "pile" below the toe. This in effect adds another section to the pile, only semi-infinitely as a boundary rather than as a continuous rod.

To see how this might work out with Equation (71), consider the acoustic speed of the soil for compressive waves, which is

$$c_s = \sqrt{\frac{E_s}{\rho_s}} \quad \text{.......... (75)}$$

where c_s = Acoustic Speed of the Soil, m/sec

which is analogous to Equation (3). Modifying Equation (12) from semi-infinite pile theory,

$$u_x(L,t) = -u_t(L,t)\frac{1}{c_s} = -u_t(L,t)\frac{1}{\sqrt{\frac{E_s}{\rho_s}}} \quad \text{.......... (76)}$$

or, rearranging,

$$\sqrt{\frac{E_s}{\rho_s}}\,u_x(L,t) = -u_t(L,t) \quad \text{.......... (77)}$$

In this model the Young's modulus is given (Holeyman, 1988) by the equation

$$E_s = \frac{G_s}{0.85(1-\nu)^2} \quad \text{.......... (78)}$$

where E_s = Soil Young's Modulus of Elasticity, Pa

Substituting this into Equation (77),

$$u_t(L,t) = -\frac{1}{1-\nu}\sqrt{\frac{G_s}{0.85\rho_s}}\,u_x(L,t) \quad \text{.......... (79)}$$

Multiplying both sides by the right hand side of Equation (71), this yields

$$\frac{3.4 r_t^2\sqrt{\rho_s G_s}}{1-\nu}u_t(L,t) = -\frac{3.4 r_t^2\sqrt{\rho_s G_s}}{1-\nu}\frac{1}{1-\nu}\sqrt{\frac{G_s}{0.85\rho_s}}\,u_x(L,t) \quad \text{.......... (80)}$$

and reduces to

$$\frac{3.4r_t^2\sqrt{\rho_s G_s}}{1-\nu}u_t(L,t) = -\frac{3.69r_t^2 G_s}{(1-\nu)^2}u_x(L,t) \qquad (81)$$

If the pile toe is assumed to be circular (which is the basis for this theory, as was the case with the shaft) and substitute A_t for the area of a circle, and solve Equation (78) for G_s and substitute,

$$\frac{3.4r_t^2\sqrt{\rho_s G_s}}{1-\nu}u_t(L,t) \cong -E_s A_t u_x(L,t) \qquad (82)$$

For use later the approximation sign is treated as an equality. It is interesting to note that Lysmer (1965) formulated his analogue without direct recourse to semi-infinite pile theory.

This is an important result; it indicates (but does not necessarily prove) that the soil under the pile toe is in fact a semi-infinite "pile" with special conditions. One of those is a surrounding body of soil. If this were not the case then the elastic portion of the dynamic toe reaction could be ignored, but the surrounding soil offers elastic resistance to the pile toe movement in addition to the distributed mass and elasticity of the soil under the pile toe.

Another of these conditions is that, when $u(L,t)$ and its derivatives are discussed, the coordinate system for the pile is being used; for the soil, z = 0 when x = L.

Substituting Equation (82) into Equation (73),

$$-EAu_x(L,t) = \frac{4G_s r_t}{1-\nu}u(L,t) - E_s A_t u_x(L,t) \qquad (83)$$

or rearranging,

$$(E_s A_t - EA)u_x(L,t) = \frac{4G_s r_t}{1-\nu}u(L,t) \qquad (84)$$

which is used in the general case for the boundary condition of the toe. Two items must be noted here:

1. With the time dependent elements removed, the boundary condition in this form satisfies the requirements of Equation (36) for orthogonality.
2. The toe is inextensible as the top is; however, given the nature of the problem, the need to consider such an event (except for very elastic soils) is not as great as it is with the pile top.

(3) Discussion of the Simplification

Even without application of the results to the solution of the differential equation, the boundary condition has been considerably simplified. Yet virtually all of the models of this system, both closed form and numerical, persist in using the time derivative rather than the distance one.

First it is important to state that there is additional work to be done with soil modelling, either in a one-dimensional simplification or in a model where the soil is modelled as a mass. Although this simplification can be done with different soil models and even empirical ones, the results may not be as "neat" as they are here.

To get back to the derivative question, most physical systems of this type are either discrete or distributed. Discrete systems are made up of (or at least can be modelled as) components with varying characteristics; in linear systems at least, the response of the system is basically the sum of the characteristics of the components and whatever initial conditions and/or forcing functions are applied to the system. There are no explicit "boundary conditions" because the system's boundary can be expressed by the characteristics of the components themselves.

Distributed systems have properties which are spread about through the system; they thus have boundaries with boundary conditions. The various parameters can vary from one part of the system to another, but the system is continuous to some degree.

Generally speaking, discrete systems are simpler to conceptualise than distributed ones, especially if they are of a low degree of freedom. Their solutions are generally obtained with ordinary differential equations, as opposed to the partial differential equations of distributed systems. The complexity of the latter has led to the conversion of distributed systems to discrete ones through the use of numerical methods.

With a system such as a pile, the mass can be concentrated at nodes or segments and these masses are connected with springs, as is the case with finite difference methods. The wave equation is then solved using the appropriate numerical integration technique. This is still an approximation because the mass and elasticity are in fact mutually distributed. With finite element techniques this is refined through the use of shape functions, which can better approximate the changes in stress and strain in an area of the system; the accuracy of the solution can be very dependent upon the level of realism of the shape functions relative to the actual response of the system.

In any case, strictly speaking the concept of strain with discrete mechanical systems is meaningless, because the springs in the system deflect and the force is computed according to Hooke's law. Velocity, though, is a quantity which is a direct product of the numerical integration and can be calculated at any mass point or node in the system. Furthermore external forces (such as soil resistance) can be computed from node velocity, thus making them possible to include in the model.

With a truly distributed system, strain is not only possible to compute, it is a fundamental property of the system. Thus strain related effects can be considered, and such effects can include soil response if the physical characteristics of the system justify such an inclusion. This is an advantage of this kind of solution over the numerical solutions.

Finally, it is noteworthy that strain becomes important again when piles are instrumented, generally with strain gages and accelerometers at the pile top.

4. Initial Conditions

Most pile models start with both displacement and velocity at zero; this is expressed in Equations (4) and (5). This type of initial condition does not take into account any residual stress considerations in the pile after the previous blow. It also does not consider the possibility that the pile may still be in motion; as a practical matter, this is only possible with a hammer with a very fast blow rate or a pile with a very long period.

Although these conditions look very simple, if one wants to use a method such as Fourier series these can pose a serious problem. As with the semi-infinite pile, Laplace transforms are better equipped to handle initial conditions such as this. This fact is used in the development of the basic solution technique which follows.

IV. PROPOSAL OF CLOSED FORM SOLUTION BY APPLICATION TO THE UNDAMPED CASE

The fundamentals of the problem established, a proposal and implementation of a solution follows. Because the actual solution of the problem is somewhat "situation specific," i.e. it is variable depending upon how the problem is formulated even with the limitations of the parameters shown above, a relatively simple system is used to both prove and demonstrate the method.

A. Outline of the Solution

1. Basic Solution Method

The basic solution method proposed in this thesis is as follows:

1. Determine the force-time or displacement-time history of the hammer at the pile top, either using semi-infinite pile theory or actual field data.
2. Using Laplace transforms, solve the wave equation for the semi-infinite pile case. This is the solution for $t < L/c$.
3. Compute the displacement and velocity functions as a function of distance at $t = L/c$. These become the initial conditions for the remainder of the problem.
4. Using the boundary conditions, compute the eigenvalues and eigenfunctions for the Fourier series. The pile top is assumed to be a free end in this case.
5. Using the displacement and velocity functions at $t = L/c$, compute the Fourier coefficients. This Fourier series is the solution for $t \geq L/c$.

As stated, this procedure assumes the transition point to be fixed at $t = L/c$. However, if the impulse force of the hammer system ends before this time, it is most advantageous to make the turnover point at the time when the impulse force becomes zero, or tension begins to develop in the pile top. This is demonstrated in the damped case.

2. Assumptions for the Solution

The following assumptions are made:

1. The solution must be reasonably simple; the solution must not require integration or other transformation once it is formulated.
2. The system is a linear system. No plasticity is taken into account in this system.
3. All properties between the boundaries are uniform. These include pile area and material, dampening, soil spring constant.
4. The soil below the pile toe can be modelled as a semi-infinite pile (see above), thus eliminating the first time derivative of the dampening portion.
5. Extensibility considerations of the pile top and toe are not significant. The validity of this assumption is dependent upon how the pile top force is formulated.

6. The force of the hammer is substantially finished before t = L/c. The use of semi-infinite pile techniques to generate the pile top force allows $t \leq 2L/c$, but $t > L/c$ is necessitated by the solution technique. However, it can be used to deal with the extensibility problem. This solution favours long piles relative to the hammer blow duration.

3. Mathematical Parameters of the Solution Example

The solution example is that of the undamped case with no pile cap; this is diagrammed in Figure 5. This simplified system was studied extensively in the early years of the wave equation for piles, and is featured in Isaacs (1931) and Glanville et. al. (1938).

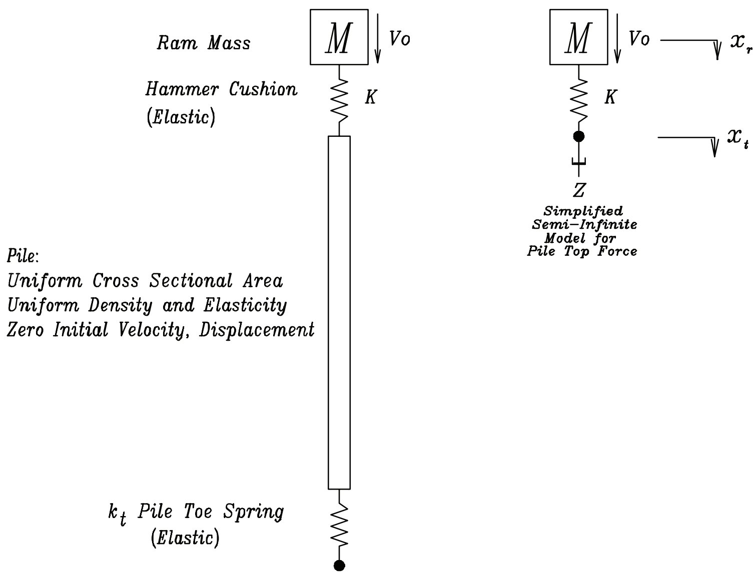


Figure 5 Simplified, Undamped Pile Model

Governing Equation: Equation (2), the undamped wave equation, is the governing equation.

Initial Conditions: The initial conditions are

$$u(x,0) = f(x) = 0 \qquad (85)$$

and

$$u_t(x,0) = g(x) = 0 \qquad (86)$$

At t = L/c, there is an intermediate initial condition for the Fourier series.

Boundary Conditions: At the pile toe, the boundary condition contains only the elasticity of the spring. This is expressed as

$$-EAu_x(L,t) = \frac{4G_s r_t}{1-\nu} u(L,t) \qquad (87)$$

which is of course a modification of Equation (73). For simplicity's sake the following substitution is defined

$$k_t A_t = \frac{4G_s r_t}{1-\nu} \quad (88)$$

and also

$$Zc = EA \quad (89)$$

from Equation (15). Substituting both of these yields

$$-Zcu_x(L,t) = k_t A_t u(L,t) \quad (90)$$

B. Solution of the Problem

1. Equation of Motion for the Pile Top

The equation of motion for the pile top, which in turn is the boundary condition for $0 \leq t < L/c$, is determined using semi-infinite pile theory. The procedure is discussed in Deeks (1992); however, the solution proposed here varies from this in that a) the ultimate objective is the displacement-time history, and b) the notation is different.

The hammer system presented here has a ram with initial velocity V_0 and mass M, and a cushion spring between the ram and the pile top of stiffness K. There is no pile cap mass considered.

The hammer impedance is defined (Warrington, 1987) as

$$Z_h = \sqrt{KM} \quad (91)$$

where Z_h = Pile Hammer Impedance, N-sec/m
K = Cushion Material Spring Constant, N/m
M = Mass of Ram, kg

The pile impedance is known from Equation (15). The impedance ratio can be then defined as

$$Z' = \frac{Z}{Z_h} \quad (92)$$

where Z' = Pile-Hammer Impedance Ratio

Using this notation, the equations of motion for this system are

$$M\ddot{x}_r + K(x_r - x_t) = 0 \quad (93)$$

and

$$Z\dot{x}_t - K(x_r - x_t) = 0 \quad (94)$$

where x_r = Ram Displacement, m
x_t = Pile Top Displacement, m

This results in a homogeneous system of ordinary differential equations. According to Rabenstein (1972), such systems of equations can be directly solved by methods similar to those used in linear algebra. Such methods, however, are more straightforward with first order equations then higher order ones, because it is necessary to use substitute equivalent systems to maintain the linear solution technique. To simplify the solution of the second order system such as is presented here, Laplace transforms were used. This method is also described in Rabenstein (1972).

Substituting Equations (91) and (92), taking the Laplace transform of each differential equation and applying the initial conditions of the hammer system, the Laplace transforms are

$$M^2\left(\mathscr{L}(x_r)s^2 - V_0\right) + \frac{Z^2\left(\mathscr{L}(x_r) - \mathscr{L}(x_t)\right)}{Z'^2} = 0 \qquad (95)$$

and

$$ZM\mathscr{L}(x_r)s - \frac{Z^2\left(\mathscr{L}(x_r) - \mathscr{L}(x_t)\right)}{Z'^2} = 0 \qquad (96)$$

If these are equated and solved for the Laplace transform of the pile top motion,

$$\mathscr{L}(x_t) = X_t(s) = \frac{ZMV_0}{s\left((Z'M)^2 s^2 + ZMs + Z^2\right)} \qquad (97)$$

where $X_t(s)$ = Laplace Transform of Pile Top Displacement

To perform the inverse transform, the method of Starkey (1954) is employed, which involves the use of complex integration. For meromorphic functions, the inverse Laplace transform of a function is given by the equation

$$x_t = \sum_{n=1}^{k} \mathrm{Res}\left\{X_t(s_n)e^{s_n t}\right\} \qquad (98)$$

To solve this equation first a) the poles of Equation (97) and b) the residues at each pole were determined; the sum of these residues was multiplied in each case by the exponential term. The poles of Equation (97) are

$$s_1 = 0 \qquad (99)$$

$$s_{2,3} = \frac{-ZM \pm \sqrt{(ZM)^2 - 4(MZ'Z)^2}}{2(MZ')^2} \qquad (100)$$

If the general form of the residue for Equation (97) is taken, then substituted into Equation (98) with the poles, then summed, solved and the exponential terms converted to hyperbolic functions,

$$x_t = \frac{V_0}{2Z^2(4Z'^2-1)}\left(\left(-ZM(4Z'^2-1)+\sqrt{(ZM)^2(4Z'^2-1)}\right)\right)\cosh\left(\frac{\left(ZM-\sqrt{(ZM)^2(4Z'^2-1)}\right)t}{2(MZ')^2}\right)$$

$$+\frac{V_0}{2Z^2(4Z'^2-1)}\left(\left(ZM(4Z'^2-1)-\sqrt{(ZM)^2(4Z'^2-1)}\right)\right)\sinh\left(\frac{\left(ZM-\sqrt{(ZM)^2(4Z'^2-1)}\right)t}{2(MZ')^2}\right)$$

$$+\frac{V_0}{2Z^2(4Z'^2-1)}\left(\left(-ZM(4Z'^2-1)-\sqrt{(ZM)^2(4Z'^2-1)}\right)\right)\cosh\left(\frac{\left(ZM+\sqrt{(ZM)^2(4Z'^2-1)}\right)t}{2(MZ')^2}\right)$$

$$+\frac{V_0}{2Z^2(4Z'^2-1)}\left(\left(ZM(4Z'^2-1)+\sqrt{(ZM)^2(4Z'^2-1)}\right)\right)\cosh\left(\frac{\left(ZM+\sqrt{(ZM)^2(4Z'^2-1)}\right)t}{2(MZ')^2}\right)$$

$$+\frac{V_0 M}{Z} \quad \text{.......... (101)}$$

Removing the real exponentials, converting the hyperbolic functions to circular ones and simplifying further, this yields

$$x_t = \frac{V_0 M}{Z} - \frac{V_0 M}{Z} e^{\frac{Z}{2MZ'^2}t}\left\{\frac{\sin\left(\frac{Z\sqrt{4Z'^2-1}t}{2MZ'^2}\right)}{\sqrt{4Z'^2-1}} + \cos\left(\frac{Z\sqrt{4Z'^2-1}t}{2MZ'^2}\right)\right\} \quad \text{.......... (102)}$$

The following now needs to be observed:

$$\lim_{t\to\infty} x_t = \frac{MV_0}{Z} \quad \text{.......... (103)}$$

This limit is the maximum elastic compression of the pile from the impact, without consideration of intermediate losses between the ram and the pile.

There are repetitious quantities in these equations. To simplify these equations the quantities

$$\alpha_0 = \frac{Z}{2MZ'^2} \quad \text{.......... (104)}$$

and

$$\beta_0 = \sqrt{4Z'^2-1} \quad \text{.......... (105)}$$

where α_0 = Pile Top Consolidation Variable, 1/sec
β_0 = Pile Top Consolidation Variable

can be defined. Equation (102) then simplifies to

$$x_t = \frac{MV_0}{Z}\left(1-e^{-\alpha_0 t}\left(\frac{\sin(\alpha_0\beta_0 t)}{\beta_0}+\cos(\alpha_0\beta_0 t)\right)\right) \quad \text{.......... (106)}$$

This is the solution for the displacement at the pile top. Although the sine and cosine functions can be combined through the use of a phase angle, there is no advantage in doing so, since these functions are used in integration to obtain Fourier coefficients.

In this form, the Laplace transform of the equation is

$$\mathscr{L}(x_t) = X_t(s) = \frac{MV_0\alpha_0^2\left(\beta_0^2+1\right)}{Zs\left(s^2+2\alpha_0 s+\alpha_0^2\left(\beta_0^2+1\right)\right)} \quad (107)$$

2. Laplace Transform for the Pile (Semi-Infinite Case)

The displacement-time history of the pile top having been determined, this can now be extended this downward in time in the pile itself. This is accomplished using a semi-infinite pile based technique. This decision needs to be discussed in detail.

a) Rationale for Semi-Infinite Pile Solution and Time Divided Solution

A great deal of time has been spent discussing semi-infinite pile theory. There is one problem here, namely that no pile is really semi-infinite, but has a length L in all cases. The reflections from this end are significant. This is a serious discussion because there exist solutions other than numerical solutions (Espinoza, 1991; Starkey, 1954; Zhou and Liang, 1996) where the pile (or electrical transmission line in Starkey's case) has ends and where both the end conditions are accounted for in the solution.

Espinoza (1991) uses a Fourier transform and spectral analysis; to invert the transformation to solution, he notes the following:

> *It is noted that to obtain the force and displacement-time history at any specified location a Fourier inverse transform must be applied. Analytical solution of the inverse transform, even for the simplest forcing function, is not possible to obtain; thus a numerical inversion has to be performed.*

Although his solution is in principle a closed form solution, is relies on a numerical method for completion. The objective of this thesis is to produce a true closed form solution, not only for historical reasons but also to make the intermediate discoveries that working with closed form solutions do. Therefore the use of this solution is not within the scope of this thesis.

In Starkey's case, he considers the solution of a balanced transmission line with a constant voltage applied at one end and an open circuit at the other. In spite of these idealized conditions, the solution is involved, and at the same time includes an infinite series similar to a Fourier series. Moreover he consolidates the exponential terms into hyperbolic functions, which obscure the inevitable Heaviside step functions (which do not appear in his solution.) Were these to be considered, there would be Heaviside step functions in every term, which complicates the summation of the terms. This in fact the result of Zhou and Liang (1996); although a comprehensive closed form solution was not their ultimate objective, their reduction to a semi-infinite solution is indicative of the difficulties associated with a pure Laplace transform solution of a pile with finite boundaries.

Since an infinite series results in any case, it seems simpler to use Fourier series, which would automatically take into consideration the reflections of the ends without resorting to the

Heaviside step functions. The main weakness to these is the difficulty with the initial conditions; however, since the reflections from the pile toe are of no effect in the time period $0 \le t < L/c$, a Laplace transform of the semi-infinite case can be used to solve the problem until t = L/c. The displacement and velocity at this time can then be used as initial conditions for the rest of the problem. The only changes necessary in the Fourier series are as follows:

a) A new time reference frame is established for the Fourier series. To generate the series the time is assumed equal to zero at t = L/c. The time t = 0 for the Fourier series can be adjusted to be the impact time with a change in variable.

b) The pile top boundary condition is changed from the displacement just computed to Equation (69).

The main weakness of this composite method is that the hammer blow must be substantially complete by the time the wavefront of the stress wave reaches the pile toe. This is attainable in a large number of conditions, although it favours longer piles. Also, it allows an opportunity to "stop" a closed form solution for the hammer blow at the pile top, since many of these solutions produce tension in the pile top after the initial peak force.

b) Laplace Transform Solution for t < L/c

Although the equations from the previous section can be taken and the solution can be derived in full, the solution to this problem was presented earlier for the general pile toe displacement case with the undamped rod. Equating

$$f(t) = x_t = \frac{MV_0}{Z}\left(1 - e^{-\alpha_0 t}\left(\frac{\sin(\alpha_0\beta_0 t)}{\beta_0} + \cos(\alpha_0\beta_0 t)\right)\right) \quad (108)$$

the solution for the displacement in the rod can be readily determined from Equation (9), which was derived using Laplace transforms (Kreyszig, 1993). The solution is

$$u(x,t) = \frac{MV_0}{Z}\left(1 - e^{-\alpha_0\left(t-\frac{x}{c}\right)}\left(\frac{\sin\left(\alpha_0\beta_0\left(t-\frac{x}{c}\right)\right)}{\beta_0} + \cos\left(\alpha_0\beta_0\left(t-\frac{x}{c}\right)\right)\right)\right)H\left(t-\frac{x}{c}\right), 0 \le t \le \frac{L}{c} \quad (109)$$

The velocity in the rod is given by

$$u_t(x,t) = \frac{MV_0}{Z}\alpha_0 e^{-\alpha_0\left(t-\frac{x}{c}\right)}\frac{\sin\left(\alpha_0\beta_0\left(t-\frac{x}{c}\right)\right)\left(1+\beta_0^2\right)}{\beta_0}H\left(t-\frac{x}{c}\right), 0 \le t \le \frac{L}{c} \quad (110)$$

and the stress

$$\sigma(x,t) = -\frac{Zc}{A}u_x(x,t) = \frac{MV_0}{A}\alpha_0 e^{-\alpha_0\left(t-\frac{x}{c}\right)}\frac{\sin\left(\alpha_0\beta_0\left(t-\frac{x}{c}\right)\right)\left(1+\beta_0^2\right)}{\beta_0}H\left(t-\frac{x}{c}\right), 0 \le t \le \frac{L}{c} \quad (111)$$

Although the derivatives of Equation (109) properly includes Dirac delta functions, these do not contribute to the solution in a meaningful way and are thus not included in the derivative. The reason for this is because they appear at the wavefront, where there is a discontinuity in the function but where the value for the displacement or velocity is nominally zero. This discontinuity creates difficulty for numerical methods such as finite element analysis (Deeks, 1992).

c) Initial Conditions at Transition Point (t = L/c)

At the end of the period, the displacement and velocity in the pile are given by the equations

$$u\left(x,\frac{L}{c}\right)=f(x)=\frac{MV_0}{Z}\left(1-e^{-\alpha_0\left(\frac{L-x}{c}\right)}\left(\frac{\sin\left(\alpha_0\beta_0\left(\frac{L-x}{c}\right)\right)}{\beta_0}+\cos\left(\alpha_0\beta_0\left(\frac{L-x}{c}\right)\right)\right)\right), t=\frac{L}{c} \quad \text{......... (112)}$$

and

$$u_t\left(x,\frac{L}{c}\right)=g(x)=\frac{MV_0}{Z}\alpha_0 e^{-\alpha_0\left(\frac{L-x}{c}\right)}\frac{\sin\left(\alpha_0\beta_0\left(\frac{L-x}{c}\right)\right)\left(1+\beta_0^2\right)}{\beta_0}, t=\frac{L}{c} \quad \text{.................................. (113)}$$

and these become the initial conditions for the next part of the problem

3. Fourier Series for t > L/c

The Fourier series can now be developed. First

$$t'=t-\frac{L}{c} \quad \text{.. (114)}$$

where t' = Time from Transition Point t = L/c or t=δ, seconds

needs to be defined. This time is used with the Fourier series. It makes the initial conditions described above take place at t' = 0, which simplifies considerations. At the end of the derivation the original time convention is reverted to for consistency by simple substitution.

a) Determination of Eigenvalues and Eigenfunctions

The solution is first assumed to be in the form

$$u(x,t')=e^{\left(\frac{\beta ct'+\lambda x}{L}\right)} \quad \text{.. (115)}$$

where $\lambda,\lambda',\lambda_m,\lambda_n$ = Constants or Eigenvalues

This form is chosen so that β and λ can be dimensionless; in any case, they can be either real or imaginary.

Substituting this into Equation (2) and solving for β,

$$\beta=\pm\lambda \quad \text{.. (116)}$$

Substituting this into Equation (115),

$$u(x,t') = e^{\left(\frac{\lambda(x \pm ct')}{L}\right)} \qquad (117)$$

Although the exponent to this equation suggested d'Alembert's solution, this was expanded into hyperbolic functions to yield

$$u(x,t') = \left(\cosh\left(\frac{\lambda x}{L}\right) + \sinh\left(\frac{\lambda x}{L}\right)\right)\left(\cosh\left(\pm\frac{\lambda ct'}{L}\right) + \sinh\left(\pm\frac{\lambda ct'}{L}\right)\right) \qquad (118)$$

or

$$u(x,t') = \left(\cosh\left(\frac{\lambda x}{L}\right) + \sinh\left(\frac{\lambda x}{L}\right)\right)\left(\cosh\left(\frac{\lambda ct'}{L}\right) \pm \sinh\left(\frac{\lambda ct'}{L}\right)\right) \qquad (119)$$

A more general expression for this is

$$u(x,t') = \left(C_1\cosh\left(\frac{\lambda ct'}{L}\right) + C_2\sinh\left(\frac{\lambda ct'}{L}\right)\right)\left(C_3\cosh\left(\frac{\lambda x}{L}\right) + C_4\sinh\left(\frac{\lambda x}{L}\right)\right) \qquad (120)$$

The distance differential of this is

$$u_x(x,t') = \frac{\lambda}{L}\left(C_1\cosh\left(\frac{\lambda ct'}{L}\right) + C_2\sinh\left(\frac{\lambda ct'}{L}\right)\right)\left(C_3\sinh\left(\frac{\lambda x}{L}\right) + C_4\cosh\left(\frac{\lambda x}{L}\right)\right) \qquad (121)$$

Substituting the values of Equation (69), which is the boundary condition at x = 0 for this portion of time,

$$u_x(0,t') = 0 = C_4\frac{\lambda}{L}\left(C_1\cosh\left(\frac{\lambda ct'}{L}\right) + C_2\sinh\left(\frac{\lambda ct'}{L}\right)\right) \qquad (122)$$

The only way to insure that this quantity works out is if

$$C_4 = 0 \qquad (123)$$

The boundary condition at the pile toe x = L is defined by Equation (90). This represents the soil spring at the toe. Equation (123) was first substituted into Equation (120) and this in turn was substituted into Equation (90). These steps took both boundary conditions into account and this yielded

$$\left(C_1\cosh\left(\frac{\lambda ct'}{L}\right) + C_2\sinh\left(\frac{\lambda ct'}{L}\right)\right)\left(-Zc\frac{\lambda}{L}C_3\sinh(\lambda) - k_tA_tC_3\cosh(\lambda)\right) = 0 \qquad (124)$$

or

$$-Zc\frac{\lambda}{L}C_3\sinh(\lambda) = k_tA_tC_3\cosh(\lambda) \qquad (125)$$

The trivial solution for this is for $C_3 = 0$. There is no real λ that satisfies this condition. However, substituting

$$\lambda' = i\lambda \quad (126)$$

and

$$C_3 = 1 \quad (127)$$

the result was

$$Zc\frac{\lambda'}{L}\sin(\lambda') = k_t A_t \cos(\lambda') \quad (128)$$

or rearranging

$$\tan(\lambda') = \frac{\sin(\lambda')}{\cos(\lambda')} = \frac{1}{\lambda'}\frac{k_t A_t L}{Zc} \quad (129)$$

This is a similar result to the one obtained in the Prescott-Laura problem. Dropping the accents and adding a subscript to note that the number of solutions of this equation is infinite, the eigenvalues for this problem are

$$\tan(\lambda_n) = \frac{\sin(\lambda_n)}{\cos(\lambda_n)} = \frac{1}{\lambda_n}\frac{k_t A_t L}{Zc}, \pi\left(n - \frac{3}{2}\right) < \lambda_n < \pi\left(n - \frac{1}{2}\right), n = 1,2,3,\ldots\infty \quad (130)$$

and the solution (without consideration of the initial conditions) is

$$u(x,t') = \sum_{n=1}^{\infty}\cos\left(\frac{\lambda_n x}{L}\right)\left(C_{1n}\cos\left(\frac{\lambda_n ct'}{L}\right) + C_{2n}\sin\left(\frac{\lambda_n ct'}{L}\right)\right) \quad (131)$$

Included with this solution are subscripts for the constants; they too are infinite. The velocity in the pile is

$$u_t(x,t') = \sum_{n=1}^{\infty}\frac{\lambda_n c}{L}\cos\left(\frac{\lambda_n x}{L}\right)\left(-C_{1n}\sin\left(\frac{\lambda_n ct'}{L}\right) + C_{2n}\cos\left(\frac{\lambda_n ct'}{L}\right)\right) \quad (132)$$

b) Computation of the Fourier Coefficients

The computation of the Fourier coefficients C_{1n} and C_{2n} was completed by first substituting t' = 0 into Equations (131) and (132);

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} C_{1n}\cos\left(\frac{\lambda_n x}{L}\right) \quad (133)$$

and

$$u_t(x,0) = g(x) = \sum_{n=1}^{\infty} C_{2n}\frac{\lambda_n c}{L}\cos\left(\frac{\lambda_n x}{L}\right) \quad (134)$$

Substituting from Equations (112) and (113),

$$\frac{MV_0}{Z}\left(1-e^{-\alpha_0\left(\frac{L-x}{c}\right)}\left(\frac{\sin\left(\alpha_0\beta_0\left(\frac{L-x}{c}\right)\right)}{\beta_0}+\cos\left(\alpha_0\beta_0\left(\frac{L-x}{c}\right)\right)\right)\right)=\sum_{n=1}^{\infty}C_{1n}\cos\left(\frac{\lambda_n x}{L}\right) \quad \text{.................} \quad (135)$$

and

$$\frac{MV_0}{Z}\alpha_0 e^{-\alpha_0\left(\frac{L-x}{c}\right)}\frac{\sin\left(\alpha_0\beta_0\left(\frac{L-x}{c}\right)\right)\left(1+\beta^2\right)}{\beta_0}=\sum_{n=1}^{\infty}C_{2n}\frac{\lambda_n c}{L}\cos\left(\frac{\lambda_n x}{L}\right) \quad \text{......................................} \quad (136)$$

Multiplying both sides by $\cos\left(\lambda_m x/L\right)$,

$$\frac{MV_0}{Z}\cos\left(\frac{\lambda_m x}{L}\right)\left(1-e^{-\alpha_0\left(\frac{L-x}{c}\right)}\left(\frac{\sin\left(\alpha_0\beta_0\left(\frac{L-x}{c}\right)\right)}{\beta_0}+\cos\left(\alpha_0\beta_0\left(\frac{L-x}{c}\right)\right)\right)\right)$$

$$\boxed{} \quad \text{..} \quad (137)$$

and

$$\frac{MV_0}{Z}\cos\left(\frac{\lambda_m x}{L}\right)\alpha_0 e^{-\alpha_0\left(\frac{L-x}{c}\right)}\frac{\sin\left(\alpha_0\beta_0\left(\frac{L-x}{c}\right)\right)\left(1+\beta_0^2\right)}{\beta_0}=C_{2n}\frac{\lambda_n c}{L}\cos\left(\frac{\lambda_n x}{L}\right)\cos\left(\frac{\lambda_m x}{L}\right) \quad \text{............} \quad (138)$$

Since the orthogonality requirements from Equations (35) and (36) are met, the Fourier series represents a complete solution because the coefficients where n≠m are zero. Substituting n for m and integrating both sides,

$$\int_0^L \frac{MV_0}{Z}\cos\left(\frac{\lambda_n x}{L}\right)\left(1-e^{-\alpha_0\left(\frac{L-x}{c}\right)}\left(\frac{\sin\left(\alpha_0\beta_0\left(\frac{L-x}{c}\right)\right)}{\beta_0}+\cos\left(\alpha_0\beta_0\left(\frac{L-x}{c}\right)\right)\right)\right)dx$$

$$=C_{1n}\int_0^L \cos\left(\frac{\lambda_n x}{L}\right)^2 dx \quad \text{..} \quad (139)$$

and

$$\int_0^L \frac{MV_0}{Z}\cos\left(\frac{\lambda_n x}{L}\right)\alpha_0 e^{-\alpha_0\left(\frac{L-x}{c}\right)}\frac{\sin\left(\alpha_0\beta_0\left(\frac{L-x}{c}\right)\right)\left(1+\beta_0^2\right)}{\beta_0}dx=C_{2n}\frac{\lambda_n c}{L}\int_0^L \cos\left(\frac{\lambda_n x}{L}\right)^2 dx \quad \text{...............} \quad (140)$$

In both cases the right hand side integration is

$$\int_0^L \cos\left(\frac{\lambda_n x}{L}\right)^2 dx = \frac{\left(\cos(\lambda_n)\sin(\lambda_n)+\lambda_n\right)L}{2\lambda_n} \qquad (141)$$

Performing all integration and solving for the Fourier coefficients,

$$C_{1n} = \frac{2V_0 M(L\alpha_0)\left(\beta_0^2+1\right)}{Z\beta_0\left(\left(\cos(\lambda_n)\sin(\lambda_n)+\lambda_n\right)\left(\left((\alpha_0 L)^2+(c\lambda_n)^2\right)+\left(\alpha_0\beta_0 L\right)^2\right)^2-\left(\alpha_0\beta_0 Lc\lambda_n\right)^2\right)}$$
$$\left(\begin{array}{l} \sin(\lambda_n)\left((L\alpha_0)^3\left(\beta_0^3+\beta_0\right)-(L\alpha_0)\beta_0(c\lambda_n)^2\right) \\ +e^{-\frac{\alpha_0 L}{c}}\sin\left(\frac{\alpha_0\beta_0 L}{c}\right)\left(-\left(L\alpha_0\beta_0\right)^2 c\lambda_n+(c\lambda_n)^3+(L\alpha_0)^2 c\lambda_n\right) \\ +\left(e^{-\frac{\alpha_0 L}{c}}\cos\left(\frac{\alpha_0\beta_0 L}{c}\right)-\cos(\lambda_n)\right)\left(2(L\alpha_0)^2\beta_0 c\lambda_n\right) \end{array}\right) \qquad (142)$$

and

$$C_{2n} = \frac{2V_0 M(L\alpha_0)^2\left(\beta_0^2+1\right)}{Z\beta_0\left(\left(\cos(\lambda_n)\sin(\lambda_n)+\lambda_n\right)\left(\left((\alpha_0 L)^2+(c\lambda_n)^2\right)+\left(\alpha_0\beta_0 L\right)^2\right)^2-\left(\alpha_0\beta_0 Lc\lambda_n\right)^2\right)} \qquad (143)$$
$$\left(\begin{array}{l} \sin(\lambda_n)\left(2(L\alpha_0)\beta_0(c\lambda_n)\right)+e^{-\frac{\alpha_0 L}{c}}\sin\left(\frac{\alpha_0\beta_0 L}{c}\right)\left(-\left(L\alpha_0\beta_0\right)^2-(c\lambda_n)^2-(L\alpha_0)^2\right) \\ +\left(e^{-\frac{\alpha_0 L}{c}}\cos\left(\frac{\alpha_0\beta_0 L}{c}\right)-\cos(\lambda_n)\right)\left(-(L\alpha_0)^2\left(\beta_0^3+\beta_0\right)+\beta_0(c\lambda_n)^2\right) \end{array}\right)$$

The final substitution is the original time frame of reference; substituting Equation (114) into Equation (131),

$$u(x,t)=\sum_{n=1}^{\infty}\cos\left(\frac{\lambda_n x}{L}\right)\left(C_{1n}\cos\left(\lambda_n\left(\frac{tc}{L}-1\right)\right)+C_{2n}\sin\left(\lambda_n\left(\frac{tc}{L}-1\right)\right)\right), t>\frac{L}{c} \qquad (144)$$

where the coefficients are defined by Equations (142) and (143). The stress in the pile is given by the equation

$$\sigma(x,t)=-\frac{Zc}{A}u_x(x,t)=\frac{Zc}{AL}\sum_{n=1}^{\infty}\lambda_n\sin\left(\frac{\lambda_n x}{L}\right)\left(C_{1n}\cos\left(\lambda_n\left(\frac{tc}{L}-1\right)\right)+C_{2n}\sin\left(\lambda_n\left(\frac{tc}{L}-1\right)\right)\right), t>\frac{L}{c} \qquad (145)$$

This is the solution for the undamped case. Because it lacks any dissipative elements in it, it is not a totally realistic case for piling; however, it can be used for comparison with numerical methods, where dissipative elements (such as dampening) tend to obscure instabilities in the numerical integration.

V. SOLUTION OF THE DAMPED CASE

A. *Outline of the Solution*

The *basic solution method* and the *assumptions for the solution* are the same as the undamped case.

The *physical system* to be modelled is shown in Figure 2.

The *governing equation* is given in Equation (55). For the purposes of this thesis the values for a and b are given in Equations (66) and (67) respectively.

The *initial conditions* are given by Equations (4) and (5).

The *boundary condition* for the pile toe is given in Equation (84). The pile top boundary condition for t > L/c (or whatever transition point is chosen) is given by Equation (69). The boundary condition before this time is discussed below.

B. *Solution of the Problem*

1. Equation of Motion for the Pile Top

In order that the damped case might be a more realistic representation of the pile behaviour, both the mass of the driving accessory and material dampening of the hammer cushion are included. Because meromorphic functions appear here as well, the solution method is the same as for the undamped case, but the introduction of these new elements makes the actual computations more difficult. As before the solution of this problem is presented in Deeks (1992) but the notation is rather different. The Maple V worksheet for the derivation is found in Appendix B.

New variables which needed to be defined are

$$m' = \frac{\hat{m}}{M} \quad \text{.......} \quad (146)$$

and

$$c' = \frac{C}{Z_h} = \frac{Z'C}{Z} \quad \text{.......} \quad (147)$$

where m' = Pile Cap/Ram Mass Ratio
$\hat{m}$ = Mass of Driving Accessory for Pile Hammer, kg
c' = Cushion Dampening/Hammer Impedance Ratio
C = Cushion Dampening Coefficient, N-sec/m

Using the same semi-infinite pile theory employed before, the equations of motion are

$$M\ddot{x}_r + K\left(x_r - x_t\right) + C\left(\dot{x}_r - \dot{x}_t\right) = 0 \quad \text{.......} \quad (148)$$

and

$$m\ddot{x}_t + Z\dot{x}_t - K\left(x_r - x_t\right) + C\left(\dot{x}_r - \dot{x}_t\right) = 0 \quad \text{.......} \quad (149)$$

As was done with the undamped case, the appropriate variable substitutions having been made and the Laplace transforms having been taken, the Laplace transform of the pile top velocity was solved for; this expression is

$$\mathscr{L}\left(\dot{x}_t\right) = V_t(s) = \frac{ZV_o M(Z - c'Z'Ms)}{Z'^2 M^3 m' s^3 + M^2 Z'Z\left(c'(m'-1) + Z'\right)s^2 + MZ^2(m'+1+Z'c')s + Z^3} \qquad (150)$$

where $V_t(s)$ = Laplace Transform for Pile Top Velocity

The same method of inverse transformation is used as before, i.e., using Equation (98). Because there is a cubic equation in the denominator, the expressions for the poles are rather involved, so it is necessary to break these up. The poles for this equation are

$$s_1 = \alpha_1 \qquad (151)$$

and

$$s_{2,3} = \alpha_2 \pm i\alpha_3 \qquad (152)$$

The coefficients for these equations are

$$\alpha_1 = \sqrt[3]{\alpha_5} + \frac{\alpha_6}{9} - \frac{Z\left(c'(m'-1) + Z'\right)}{3MZ'm'} \qquad (153)$$

$$\alpha_2 = -\frac{1}{2}\sqrt[3]{\alpha_5} - \frac{\alpha_6}{18} - \frac{Z\left(c'(m'-1) + Z'\right)}{3MZ'm'} \qquad (154)$$

$$\alpha_3 = \frac{\sqrt{3}}{2}\sqrt[3]{\alpha_5} - \frac{\sqrt{3}\alpha_6}{18} \qquad (155)$$

$$\alpha_4 = m'c'Z' \qquad (156)$$

$$\alpha_5 = \frac{Z^3}{18M^3m'^2Z'^3}\sqrt{\begin{pmatrix} 12m' + 12Z'^4 + 36m'^2 + 6c'^2m'^2 - 3Z'^2 - 3c'^2 + 96\alpha_4 \\ -24m'^3c'Z' + 66m'^2c'Z' + 6\alpha_4^2 + 6m'^3c'^3Z' - 30m'^2c'^3Z \\ -24m'c'Z'^3 + 18m'c'^2Z'^2 + 42m'c'^3Z' + 6c'Z' + 12m'^4 \\ +36m'^3 - 3m'^4c'^2 + 24m'^2Z'^2 - 60m'Z'^2 - 42Z'^3c' \\ +48Z'^2c'^2 - 18c'^3Z' - 3m'^2c'^4Z'^2 + 6m'c'^3Z'^3 \\ +6m'c'^4Z'^2 - 3Z'^2c'^4 - 3Z'^4c'^2 + 6Z'^3c'^3 \end{pmatrix}} - \frac{Z^3}{54M^3Z'^3m'^3}\begin{pmatrix} -9c'm'^3 + 18Z'm'^2 - 3c'^2Z'm'^2 - 3m'c'Z'^2 - 3c'^2Z'm' \\ -9m'Z' + 9c'm' + 2c'^3m'^3 - 6c'^3m'^2 + 6c'^3m' + 2Z'^3 - 6c'Z'^2 \\ +6c'^2Z' - 2c'^3 \end{pmatrix} \qquad (157)$$

$$\alpha_6 = \frac{Z^2}{M^2Z'^2m'^2\sqrt[3]{\alpha_5}}\left(-3m'^2 - \alpha_4 - 3m' + c'^2m'^2 - 2c'^2m' + Z'^2 - 2c'Z' + c'^2\right) \qquad (158)$$

where $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6$ = Consolidation Constants for Pile Top Forces

Applying Equation (98) to the Laplace transform and solving with simplification and elimination of the complex terms, this yields

$$u_t(0,t) = \frac{V_0 Z}{\left(\alpha_1^2 - 2\alpha_1\alpha_2 + \alpha_2^2 + \alpha_3^2\right) M^2 m' Z'^2 \alpha_3}$$

$$\left(\alpha_3 e^{\alpha_1 t}\left(Z - \alpha_1 c' Z' M\right) + e^{\alpha_2 t}\left(\begin{array}{l}\alpha_3 \cos(\alpha_3 t)\left(\alpha_1 c' Z' M - Z\right) \\ + \sin(\alpha_3 t)\left(Z(\alpha_2 - \alpha_1) + c' Z' M\left(\alpha_2\alpha_1 - \alpha_2^2 - \alpha_3^2\right)\right)\end{array}\right)\right) \quad (159)$$

Further simplification is possible if

$$\gamma_1 = \frac{V_0 Z}{\left(\alpha_1^2 - 2\alpha_1\alpha_2 + \alpha_2^2 + \alpha_3^2\right) M^2 m' Z'^2 \alpha_3} \quad (160)$$

$$\gamma_2 = \alpha_3\left(\alpha_1 c' Z' M - Z\right) \quad (161)$$

$$\gamma_3 = Z(\alpha_2 - \alpha_1) + c' Z' M\left(\alpha_2\alpha_1 - \alpha_2^2 - \alpha_3^2\right) \quad (162)$$

where $\gamma_1, \gamma_2, \gamma_3$ = Consolidation Constants for Pile Top Forces

are substituted. Substituting these into Equation (159) yields

$$u_t(0,t) = \gamma_1\left(\gamma_2 e^{\alpha_1 t} - e^{\alpha_2 t}\left(\gamma_2 \cos(\alpha_3 t) - \gamma_3 \sin(\alpha_3 t)\right)\right) \quad (163)$$

There are two things that need to be noted about this solution.

First, this solution is generally valid unless α_3=0. In this case s_2=s_3 and there is a multiple pole as opposed to two single poles. This alters the complex integration. However, in view of the fact that the known values of the variables are more precise than accurate (especially the values for the cushion material properties,) the possibility that one would need to use the solution when α_3=0 is rather remote.

Second, the equation is valid in the form presented if α_1, α_2, and α_3 are real. If these are imaginary, the transformations necessitated by the imaginary values of the coefficients are relatively simple. If they are complex, then these transformations may become more difficult. This means that, if all cases for this solution be considered, solutions in all parts of the complex plane must be included and not simply solutions that are only real.

2. General Solution for a Semi-Infinite Damped Pile

Before the damped wave equation is solved, it would be helpful to consider the solution of the semi-infinite pile in the damped case, just as it was with the undamped case.

a) Theory of the Damped Solution

Beginning with Equation (55), the Laplace transform of this equation with respect to time is

$$c^2 U_{tt}(x,s) = \left(sU(x,s) - u(x,0)\right)s - u_t(x,0) + aU(x,s) + 2b\left(sU(x,s) - u(x,0)\right) \quad (164)$$

Substituting the initial conditions of Equations (4) and (5), the solution for this differential equation is

$$U(x,s) = C_1 e^{\sqrt{s^2+2bs+a}\frac{x}{c}} + C_2 e^{-\sqrt{s^2+2bs+a}\frac{x}{c}} \quad (165)$$

In order to prevent an unbounded condition,

$$C_1 = 0 \quad (166)$$

Substituting this yields

$$U(x,s) = C_2 e^{-\sqrt{s^2+2bs+a}\frac{x}{c}} \quad (167)$$

Consider a generalized forcing function $F_0(t)$ acting on the pile top. The boundary condition for the pile top is given by the equation

$$F_0(t) = -Zcu_x(0,t) \quad (168)$$

The Laplace transform of this equation is

$$\mathscr{L}(F_0(t)) = P(s) = -ZcU_x(0,s) \quad (169)$$

where $P(s)$ = Laplace Transform for Pile Top Force

Substituting Equation (167) and then x = 0,

$$P(s) = ZC_2\sqrt{s^2+2bs+a} \quad (170)$$

Solving for C_2, this yields

$$C_2 = \frac{P(s)}{Z\sqrt{s^2+2bs+a}} \quad (171)$$

Substituting this back into Equation (167),

$$U(x,s) = \frac{P(s)}{Z}\frac{e^{-\sqrt{s^2+2bs+a}\frac{x}{c}}}{\sqrt{s^2+2bs+a}} \quad (172)$$

This corresponds with Equation (8) for the undamped case.

This equation is deceptively simple, because now, instead of the meromorphic equations seen earlier, there is a polymorphic equation whose complex integration requires integrating around branch cuts instead of simply poles. So the method of Equation (98) cannot be applied in a simplistic fashion to the inverse Laplace transform of this equation.

The alternative is to use Borel's theorem. In this case the equation can be divided into the expressions

$$\hat{F}(s) = \frac{P(s)}{Z} \quad (173)$$

and

$$\hat{G}(s) = \frac{e^{-\sqrt{s^2+2bs+a}\frac{x}{c}}}{\sqrt{s^2+2bs+a}} \quad (174)$$

where $\hat{F}(s)$ = Laplace Transform of Pile Top Forcing Function

$\hat{G}(s)$ = Laplace Transform of Pile Response Function

The inverse Laplace transforms of these expressions are, respectively,

$$\hat{f}(t) = \frac{F_0(t)}{Z} \quad (175)$$

for the forcing function and (Oberhettinger and Badii, 1973)

$$\hat{g}(t) = e^{-bt} I_0\left(\sqrt{(b^2-a)\left(t^2 - \left(\frac{x}{c}\right)^2\right)}\right), t > \frac{x}{c} \quad (176)$$

$$\hat{g}(t) = 0, t \leq \frac{x}{c} \quad (177)$$

where $\hat{f}(t)$ = Inverse Laplace Transform of Pile Top Forcing Function

$\hat{g}(t)$ = Inverse Laplace Transform of Pile Response Function

for the response function. The inverse Laplace transform of Equation (172) can be expressed as

$$u(x,t) = \hat{f}(t) * \hat{g}(t) = \frac{1}{Z}\int_{\frac{x}{c}}^{t} e^{-b\tau} I_0\left(\sqrt{(b^2-a)\left(\tau^2 - \left(\frac{x}{c}\right)^2\right)}\right) F_0(t-\tau)\,d\tau, t > \frac{x}{c} \quad (178)$$

where τ = Dummy Variable for Borel's Theorem, sec.

and zero for other times. This is identical to the result of Van Koten et. al (1980) except for changes in the notation. It is similar to the solution of Webster (1960); however, he assumes non-zero initial conditions.

b) Discussion of the Solution

This solution has a number of important results, which need to be understood completely.

First, this equation has no straightforward closed form solution. The most direct method of solving this equation is to substitute a power series or polynomial approximation for the Bessel function and perform termwise integration. How this is performed depends upon the values of the argument of the Bessel function and the desired complexity of the resulting algebra.

Second, for the pile top,

$$u(x,t) = \hat{f}(t) * \hat{g}(t) = \frac{1}{Z}\int_{0}^{t} e^{-b\tau} I_0\left(\sqrt{(b^2-a)(\tau^2)}\right) F_0(t-\tau)\,d\tau, t > 0 \quad (179)$$

This result also appears in Zhou and Liang (1996).

Although this is obviously simpler than Equation (178), it means that Equation (17) and its related equations do not apply to this problem. Thus, any force-time relationships that are computed for semi-infinite piles with distributed soil elasticity and dampening along the pile shaft should be computed with this equation and not Equation (17) in the pile top model. The reason for this of course is that piles with distributed soil elasticity and dampening are reflecting stress waves back to the pile top, which interact with whatever is forcing the pile down during impact.

This, however, relates to the third observation, namely that any function used as the forcing function results in very difficult integration depending upon what kind of function is used. For the functions derived for pile top force, this can be potentially overwhelming – especially if one considers that these equations are strictly speaking inapplicable.

For this thesis this problem is dealt with by substituting a constant force that acts for a time δ after impact. The force is zero afterward. This is expressed as

$$F_0(t) = F_0, t < \delta, \delta \leq \frac{L}{c} \quad (180)$$

This is essentially the same forcing function as used by Van Koten et. al. (1980). The difference in this solution is twofold. First, the force-time curve used is matched with the semi-infinite solution by having the two force-time curves have the same impulse and maximum force (the latter to match the pile stresses.) *Second, the time used to begin the Fourier series solution is altered to t = δ. rather than t = L/c.* This is as opposed to Van Koten's solution of using an equal negative forcing function after the end of the impulse to simulate a zero pile top forcing function. The method of dividing the solution makes this possible.

The problem of matching the force-time curves is discussed in more detail with the numerical comparison.

c) Analysis of the Bessel Function and its Argument

Since the Bessel function represents the central difficulty in the analysis of this problem, it was considered first. The square of the argument is first defined as

$$\hat{z} = \left(b^2 - a\right)\left(t^2 - \left(\frac{x}{c}\right)^2\right) \quad (181)$$

where $\hat{z}$ = Bessel Function Argument for Damped Case

The first parentheses have the dimensions of inverse time and the second of time. For simplicity's sake the quantities were rearranged so that both of the parenthetical terms were dimensionless. With judicious rearranging and substitution,

$$\hat{z} = \left(\left(b^2 - a\right)\left(\frac{L}{c}\right)^2\right)\left(\left(\frac{tc}{L}\right)^2 - \left(\frac{x}{L}\right)^2\right) \quad (182)$$

Now the quantity

$$\hat{d} = \left(b^2 - a\right)\left(\frac{L}{c}\right)^2 \quad (183)$$

where $\hat{d}$ = Pile Shaft Damping and Elasticity Ratio

is defined. Then Equation (182) can be rewritten as

$$\hat{z} = \hat{d}\left(\left(\frac{tc}{L}\right)^2 - \left(\frac{x}{L}\right)^2\right) \qquad (184)$$

To analyse the argument, the remaining parenthetical expression is basically a dimensionless time quantity in "units" of L/c. Thus, the maximum value for this quantity takes place at the pile top (x = 0). For the first "semi-infinite" phase of the analysis, since the maximum time is L/c, the maximum value for this quantity is unity.

For a case defined in this way, the maximum value of $\hat{z}$ is thus completely dependent upon $\hat{d}$. There are three basic cases for this variable, which depend upon a and b since L and c are both positive.

1. $b^2 > a, \hat{d} > 0$. In this case the I_0 Bessel function remains, which is unbounded as the argument increases. This would create difficulties except for the exponential, which approaches zero as I_0 approaches infinity with increasing time.
2. $b^2 = a, \hat{d} = 0$. The Bessel function is valued at unity. This is analogous to the "balanced line" condition which appears in transmission line problems and which simplifies the analysis considerably. Unfortunately this cannot be counted on taking place in piling.
3. $b^2 < a, \hat{d} < 0$. In this case the J_0 Bessel function is used for the negative value of the argument. This results in oscillatory response.

More importantly this variable defines in large part (except for the exponential decay, which is a function of b) the response of the pile to excitation at the top, not only in quantity but in its nature as well. To obtain variables such as this is one of the objects of closed form analysis and thus it is an important result even without a subsequent solution.

Using the notation for the argument developed earlier, the power series representation for the Bessel function is

$$I_0\left(\sqrt{\hat{z}}\right) = \sum_{m=0}^{\infty} \frac{\left(\sqrt{\hat{z}}\right)^{2m}}{2^{2m} m!^2} \qquad (185)$$

This can be simplified to

$$I_0\left(\sqrt{\hat{z}}\right) = \sum_{m=0}^{\infty} \frac{\left(\frac{\hat{z}}{4}\right)^m}{m!^2} = 1 + \frac{\hat{z}}{4} + \frac{\hat{z}^2}{64} + \frac{\hat{z}^3}{2304} \cdots \qquad (186)$$

The series is valid for all values of $\hat{z}$, and furthermore automatically changes the nature of the Bessel function with the changes in sign of the argument.

As is the case with many functions of this type, the function converges everywhere, but how many terms are needed for convergence? This depends on the value of the argument. As $\hat{z}$ increases, the number of terms required for convergence also increases. It is necessary to analyse possible values for the argument to determine the number of terms necessary for convergence.

d) Determination of Maximum Values for $\hat{d}$

The method chosen for the closed form solution has a maximum time of transition from the semi-infinite pile method to the Fourier series to be t = L/c. In the case of the damped solution the turnover time has been altered to be t = δ < L/c.

Therefore,

$$\left|\hat{z}_{\max}\right| \leq \left|\hat{d}_{\max}\right| \quad \text{.......... (187)}$$

The inequality is significant because, in an actual case, the maximum value of d_{max} may not be encountered in a Bessel function argument. In any case absolute values are used because the convergence requires more terms of the series to be used as the value of the argument gets further and further away from zero, irrespective of whether this is in the positive or negative direction. Negative values (J_0 Bessel function) are more critical as convergence on an oscillating function is generally more difficult than on one that approaches infinity.

Substituting Equations (66) and (67) into Equation (183),

$$\hat{d} = \left(\frac{\rho_s G_s}{4\rho^2 A r_g} - \frac{\pi G_s}{\rho A}\right)\left(\frac{L}{c}\right)^2 \quad \text{.......... (188)}$$

and this can be reduced to

$$\hat{d} = \frac{G_s}{\rho A}\left(\frac{\rho_s}{4\rho r_g} - \pi\right)\left(\frac{L}{c}\right)^2 \quad \text{.......... (189)}$$

It is interesting to note at this point that the "balanced line" condition is achieved when

$$r_g = \frac{\rho_s}{\rho}\frac{1}{4\pi} \quad \text{.......... (190)}$$

This illustrates the importance of the geometry ratio in these calculations. For any given pile material and soil combination, the geometry of the pile determines the relationship between the transmissibility of the pile and the effect on that transmissibility by the soil.

It is evident from Equation (189) that a large geometry ratio results in larger, negative values of the argument. Equation (59) shows that such a ratio can be achieved if the area is at a maximum value relative to the perimeter, or the perimeter at a minimum relative to the area. This last condition can be achieved with the solid circle; the geometry ratio for this shape is

$$r_{g_{circle}} = \frac{1}{4\pi} \approx .0796 \quad \text{.......... (191)}$$

Except for wood piles, solid circular piles are rare. With pipe piles, there is a hollow area in the centre; both area and geometry ratios decrease. With concrete piles, there is an increase in the perimeter relative to the area due to the use of flat sides; again the geometry ratio increases.

This effect is augmented with a hollow area in the centre of the pile, as is common with concrete piles. With H-beams and sheet piles, there is pretty much the same effect. So this can be used to determine the maximum negative value of the argument.

The expression can be further simplified for search purposes if by assuming the area in Equation (189) to be a solid circle; additionally eliminating the acoustic velocity, this equation then can be reduced to

$$\hat{d} = \frac{4G_s}{E}\left(\frac{\rho_s}{\rho} - 1\right)\left(\frac{L}{D}\right)^2 \quad (192)$$

where D = Pile Outside Diameter, m

The result of this equation is very dependent upon the density ratio of the soil to the pile. For steel piles, this may yield unrealistically low values. Defining

$$d' = \frac{d}{D} \quad (193)$$

where d' = Pipe Pile Diameter Ratio
d = Pile Inside Diameter, m

for hollow piling this equation becomes

$$\hat{d} = \frac{4G_s}{E\left(1 - d'^2\right)}\left(\frac{\rho_s}{\rho}\frac{1}{1 - d'^2} - 1\right)\left(\frac{L}{D}\right)^2 \quad (194)$$

This equation gives a relatively simple expression, which can be used to compute extreme values for the Bessel function argument, and thus determine how many terms in the power series are necessary.

To obtain a range of values for the argument, soil Young's modulus and Poisson's ratio values were taken from Das (1984) and the necessary values for the soil shear modulus and density were computed. Typical values for the pile material properties were assumed. The process then proceeded as follows:

1) Extreme values of the soil shear modulus were computed using standard mechanics of materials methods. A high value of 50 MPa was chosen for a maximum value of the argument; however, this was largely driven by one soil type and probably can be reduced for practical cases.
2) Density ratios between soil and pile were computed for the three pile materials. An average value was computed. There was relatively little variance of this ratio with a given pile material.
3) Two ratios were assumed: the diameter ratio and the L/D ratio. For wood and concrete piles, a diameter ratio of zero (solid piles) was assumed and for steel a ratio of 0.8, which represents a very thick walled steel pile. As for the L/D, this is a relatively common quantity in pile analysis; an L/D of 50 was assumed. This is the submerged L/D in the soil and not the total L/D for the pile; however, in this model the entire pile was submerged in the soil.

The results are shown in Table 1. The minimum value occurs with the concrete pile; in this case $\hat{d} > -6$. This would indicate that four terms of the series in Equation (182) would be needed for an ideal approximation. However, the complete argument of the Bessel function is in fact a polynomial; each power of this produces yet a higher value and more involved polynomial. However it is probable that attainment of this low value of $\hat{d}$ is in fact unlikely. Therefore, for simplicity's sake, the first three terms of the series are used.

Table 1 Soil Properties Survey and Values of $\hat{d}$

	Soil Young's Modulus, MPa		Soil Poisson's Ratio		Calculated Soil Shear Modulus, MPa		Density kg/m^3	Density Ratios		
Soil Type	Min.	Max.	Min.	Max.	"Min. G"	"Max. G"		Wood	Concrete	Steel
Loose Sand	10.35	24.15	0.2	0.4	3.70	10.06	1500	2.307	0.652	0.192
Medium Dense Sand	17.25	27.6	0.25	0.4	6.16	11.04	1600	2.461	0.696	0.205
Dense Sand	34.5	55.2	0.3	0.45	11.89	21.23	1600	2.461	0.696	0.205
Silty Sand	10.35	17.25	0.2	0.4	3.70	7.19	1600	2.461	0.696	0.205
Sand and Gravel	69	172.5	0.15	0.35	25.55	75	2000	3.076	0.869	0.256
Soft Clay	2.07	5.18					1700	2.615	0.739	0.217
Medium Clay	5.18	10.35	0.2	0.5	1.73	4.31	1700	2.615	0.739	0.217
Stiff Clay	10.35	24.15					1700	2.615	0.739	0.217
Average/Value Used						50		2.576	0.728	0.214
								Diameter Ratio		
								0	0	0.8
								Pile Density, kg/m^3		
								650	2300	7800
								Pile Modulus of Elasticity, GPa		
								13.8	25	210
								L/D		
								50	50	50
								$\hat{d}$		
								57.1	-5.43	-2.69

Note: Values for soil properties are given in Das (1984).

It is interesting to note that Van Koten et. al. (1980) use only two terms of this series, which gives them a linear equation. This is because they assume that the minimum value of the argument to be -1. Zhou and Liang (1996) use four terms of this series, but because their ultimate objective is to analyse pile top signals x = 0 and the resulting polynomial is much simpler.

e) *Practical Statement of the Solution for t<δ*

The objective in this analysis is to provide a relatively simple solution for this problem. It is now possible to finalize this solution for the first portion of time.

If the Bessel Function in Equation (178) is changed into the series of Equation (186) for three terms and substitute Equation (180) for the forcing function,

$$u(x,t) = \frac{F_0}{Z}\int_{\frac{x}{c}}^{t} e^{-b\tau}\left(1+\frac{\left((b^2-a)\left(\tau^2-\left(\frac{x}{c}\right)^2\right)\right)}{4}+\frac{\left((b^2-a)\left(\tau^2-\left(\frac{x}{c}\right)^2\right)\right)^2}{64}\right)d\tau, \frac{x}{c}<t<\delta \quad \text{.......... (195)}$$

where δ = Time of Square Wave Simplified Impulse, sec.

Integration (with appropriate substitutions) of this yields

$$u(x,t) = \frac{F_o}{b^5L^4Z}\left(\left(-\frac{\hat{d}^2c^4b^4}{64}t^4-\frac{\hat{d}^2c^4b^3}{16}t^3+\left(\frac{\hat{d}^2c^2x^2b^4}{32}-\frac{\hat{d}c^2b^4L^2}{4}-\frac{3\hat{d}^2c^4b^2}{16}\right)t^2 + \left(-\frac{\hat{d}c^2b^3L^2}{2}-\frac{3\hat{d}^2c^4b}{8}+\frac{\hat{d}^2c^2x^2b^3}{16}\right)t+\frac{\hat{d}^2c^2x^2b^2}{16}+\frac{\hat{d}L^2x^2b^4}{4}-\frac{\hat{d}^2x^4b^4}{64}-b^4L^4-\frac{3\hat{d}^2c^4}{8}-\frac{\hat{d}c^2L^2b^2}{2}\right)e^{-bt} + \left(\frac{\hat{d}c^2b^2L^2}{2}+\frac{\hat{d}^2c^2x^2b^2}{8}+\frac{3\hat{d}^2bc^3x}{8}+b^4L^4+\frac{\hat{d}b^3L^2cx}{2}+\frac{3\hat{d}^2c^4}{8}\right)e^{-\frac{bx}{c}}\right) \quad \text{.............................. (196)}$$

The velocity in the pile is

$$u_t(x,t) = \frac{F_0\left(\hat{d}t^2c^2-\hat{d}x^2+8L^2\right)^2 e^{-bt}}{64ZL^4} \quad \text{.. (197)}$$

and the pile stress is

$$\sigma(x,t)=\frac{F_o}{b^5L^4A}$$
$$\left(\begin{array}{l}\left(\left(\frac{\hat{d}^2c^2xb^4}{16}\right)t^2\left(\frac{\hat{d}^2c^2xb^3}{8}\right)t+\frac{\hat{d}^2c^2xb^2}{8}+\frac{\hat{d}L^2xb^4}{2}-\frac{\hat{d}^2x^3b^4}{16}\right)ce^{-bt}\\+\left(\frac{\hat{d}^2c^2xb^2}{4}+\frac{3\hat{d}^2bc^3}{8}+\frac{\hat{d}b^3L^2c}{2}\right)ce^{-\frac{bx}{c}}\\-\left(\frac{\hat{d}c^2b^2L^2}{2}+\frac{\hat{d}^2c^2x^2b^2}{8}+\frac{3\hat{d}^2bc^3x}{8}+b^4L^4+\frac{\hat{d}b^3L^2cx}{2}+\frac{3\hat{d}^2c^4}{8}\right)be^{-\frac{bx}{c}}\end{array}\right) \quad (198)$$

At the turnover time t=δ, the displacement and velocity are

$$u(x,\delta)=f(x)=\frac{F_0}{b^5L^4Z}$$
$$\left(\begin{array}{l}\left(\begin{array}{l}-\frac{\hat{d}^2c^4b^4}{64}\delta^4-\frac{\hat{d}^2c^4b^3}{16}\delta^3+\left(\frac{\hat{d}^2c^2x^2b^4}{32}-\frac{\hat{d}c^2b^4L^2}{4}-\frac{3\hat{d}^2c^4b^2}{16}\right)\delta^2\\\left(-\frac{\hat{d}c^2b^3L^2}{2}-\frac{3\hat{d}^2c^4b}{8}+\frac{\hat{d}^2c^2x^2b^3}{16}\right)\delta+\frac{\hat{d}^2c^2x^2b^2}{16}+\frac{\hat{d}L^2x^2b^4}{4}-\\\frac{\hat{d}^2x^4b^4}{64}-b^4L^4-\frac{3\hat{d}^2c^4}{8}-\frac{\hat{d}c^2L^2b^2}{2}\end{array}\right)e^{-b\delta}\\+\left(\frac{\hat{d}c^2b^2L^2}{2}+\frac{\hat{d}^2c^2x^2b^2}{8}+\frac{3\hat{d}^2bc^3}{8}+b^4L^4+\frac{\hat{d}b^3L^2cx}{2}+\frac{3\hat{d}^2c^4}{8}\right)e^{-\frac{bx}{c}}\end{array}\right) \quad (199)$$

and

$$u_t(x,\delta)=g(x)=\frac{F_0\left(\hat{d}t^2c^2-\hat{d}x^2+8L^2\right)^2e^{-b\delta}}{64ZL^4} \quad (200)$$

These last equations are used for the computation of the Fourier coefficients.

3. Fourier Series Solution for $t>\delta$

Now that the first part of the solution (with the transitional values) is complete, the solution for the time after the step load has ceased can be derived as well.

a) Determination of the Eigenvalues and Eigenfunctions

To begin Equation (114) is modified to

$$t'=t-\delta \quad (201)$$

and Equation (115) to

$$u(x,t')=e^{\frac{\beta ct'+i\lambda x}{L}} \quad (202)$$

Substituting this into Equation (55) and solving for β yields

$$\beta = \hat{\alpha} \pm i\hat{\omega} \qquad (203)$$

where

$$\hat{\alpha} = -\frac{bL}{c} \qquad (204)$$

and

$$\hat{\omega} = i\sqrt{\left(\frac{L}{c}\right)^2 \left(b^2 - a\right) - \lambda^2} = \sqrt{\lambda^2 - \hat{d}} \qquad (205)$$

This is an important result because it relates the previous results to those in this phase of the analysis.

Substituting these results in to Equation (202),

$$u(x,t') = e^{\frac{\alpha c t' + i\lambda x \pm i\omega c t'}{L}} \qquad (206)$$

and this expands to

$$u(x,t') = e^{-bt'}\left(C_1 \cos\left(\frac{\sqrt{\lambda^2 - \hat{d}}\,c}{L} t'\right) + C_2 \sin\left(\frac{\sqrt{\lambda^2 - \hat{d}}\,c}{L} t'\right)\right)\left(C_3 \cos\left(\frac{\lambda x}{L}\right) + C_4 \sin\left(\frac{\lambda x}{L}\right)\right) \qquad (207)$$

This is similar to Equation (121) in the undamped case, except that the circular functions are used as opposed to the hyperbolic ones.

Since the pile top boundary condition is the same as the undamped case, Equations (123) and (127) apply and the expression reduces to

$$u(x,t') = e^{-bt'} \cos\left(\frac{\lambda x}{L}\right)\left(C_1 \cos\left(\frac{\sqrt{\lambda^2 - \hat{d}}\,c}{L} t'\right) + C_2 \sin\left(\frac{\sqrt{\lambda^2 - \hat{d}}\,c}{L} t'\right)\right) \qquad (208)$$

Turning to the pile toe, first Equation (88) is substituted into (84) and this result is then substituted into the previous equation. Solving for λ,

$$\tan(\lambda) = \frac{\sin(\lambda)}{\cos(\lambda)} = \frac{1}{\lambda}\frac{k_t A_t L}{Zc - E_s A_t} \qquad (209)$$

Although this is very similar to Equation (129), the main difference between the damped and the undamped case is that the former contains many more "options" that complicate the results. Eigenvalues cannot be blindly extracted from this equation unless the possibilities of the values of the coefficients are considered, and specifically the denominator of the right hand side.

There are three possibilities for this equation.

1) $Zc > E_s A_t$. In this case the right hand side is positive and thus a similar result to Equation (131) is found, namely

$$\tan(\lambda_n) = \frac{\sin(\lambda_n)}{\cos(\lambda_n)} = \frac{1}{\lambda_n}\frac{k_t A_t L}{Zc - E_s A_t},$$
$$\left(n - \frac{3}{2}\right)\pi \le \lambda_n \le \left(n - \frac{1}{2}\right)\pi, n = 1,2,3,\ldots\infty, Zc > E_s A_t \quad (210)$$

2) $Zc < E_s A_t$. Here a transformation similar to Equation (126) is possible to obtain a unique value for λ,

$$\tanh(\lambda) = \frac{\sinh(\lambda)}{\cosh(\lambda)} = \frac{1}{\lambda}\frac{k_t A_t L}{E_s A_t - Zc}, Zc < E_s A_t \quad (211)$$

3) $Zc = E_s A_t$. The denominator on the right side is zero. The same result on the left hand side can be obtained if

$$\lambda_n = \left(n - \frac{1}{2}\right)\pi, n = 1,2,3,\ldots\infty \quad (212)$$

and these are of course very regular eigenvalues.

Generally speaking for piles Case (1) applies and thus it is considered to be the "normative" case, although Case (2) is readily conceivable for closed ended piling. In this thesis active consideration to Case (3) is not given.

Assuming Case (1) to be true, the solution for the displacement is

$$u(x,t') = e^{-bt'}\sum_{n=1}^{\infty}\cos\left(\frac{\lambda_n x}{L}\right)\left(C_{1n}\cos\left(\frac{\sqrt{\lambda_n^2 - \hat{d}c}}{L}t'\right) + C_{2n}\sin\left(\frac{\sqrt{\lambda_n^2 - \hat{d}c}}{L}t'\right)\right) \quad (213)$$

the velocity

$$u_{t'}(x,t') = e^{-bt'}$$
$$\sum_{n=1}^{\infty}\left(\begin{array}{l} -b\cos\left(\frac{\lambda_n x}{L}\right)\left(C_{1n}\cos\left(\frac{\sqrt{\lambda_n^2 - \hat{d}c}}{L}t'\right) + C_{2n}\sin\left(\frac{\sqrt{\lambda_n^2 - \hat{d}c}}{L}t'\right)\right) \\ + \frac{\sqrt{\lambda_n^2 - \hat{d}c}}{L}\cos\left(\frac{\lambda_n x}{L}\right)\left(-C_{1n}\sin\left(\frac{\sqrt{\lambda_n^2 - \hat{d}c}}{L}t'\right) + C_{2n}\cos\left(\frac{\sqrt{\lambda_n^2 - \hat{d}c}}{L}t'\right)\right)\end{array}\right) \quad (214)$$

and the stress

$$\sigma(x,t') = \frac{Zc}{AL}e^{-bt'}\sum_{n=1}^{\infty}\lambda_n\sin\left(\frac{\lambda_n x}{L}\right)\left(C_{1n}\cos\left(\frac{\sqrt{\lambda_n^2 - \hat{d}c}}{L}t'\right) + C_{2n}\sin\left(\frac{\sqrt{\lambda_n^2 - \hat{d}c}}{L}t'\right)\right) \quad (215)$$

As was the case with the eigenvalues, the form of these expressions depends upon the values of the existing constants. In this case the critical constant is $\hat{\omega}$. In their present form these expressions are only valid if $\hat{\omega}$ is real. If $\hat{\omega}$ is imaginary, the equations for displacement, velocity, and stress respectively are

$$u(x,t') = e^{-bt'} \sum_{n=1}^{\infty} \cos\left(\frac{\lambda_n x}{L}\right)\left(C_{1n} \cosh\left(\frac{\sqrt{\hat{d}-\lambda_n^2}\,c}{L} t'\right) + C_{2n} \sinh\left(\frac{\sqrt{\hat{d}-\lambda_n^2}\,c}{L} t'\right)\right) \quad \text{(216)}$$

$$u_{t'}(x,t') =$$

$$e^{-bt'} \sum_{n=1}^{\infty} \left(\begin{array}{l} -b \cos\left(\frac{\lambda_n x}{L}\right)\left(C_{1n} \cosh\left(\frac{\sqrt{\hat{d}-\lambda_n^2}\,c}{L} t'\right) + C_{2n} \sinh\left(\frac{\sqrt{\hat{d}-\lambda_n^2}\,c}{L} t'\right)\right) \\ + \frac{\sqrt{\hat{d}-\lambda_n^2}\,c}{L} \cos\left(\frac{\lambda_n x}{L}\right)\left(C_{1n} \sinh\left(\frac{\sqrt{\hat{d}-\lambda_n^2}\,c}{L} t'\right) + C_{2n} \cosh\left(\frac{\sqrt{\hat{d}-\lambda_n^2}\,c}{L} t'\right)\right) \end{array} \right) \quad \text{(217)}$$

$$\sigma(x,t') = \frac{Zc}{AL} e^{-bt'} \sum_{n=1}^{\infty} \lambda_n \sin\left(\frac{\lambda_n x}{L}\right)\left(C_{1n} \cosh\left(\frac{\sqrt{\hat{d}-\lambda_n^2}\,c}{L} t'\right) + C_{2n} \sinh\left(\frac{\sqrt{\hat{d}-\lambda_n^2}\,c}{L} t'\right)\right) \quad \text{(218)}$$

It is important to note that these two sets of equations are not mutually exclusive; it is entirely possible that in the progression of eigenvalues, the value of $\hat{\omega}$ may change from imaginary to real. In this case the Fourier series becomes a mixture of hyperbolic and circular functions.

The case of $\hat{\omega} = 0$ is not considered as it is unlikely that this case would be encountered for reasons stated before.

b) Computation of the Fourier Coefficients

For either the circular or hyperbolic function series, the initial displacement and velocity can be represented by the Fourier series

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} C_{1n} \cos\left(\frac{\lambda_n x}{L}\right) \quad \text{(219)}$$

and

$$u_t(x,0) = g(x) = \sum_{n=1}^{\infty} \left(-bC_{1n} + \frac{\sqrt{\lambda_n^2 - d}\,c}{L} C_{2n}\right) \cos\left(\frac{\lambda_n x}{L}\right) \quad \text{(220)}$$

This is significant because the Fourier coefficients are the same for both the circular and hyperbolic series. A similar technique to the one for the undamped case was used; however, because there are some important differences in the procedure, the derivation is discussed completely. The function f(x) is given by Equation (199). Multiplying both sides of Equation (219) by $\cos(\lambda_m x/L)$, for each term (and coefficient,)

$$C_{1n} \cos\left(\frac{\lambda_n x}{L}\right) \cos\left(\frac{\lambda_m x}{L}\right) = \cos\left(\frac{\lambda_m x}{L}\right) f(x) \quad \text{(221)}$$

Integrating both sides,

$$C_{1n}\int_{o}^{L}\cos\left(\frac{\lambda_n x}{L}\right)\cos\left(\frac{\lambda_m x}{L}\right)dx=\int_{0}^{\delta c}\cos\left(\frac{\lambda_m x}{L}\right)f(x)dx \quad \text{.......} (222)$$

The right hand side is not integrated to L but only to δc. This is because the displacement is zero from this point to the pile toe; thus, this integration is not shown. However, the left hand side is integrated for the full length of the pile.

The left hand side integral is given in Equation (141). As before orthogonality allows integration only for the case when m=n. Substituting Equation (199) for f(x), performing all integration and solving for the first Fourier coefficient,

$$C_{1n}=\frac{2\lambda_n F_0}{Z\left((bL)^2-(c\lambda_n)^2\right)^3\left(\cos(\lambda_n)\sin(\lambda_n)+\lambda_n\right)}$$
$$\left(\begin{array}{l}\dfrac{e^{-b\delta}}{8\lambda_n^5}\left(\left(\begin{array}{l}2\hat{d}^2\lambda_n^4 b^3 L^3\delta^2 c^4-3\hat{d}^2 b^5 L^7-6\hat{d}^2 c^4\delta\lambda_n^4 b^2 L^3\\ -4L^7\hat{d}\lambda_n^2 b^5-15\hat{d}^2 c^4\lambda_n^4 bL^3-12L^3\hat{d}\lambda_n^6 bc^4\\ -16L^5\hat{d}\lambda_n^4 b^3 c^2+\left(\hat{d}c\delta\lambda_n\right)^2(bL)^5+\left(\hat{d}\delta\right)^2\left(c\lambda_n\right)^6 bL\\ -10\left(\hat{d}c\lambda_n\right)^2 b^3 L^5-\left(\hat{d}c\lambda_n\right)^2\delta b^4 L^5-8bc^4 L^3\lambda_n^8\\ -5\hat{d}^2 c^6\lambda_n^6 L\delta-8c^4 L^3\hat{d}\lambda_n^6 b^2\delta-4c^6 L\hat{d}\lambda_n^8\delta\\ -16c^2 L^5\lambda_n^6 b^3-4\hat{d}c^2\delta\lambda_n^4 L^5 b^4-8L^7\lambda_n^4 b^5\end{array}\right)\sin\left(\frac{\lambda_n\delta c}{L}\right)\right.\\ \left.\qquad+\left(\begin{array}{l}7\hat{d}^2 c^5\lambda_n^5\delta bL^2+10\hat{d}^2 c^3\lambda_n^3\delta b^3 L^4+4\hat{d}c^5\lambda_n^7\delta bL^2\\ +4\hat{d}c\lambda_n^3\delta b^5 L^6-8\hat{d}^2 c^5\lambda_n^5 L^2+8\hat{d}c^3\lambda_n^5\delta b^3 L^4\\ -8\hat{d}c^3\lambda_n^5 b^2 L^4-8\hat{d}c^5\lambda_n^7 L^2+\hat{d}^2 c^3\lambda_n^3\delta^2 b^4 L^4\\ +2\hat{d}^2 c^5\lambda_n^5\delta^2 b^2 L^2+\hat{d}^2 c^7\lambda_n^7\delta^2+3\hat{d}^2 c\lambda_n^1\delta b^5 L^6\\ -8c^5\lambda_n^9 L^2-8c\lambda_n^5 b^4 L^6-16c^3\lambda_n^7 b^2 L^4\end{array}\right)\cos\left(\frac{\lambda_n\delta c}{L}\right)\right)\\ +cL^2\left(2c^2\lambda_n^2 b^2 L^2+b^4 L^4+\hat{d}^2 c^4+\hat{d}c^2 b^2 L^2+\hat{d}c^4\lambda_n^2+c^4\lambda_n^4\right)\end{array}\right) \quad \text{.......} (223)$$

Turning to the second Fourier coefficient, this is a little more complicated than the first because it is dependent on the first. The function g(x) is given by Equation (200). Taking Equation (220) and multiplying both sides by cos(λ_mx/L),

$$g(x)\cos\left(\frac{\lambda_m x}{L}\right)=\sum_{n=1}^{\infty}\left(-bC_{1n}+\frac{\sqrt{\lambda_n^2-dc}}{L}C_{2n}\right)\cos\left(\frac{\lambda_n x}{L}\right)\cos\left(\frac{\lambda_m x}{L}\right) \quad \text{.......} (224)$$

If one term at a time is considered and both sides are integrated, this yields

$$\int_{0}^{\delta c}g(x)\cos\left(\frac{\lambda_m x}{L}\right)dx=\left(-bC_{1n}+\frac{\sqrt{\lambda_n^2-dc}}{L}C_{2n}\right)\int_{0}^{L}\cos\left(\frac{\lambda_n x}{L}\right)\cos\left(\frac{\lambda_m x}{L}\right)dx \quad \text{.......} (225)$$

Solving for C_{2n},

$$C_{2n} = \frac{L}{\sqrt{\lambda_n^2 - d\,c}} \left(\frac{\int_0^{\delta c} g(x)\cos\left(\frac{\lambda_m x}{L}\right)dx}{\int_0^L \cos\left(\frac{\lambda_n x}{L}\right)\cos\left(\frac{\lambda_m x}{L}\right)dx} + bC_{1n} \right) \quad (226)$$

Substituting Equation (200) for g(x) and integrating,

$$C_{2n} = \frac{1}{4cL\lambda_n^4 Z\left(\cos(\lambda_n)\sin(\lambda_n)+\lambda_n\right)\sqrt{\lambda_n^2 - \hat{d}}} \left(F_0 e^{-b\delta} \left(\begin{array}{l} \left(3\hat{d}^2 + 8\lambda_n^4 + 4\hat{d}\lambda_n^2 - \left(\frac{\hat{d}c\delta\lambda_n}{L}\right)^2\right)L^2 \sin\left(\frac{\lambda_n \delta c}{L}\right) \\ -\left(3\hat{d} + 4\lambda_n^2\right)\hat{d}\lambda_n L\delta c\cos\left(\frac{\lambda_n \delta c}{L}\right) \end{array} \right) + 4bC_{1n}L^2\lambda_n^4 Z\left(\cos(\lambda_n)\sin(\lambda_n)+\lambda_n\right) \right) \quad (227)$$

Knowing these coefficients, and transforming the time reference to the original one using Equation (201), the Fourier series with circular functions for the displacement and stress are given by the equations

$$u(x,t) = e^{-bt'} \sum_{n=1}^{\infty} \cos\left(\frac{\lambda_n x}{L}\right)\left(C_{1n}\cos\left(\frac{\sqrt{\lambda_n^2 - \hat{d}}\,c}{L}(t-\delta)\right) + C_{2n}\sin\left(\frac{\sqrt{\lambda_n^2 - \hat{d}}\,c}{L}(t-\delta)\right)\right) \quad (228)$$

$$\sigma(x,t) = \frac{Zc}{AL} e^{-bt'} \sum_{n=1}^{\infty} \lambda_n \sin\left(\frac{\lambda_n x}{L}\right)\left(C_{1n}\cos\left(\frac{\sqrt{\lambda_n^2 - \hat{d}}\,c}{L}(t-\delta)\right) + C_{2n}\sin\left(\frac{\sqrt{\lambda_n^2 - \hat{d}}\,c}{L}(t-\delta)\right)\right) \quad (229)$$

For the hyperbolic functions, the displacement and stress are

$$u(x,t) = e^{-bt'} \sum_{n=1}^{\infty} \cos\left(\frac{\lambda_n x}{L}\right)\left(C_{1n}\cosh\left(\frac{\sqrt{\hat{d} - \lambda_n^2}\,c}{L}(t-\delta)\right) + C_{2n}\sinh\left(\frac{\sqrt{\hat{d} - \lambda_n^2}\,c}{L}(t-\delta)\right)\right) \quad (230)$$

$$\sigma(x,t) = \frac{Zc}{AL} e^{-bt'} \sum_{n=1}^{\infty} \lambda_n \sin\left(\frac{\lambda_n x}{L}\right)\left(C_{1n}\cosh\left(\frac{\sqrt{\hat{d} - \lambda_n^2}\,c}{L}(t-\delta)\right) + C_{2n}\sinh\left(\frac{\sqrt{\hat{d} - \lambda_n^2}\,c}{L}(t-\delta)\right)\right) \quad (231)$$

This is the solution for the damped case using a uniform intensity impulse function of force F_0 and time duration from impact δ. These of course are only valid after time δ.

VI. COMPARISON OF RESULTS WITH NUMERICAL METHODS

Given the large number of possible cases that exist for hammer/pile/soil combinations, the possibility for comparison of the closed form solutions described above are literally endless. The central purpose of this part of the thesis is to illustrate the possible application of these methods and to compare them with existing numerical methods, both to verify the basic soundness of the closed form solution and to further explore the relationship between numerical methods and the closed form solution.

A. Computer Implementations of the Calculations

In his exposition of the rod or cable with a mass at one end suddenly stopped at another, Prescott (1924) noted the following:

> *It will be seen that the actual calculation of tension in the rod at any time involves a considerable amount of labour, and the calculation of the maximum tension involves still more labour.*

It was fifty years later that Laura et. al. (1974) were able to have the computational power to apply to a problem such as this. The closed form solution for piles presented here, whether in undamped or damped form, is considerably more involved than the Prescott-Laura problem.

It is evident from both the solutions proposed here and a review of those who have gone on before that any viable use of any solution of the wave equation for piles involves computer solutions of some kind.

1. Closed form Solution using Maple V Release 3

Advances in computer software made it possible to consider closed form solutions that would have been impossible or impractical in the past. For the purposes of this thesis this means Maple V Release 3, which is a general-purpose mathematical software package capable of both symbolic solution and numerical computation. A detailed description of this software is given in Abell and Braselton (1994a, 1994b). Much of the derivations given earlier, although possible by hand, were in fact done with Maple V, especially as they relate to integration, algebra and complex analysis. Although Maple's capabilities with Laplace transforms were used, limitations with inverse transforms and other areas required occasional "intervention" in the calculation sequence.

With the implementation of these solutions, Maple V was used in a different way. Although it is possible in principle to use the same routines to make numerical computations as were used with derivation, both limitations in both software and hardware and the need for a relatively efficient code suggested the division of the code into a program most suited for symbolic manipulation and one for numerical computation. Example solutions for the damped case are shown in Appendix C (symbolic) and Appendix D (numerical).

2. Direct Stiffness Solution using Maple V Release 3

For most engineering problems such as this, when one considers the use of a numerical method, the first idea that comes to mind is the finite element method. For a variety of reasons this has not taken place with stress waves in piles. Some of these are historical but there are some real difficulties in using finite element direct stiffness techniques in this application (Deeks, 1992). To examine some of these Maple V was employed to construct and use a direct stiffness model of the pile for the undamped case. This model was constructed using Newmark's method as described by Logan (1992). The semi-infinite pile model was used to generate the force-time function at the pile top, as also for the closed form solution. Although Maple is not the most efficient code for this application, its matrix manipulation capabilities (it can do this symbolically in some cases) make this code relatively simple to use for the purpose. The Maple V worksheet for this is found in Appendix E. (For a more versatile example of a direct stiffness solution of this problem, see Bossard and Corté (1983).)

3. Direct Stiffness Solution using ANSYS-ED 5.0-56

One interesting concept that has not been widely pursued either by researchers or practitioners has been the use of general-purpose finite element codes for stress wave analysis of piles. For both undamped and damped cases the closed form solution was compared with results from the ANSYS general purpose computer program. The pile top force can be simulated either by applying a force-time relationship or simulating the drop of a mass onto the hammer cushion. Although the educational version is limited as to the number of nodes and elements, by finite element standards this is a relatively simple problem, so this limitation does not pose any problem here, because here the soil is modelled using visco-elastic elements and not an axisymmetric solid around the pile. The Maple V worksheet for the damped case using ANSYS is found in Appendix F.

4. Finite Difference Solution using WEAP87

From both an historical and a practical standpoint, the most important comparison is with the finite difference techniques that have been the industry standard since the days of Smith (1960). For this purpose the WEAP87 program was used. This is similar to the WEAP86 program as described by Goble and Rausche (1986). This program has a relatively undemanding personal computer implementation and many options for input and output. These are necessary in this case as the entry of soil parameters that are similar to those used in the closed form solution require some care because their theoretical basis is different. This program, however, can only be used to compare the damped case, not because it does not analyse undamped piles but because the undamped case uses a hammer system without a cap, which is not permitted by this program. The WEAP87 results for the damped case are shown in Appendix G.

B. Solution Implementation using the Example Case

1. Statement of the Problem

The basic problem under consideration is the driving of a 1000 mm diameter steel pipe pile, 50 m long, with a wall thickness of 40 mm. The pile is driven open ended into medium dense sand. The hammer used has a ram mass of 15 metric tons; it has an equivalent stroke of 1.5 m and a mechanical efficiency of 80%. The cushion block has a stiffness of 2.45 GN/m and has no damping (this is to avoid comparing the static hysteresis cushion concept of WEAP87 with the viscous material damping concept of the closed form solution and ANSYS.) This example was analysed for 0 < t < 4L/c for displacement-time and stress-time histories at the pile top (L = 0 m), pile middle (L = 25 m) and pile toe (L = 50m).

Values for the variables of the solution are shown in Table 2. These are either given variables or computed using the appropriate equations given earlier. Variables marked with an asterisk (*) are used in the damped solution only.

Table 2 Variables for Example Case

Variable Designation	Nomenclature	Value of Variable in Example Case
a *	Pile Shaft Elasticity Constant	36858.97436 1/sec²
b *	Pile Shaft Dampening constant	221.8649536 1/sec
c	Acoustic Speed of Pile Material	5188.745215 m/sec
c' *	Cushion Dampening/Hammer Impedance Ratio	0
d	Pile Inside Diameter	920 mm
d'	Pipe Pile Diameter Ratio	0.92
$\hat{d}$ *	Pile Shaft Damping and Elasticity Ratio	1.148186304
k *	Soil Shaft Spring or Elastic Constant per Unit Area	11.04 MPa
k_t	Soil Toe Spring or Elastic Constant per Unit Area	11.08 MN/m³
$\hat{m}$ *	Mass of Driving Accessory for Pile Hammer	3000 kg
m' *	Pile Cap/Ram Mass Ratio	0.2
r_g *	Geometry Ratio of Pile	0.0122
r_t	Pile Toe Radius	500 mm
A	Cross-Sectional Area of Pile	0.12064 m²
A_t	Pile Toe Area	0.12064 m²
C *	Cushion Dampening Coefficient	0 N-sec/m
D	Pile Outside Diameter	1000 mm
E	Pile Young's Modulus of Elasticity	210 GPa
E_s	Soil Young's Modulus of Elasticity	27.6 MPa
G_s	Soil Shear Modulus of Elasticity, Pa	11.04 MPa
K	Cushion Material Spring Constant	2.45 GN/m
L	Length of Pile	50 m

Table 2 (continued)

Variable Designation	Nomenclature	Value of Variable in Example Case
L/c	Time Length for Wave Transmission from Top to Toe	9.636 msec
M	Mass of Pile Hammer Ram	15,000 kg
$P*$	Pile Surface Perimeter	3142 mm
V_0	Initial Velocity of Pile Hammer Ram	4.85 m/sec
Z	Pile Impedance	4.882 MN-sec/m
Z_h	Pile Hammer Impedance	6.062 MN-sec/m
Z'	Pile-Hammer Impedance Ratio	0.8054
α_0	Pile Top Consolidation Variable	250.8985 1/sec
α_1*	Consolidation Constant for Pile Top Forces	-835.176
α_2*	Consolidation Constant for Pile Top Forces	-396.154
α_3*	Consolidation Constant for Pile Top Forces	401.678
β_0	Pile Top Consolidation Variable	1.26279
$\delta*$	Time of Square Wave Simplified Impulse	4.673 msec
$\mu*$	Shaft Soil Dampening Coefficient per Unit Area	132.906 kN.-sec/m^3
μ_t*	Soil Toe Dampening Constant per Unit Area	191.785 kN.-sec/m^3
$\nu*$	Poisson's Ratio of Soil	0.25
ρ	Pile Density	7800 kg/m^3
ρ_s	Soil Density	1600 kg/m^3
*	Time Step for Newmark's method using Maple V	46.4 µsec
*	Default Time Step for ANSYS	12.05 µsec
*	Pile Shaft Surface Area	157.08 m^2
*	Total Shaft Spring Constant	1.74 GN/m
*	Total Shaft Dampening Constant	20.88 MN-sec/m
*	Total Toe Spring Constant	29.44 MN/m
*	Total Toe Dampening Constant	23.137 kN-sec/m
*	Assumed Quake for WEAP87	½"
*	Total Pile Capacity for WEAP87	5052.2 kips
*	Percentage of Capacity at Shaft for WEAP87	98%
*	Smith Shaft Dampening Constant for WEAP87	0.288 sec/ft
*	Smith Toe Dampening Constant for WEAP87	0.009412 sec/ft

2. Computation of Pile Top Force

The closed form solution is dependent upon the function of the pile top force, as is the Newmark method using Maple V and most of the ANSYS runs.

The undamped problem uses the same hammer configuration without considering the effect of the driving accessory. This is purely a distinction of convenience and simplicity in the derivation as shown; the cap, along with cushion material dampening, could be very easily considered in this case. The displacement-time history for this is found in Equation (106). If this is differentiated with respect to time, Equation (17) applied and the example variables substituted, the force-time curve (in Newtons) for the hammer-pile system without considering the cap is

$$F_0(t) = 37.5\times 10^6 e^{(-250.8985t)}\sin(316.833t) \qquad (232)$$

This equation is directly applied to both Newmark's method using Maple V and ANSYS for the entire time of analysis. Equation (106) with the appropriate substitutions is applied to the closed form solution; this relationship, however only applies for times less than L/c.

Including the cap, the velocity-time relationship is given in Equation (163). Multiplying this by the impedance and substituting the example variables gives the force-time relationship (again in Newtons) of

$$F_0(t) = 54.61\times 10^6\left(e^{-835.176t} - e^{-396.154t}\cos(401.6678t)\right) + 59.69\times 10^6 e^{-396.154t}\sin(401.6678t) \qquad (233)$$

However, this cannot be directly applied to the damped case because it is only valid for piles without dampening. The solution to this problem is first to note that the maximum force is 15.566 MN at a time of 3.2 msec; this becomes the constant force for the assumed duration of the square wave pulse. That duration was computed by multiplying the hammer mass and its impact velocity to compute the hammer's total impulse, than dividing this by the maximum force. Because there is no dampening in the cushion material and (as it turns out) not a great deal of rebound from the pile top, this approximation can be made with little error. The impulse time calculates to 9.636 msec.

All three of these are plotted in Figure 6.

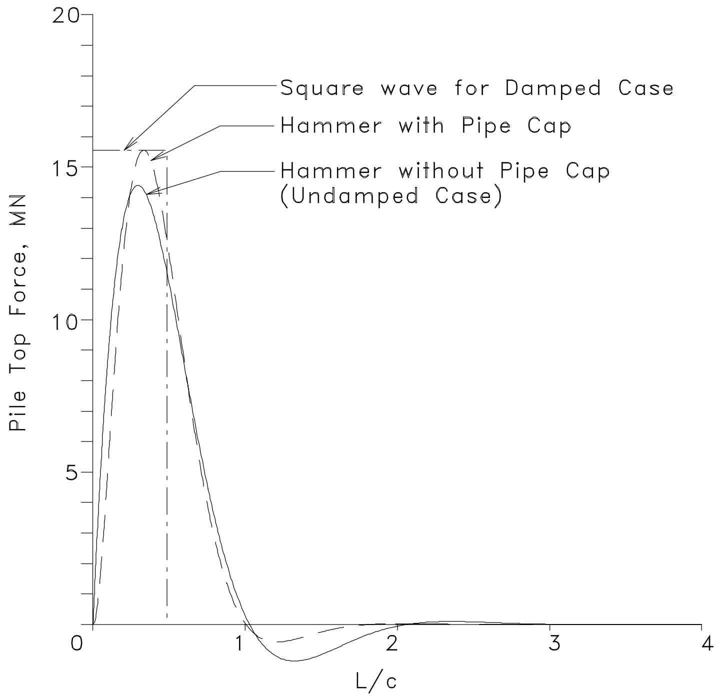


Figure 6 Pile Force-Time Relationships, Example Case

Several items need to be noted in this case:

- The semi-infinite pile top force-time curves were both substantially complete at about t = L/c, so this criterion of the closed form solution is met.
- The negative portions are physically impossible because the hammer cushion is inextensible.
- All of the consolidation coefficients are real in this case.
- The peak force is greater when the effects of the cap are included. Intuitively one would think that the inertia of the cap would diminish peak forces, but this illustrates that simple generalizations about these solutions are difficult.

3. Aspects of the Different Solutions

a) Closed Form Solutions

The closed form solutions are arrived at by substituting the variables into the appropriate equations. The following observations need to be made:

- To insure accuracy, in the Maple V routines the Fourier coefficients were always derived each time rather than to enter them from the formulae.
- The number of Fourier series terms in both undamped and damped case was limited to sixty-five (65). This was mainly a function of limitations in Maple V. Successive trial cases were performed to insure that the approximation was reasonable. With the undamped case, no difficulties were noted; the square wave forcing function created some convergence problems with the damped case but these were minor at this number of terms.

- The undamped case transferred from the Laplace transform solution to the Fourier series at $t = L/c$; the damped case made the transformation at $t = \delta$.
- The results were output to a file of x-y coordinates for each displacement or stress and pile location; these were input into a CAD program for plotting.
- As the theory suggested, the undamped case always used circular time functions. For the damped case, the point where $\omega = 0$ took place at $\lambda = 1.072$. Since $\lambda_1 = .239$ and $\lambda_2 = 3.16$, this meant that the first time functions in the Fourier series were hyperbolic and the rest were circular.

b) *Newmark's Method (Maple V)*

As stated before, Newmark's integration technique was used, as described in Logan (1992). The one-dimensional nature of the problem made for a straightforward assemblage of the stiffness and mass matrices and solution of the problem.

- The pile was divided into twenty (20) spar elements of 2.5 m length each. These elements include both stiffness and mass. A final element representing the soil elasticity connected the pile toe node with a static soil node; this element had stiffness but no mass.
- The time step was one-tenth of the time defined by dividing the element length by the acoustic speed of the material. This was recommended by the reference source.
- The Newmark beta for this was 1/6 and the gamma 1/2.
- Lumped mass matrices were used for the elements, i.e., equally dividing the mass between the two nodes.
- Partitioning the matrix at the soil node, since its displacement was always zero eliminated the singularities in the mass and stiffness matrices.
- The output format was the same as for the closed form solutions.
- In some cases it was necessary to reverse the sign convention on some of the results for consistency.

c) *ANSYS*

Variables in capital letters refer to ANSYS internal variable names; these are explained as they are stated.

Although there were only two example cases (undamped and damped), there were three ANSYS runs performed:

1. The undamped case, where the pile top force-time curve was divided into forty (40) load steps and the load was defined at each time and ramped from one load to another (KBC = 0).
2. The damped case with applied square wave load at the top, configured to be exactly comparable to the closed form solution. In this case there were only two load steps with step loads (KBC = 1); this meant that the pile top was loaded either by the full step load or not loaded at all.

3. The damped case with a mass-spring system at the top to simulate the hammer. The mass was brought up to full velocity by applying a large force uniformly during the first time step.

   The following assumptions and techniques were used to construct the ANSYS model:

- ANSYS uses Newmark's method of integration for the type of time dependent problem. Many of the changes that took place between the Maple V implementation of this and ANSYS were done to correct shortcomings in the former, using ANSYS's vastly superior element library, much greater computational speed and other features.
- The pile was divided into forty (40) spar (LINK1) elements, using lumped mass matrices (LUMPM = ON).
- The pile toe and shaft soil elements were visco-elastic (COMBIN14) elements. For the undamped case, only soil spring was included at the toe; for the damped case, both spring and "viscosity" were included at the toe, and the shaft soil effects were included by linking each pile node with the soil using these elements. At the pile top and toe nodes the spring rate and viscosity were 1.25% of the total shaft value and 2.5% for the rest of the pile nodes.
- A small amount of dampening (ALPHAD = BETAD = .0001) was introduced to minimize spurious oscillations in the system.
- The minimum time step was one-twentieth of the time defined by dividing the element length by the acoustic speed of the material, as recommended by ANSYS. However, ANSYS has an automatic time stepping option, and this was employed (AUTOTS = ON).
- All time steps were stored for later retrieval in postprocessing (NSTORE = 1).
- Although there was some printed output (Appendix F,) the main output was graphical for the six different curves. These were output to ANSYS's own graphical format during postprocessing and converted to HPGL using the Display program that accompanied ANSYS-ED. This in turn was imported into the CAD program, properly scaled, and plotted with the rest of the results.
- In some cases it was necessary to reverse the sign convention on some of the results for consistency.

d) WEAP87

This program proved to be the most challenging to adapt to the comparison. Important notes on this include the following:

- WEAP87 uses English units, while all the other methods were computed in SI. So all of the parameters had to be converted for units.
- The hammer was a special hammer, and a new one was created for the analysis; there is no existing pile hammer with the configuration of the example problem.
- All coefficients of restitution of impacting surfaces were set to unity, but internal stability dampening was allowed to go to default. WEAP87 introduces a small amount of dampening for stability purposes, just as ANSYS does.

- To enter the shaft and toe spring constant into the program, a quake was chosen so large that it would not be exceeded by the expected deflections of the pile, in this case ½". The quake represents the plastic limit of the soil; the shaft or toe ultimate capacity is analogous to the yield strength for normal engineering materials. The known shaft and toe spring constants were then multiplied by this quake to yield the ultimate shaft or toe capacity of the pile.
- Regular viscous dampening was chosen to best correspond with the other models. The difference between regular Smith dampening and true viscous dampening is explained in Goble and Rausche (1986). The Smith dampening factor for the shaft or toe (which is also used with regular viscous dampening, albeit differently) is the total viscous dampening factor μ for the shaft or toe divided by the ultimate capacity for the shaft or toe.
- Output from WEAP87 was in tabular form to a text file (Appendix G.) The detailed displacement-time and force-time outputs were imported into a spreadsheet, where the forces were converted to stresses and all results converted to SI units. The various outputs were then divided up and output to comma delineated text files, where they were imported into the CAD system. They were scaled properly and inserted in the graphs with the other results. Only the pile top and toe results were analyzed from WEAP87; the program would not yield results for the pile middle even when instructed to do so.

4. Presentation and Discussion of the Results

The parameters and special aspects of all of the solution types having been detailed, it is possible to proceed to the presentation and discussion of the results.

Because of the nature of the results, graphical comparison is the most expedient method to view these results. They are compared in two ways: a) between differing places on the pile for a single method, and b) between methods for given points on the pile.

a) Undamped Solution

(1) Displacements

Figure 7 shows the displacement-time histories for the undamped case by comparing the three pile locations using the same method for each graph, and Figure 8 shows these histories by comparing the methods with each other at each pile location.

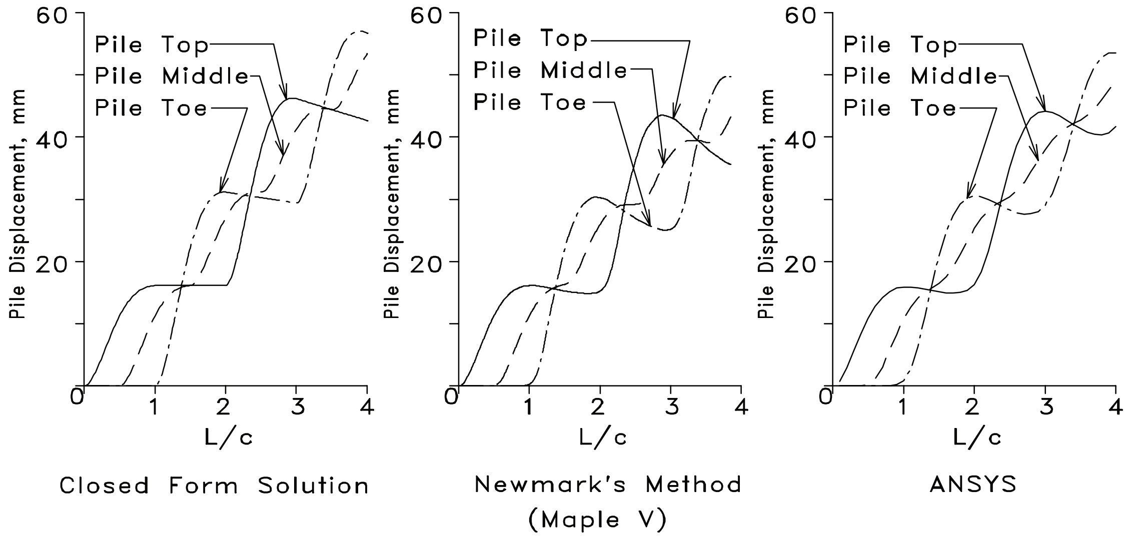


Figure 7 Undamped Case, Comparison of Pile Locations, Displacements

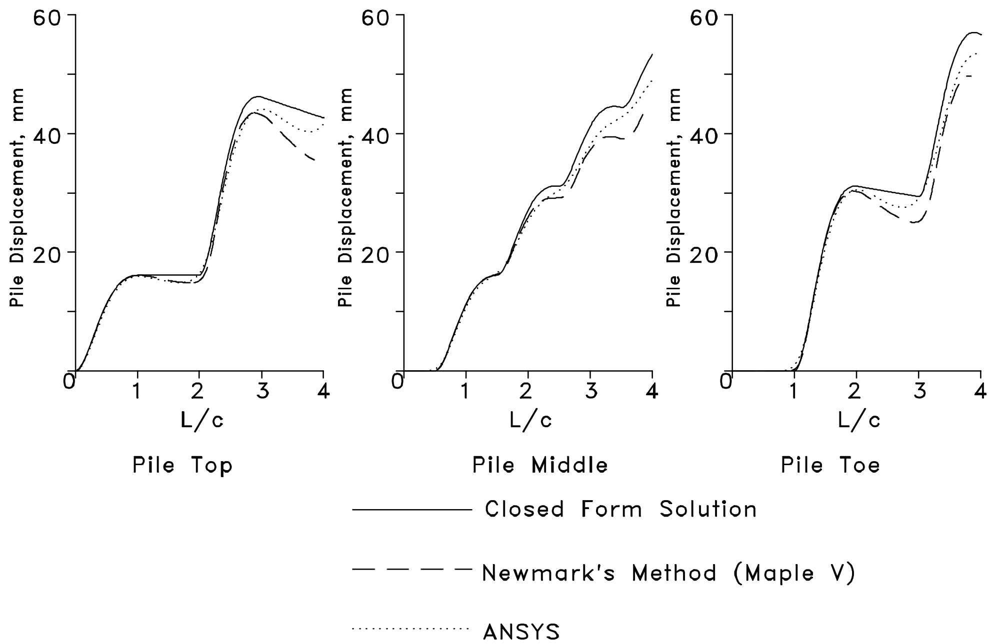


Figure 8 Undamped Case, Comparison of Methods, Displacements

These plots first show the classic pattern of wave propagation: pile top movement begins at time t = 0, a similar displacement pattern begins for the pile middle at t = L/2c, and movement at the toe begins at t = L/c. At this point the closed form solution force is taken off of the pile and the displacement at the pile top becomes constant until t = 2L/c. For the new numerical solutions, the pile top force continues in a slightly negative direction and there is a little fading of the displacement. In the meanwhile the pile toe, unencumbered by a strong spring at the toe, displaces well beyond the elastic compression of the pile and essentially “pulls” the pile along;

this effect becomes evident at t = 2L/c for the pile toe. The velocity trajectory for the pile middle is nearly constant with some ripples caused by the travelling waves in the pile.

The methods compare well in this case; the addition of elements to the ANSYS solution, along with the other improvements of the ANSYS program, makes for a better solution. Both of the finite element solutions are slightly below the closed form; this is caused by a) the inclusion of the slight negative force at t > L/c for the pile top and b) discretization errors, especially in the later times. More elements give a more accurate solution.

(2) Stresses

Figure 9 shows the stress-time histories for the undamped case by comparing the three pile locations using the same method for each graph, and Figure 10 shows these histories by comparing the methods with each other at each pile location.

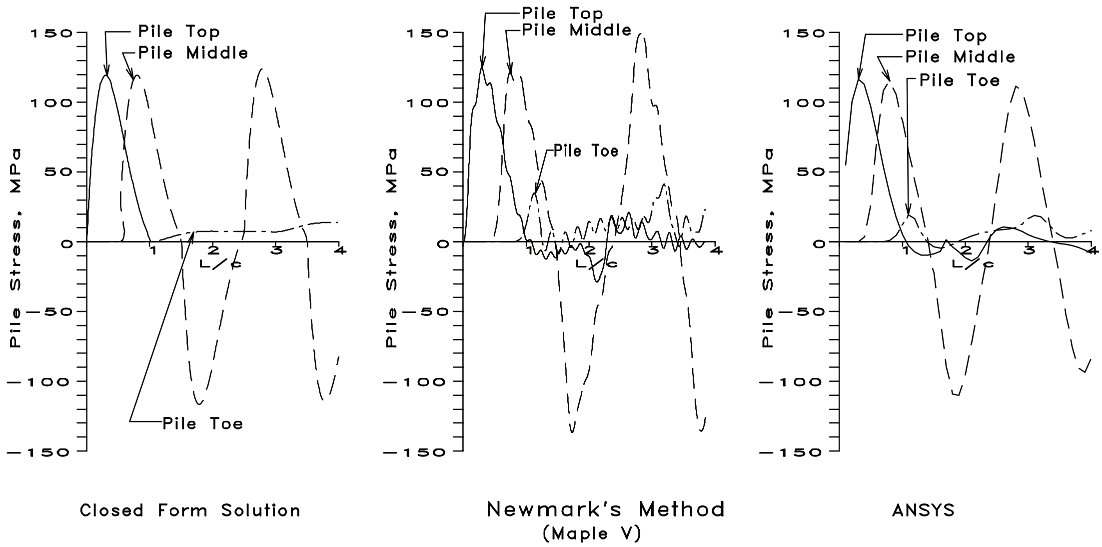


Figure 9 Undamped Case, Comparison of Pile Locations, Stresses

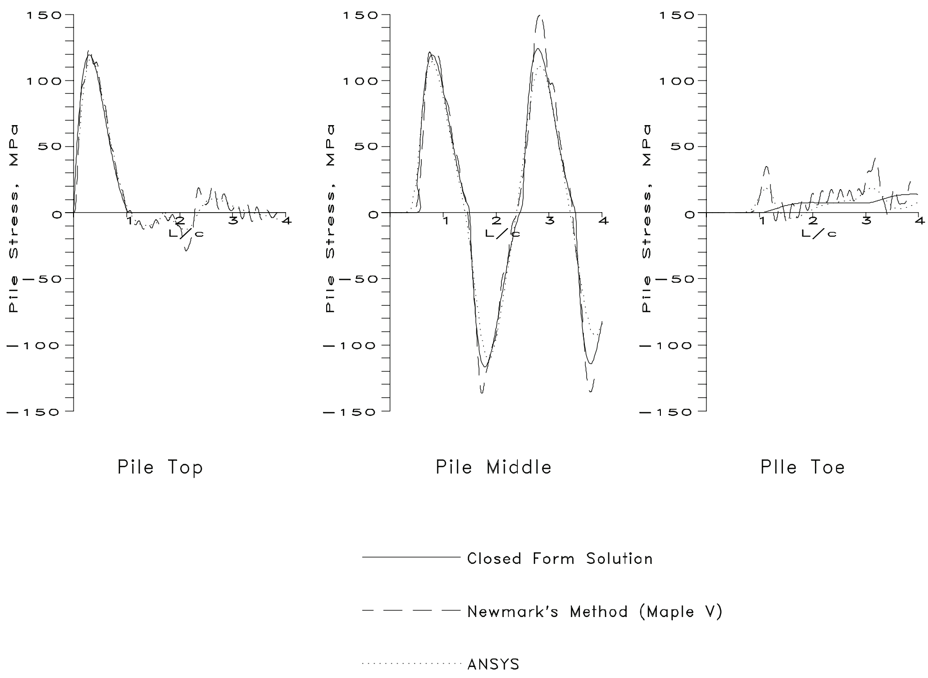


Figure 10 Undamped Case, Comparison of Methods, Stresses

The closed form solution adheres to the boundary conditions of the problem. There is no stress in the pile top after t = L/c. The pile toe stress likewise is zero until t = L/c, but not very high after that because the pile toe is very soft relative to the pile itself. The highest stresses – both tensile and compressive – take place in the middle of the pile, where the effect of the boundary conditions is minimized. In this case there is an example of "pile run" where there is very little resistance to the pile and no skin friction (in actual cases because the pile is not very far into the ground at the beginning of driving.) The closed form solution predicts that the stresses are minimal at the boundaries and highest in the middle. The main difference in an actual case is the presence of the hammer at the pile top, and this only serves to increase the compressive stresses in the pile, not the tensile ones. For concrete piles tensile stresses is an important parameter to control during driving and this result is important.

The first thing with Newmark's method in Maple V is the presence of spurious oscillations due to the discretization of the pile. These are inevitable with any division of a continuous wave propagating medium into finite sections. Each element becomes a harmonically vibrating system; the significance of the vibrations depends upon the length of the elements but it also depends upon the actual, physical stresses taking place. At the pile middle, Newmark's method in Maple V tends to exaggerate the stresses at the peaks, but the results are otherwise reasonable. At the pile top and toe the low stresses make the oscillations the main result except during the initial hammer force at the pile top. These oscillations degrade the results considerably.

With ANSYS, the combination of the internal dampening and the halving of the element size enhance the results. The pile middle stress peaks are a little lower than closed form; this is yet another discretization error that appears again with ANSYS in the damped case. The pile top and toe results mostly eliminate the oscillation errors but they tend to drift around the closed form solution rather than truly correlate with it.

The boundary conditions were the most challenging aspect of arriving at a closed form solution. This is also true with numerical methods, but for a different reason. Although numerical methods can more easily accommodate extensibility considerations and loading condition changes than closed form solution, boundaries represent discontinuities in the system, where numerical integration techniques can break down and render spurious results. Furthermore it is both theoretically and practically impossible for a numerical method to render an exact result at a specific boundary location when elements of a finite length are being employed. This is an important advantage of closed form solutions.

b) Damped Solution

(1) Displacements

Figure 11 shows the displacement-time histories for the damped case by comparing the three pile locations using the same method for each graph, and Figure 12 shows these histories by comparing the methods with each other at each pile location.

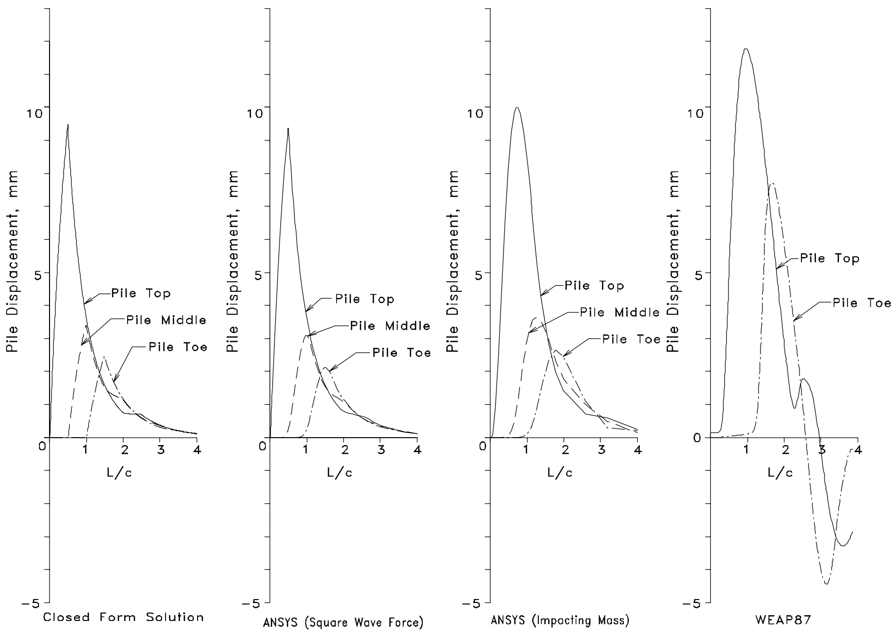


Figure 11 Damped Case, Comparison of Pile Locations, Displacements

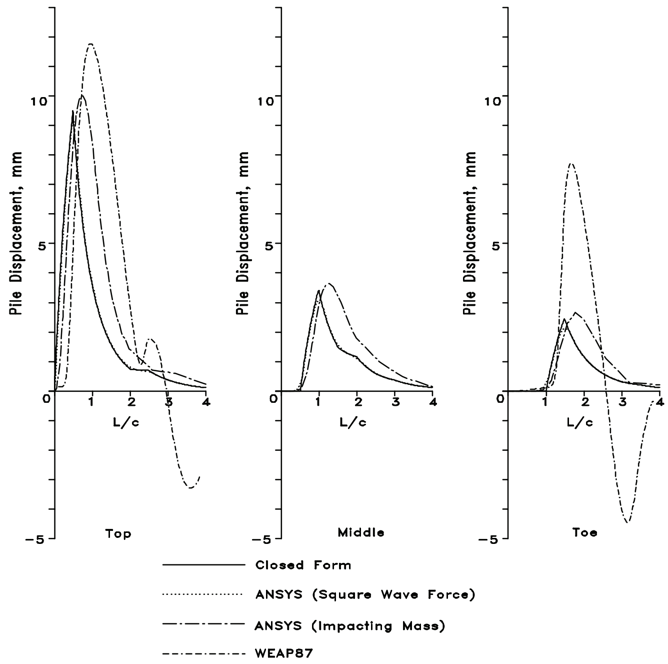


Figure 12 Damped Case, Comparison of Methods, Displacements

Although the time delay characteristics are the same as the undamped case, the dissipative effects of the shaft soil are very evident, as the displacements diminish with distance from the pile top. The closed form solution and the ANSYS solutions show a similar dissipation pattern, even with different pile top loading. There is not a great deal of pile toe rebound; only the "knee" in the displacement curves around t = 2L/c is evidence of this. This is because most of the soil "resistance" here is along the shaft. Given the length of the pile and the fact that it is being driven open ended, this is to be expected.

With the closed form solutions, it should be noted that the transition from Laplace transform solution to Fourier series takes place around t = L/2c; therefore, almost all of the results for the pile middle and toe are derived from the Fourier series. This indicates the complicated nature of these series and why a large number of terms is necessary to obtain a reasonable solution.

Except for "rounding" at the corners, the ANSYS results for the square wave pile top force are virtually indistinguishable from the closed form solution. This is a major result; in addition to confirming both solutions (since they are obtained using very different methods,) it shows that the difficulties with finite element methods can be overcome depending upon how the solution is set up. This last point is true with virtually any finite element solution.

The ANSYS results with the impacting mass is similar to the square wave solution but shows that the use of the undamped semi-infinite pile solution to determine a substitute force-time curve has its limitations. Differences in the timing of the peak displacement were expected because of the nature of the approximation. The slightly higher displacements, however, indicate that, in order to accurately determine the peak force and displacement, a solution of the damped semi-infinite pile at the pile top would be needed.

The WEAP87 results, however, are very different from the other methods. Both the pile top and especially the pile toe have much higher displacements than the other methods and the "knee" is much more pronounced than the other methods as well. Since this program, its predecessors and its successors are important in the actual analysis of wave propagation in piles, some reasonable explanation is necessary.

The most important things to keep in mind about WEAP87 is that it has been developed a) largely without the benefit of closed form solution for comparison on a theoretical basis and b) with the aim of correlation with field results. This latter point includes the very important consideration that virtually all actual pile driving problems involve plastic displacement of the soil; without it there can be no penetration of the pile. This fact is underscored by observing that the displacements for the other solutions approach zero for all pile points as time advances. In this case WEAP87 is asked to analyse a pile driving problem completely devoid of plastic deformation, something it was not really intended to do. Moreover, although every attempt was made to input the parameters of the problem into WEAP87 to have the same meaning as they did with the other problems, the necessary inclusion of empirical factors into the program (the source code was not available for this study) may make an exact comparison impossible and thus alter the results.

One option considered at the start was the use of a finite difference program for which there is available source code. This was rejected because a) most of these programs are at least twenty years old, and thus may not be very relevant to programs currently in use, and b) would have had the result of a new theoretical method to the problem, a role which ANSYS is well suited for.

(2) Stresses

Figure 9 shows the stress-time histories for the undamped case by comparing the three pile locations using the same method for each graph, and Figure 10 shows these histories by comparing the methods with each other at each pile location.

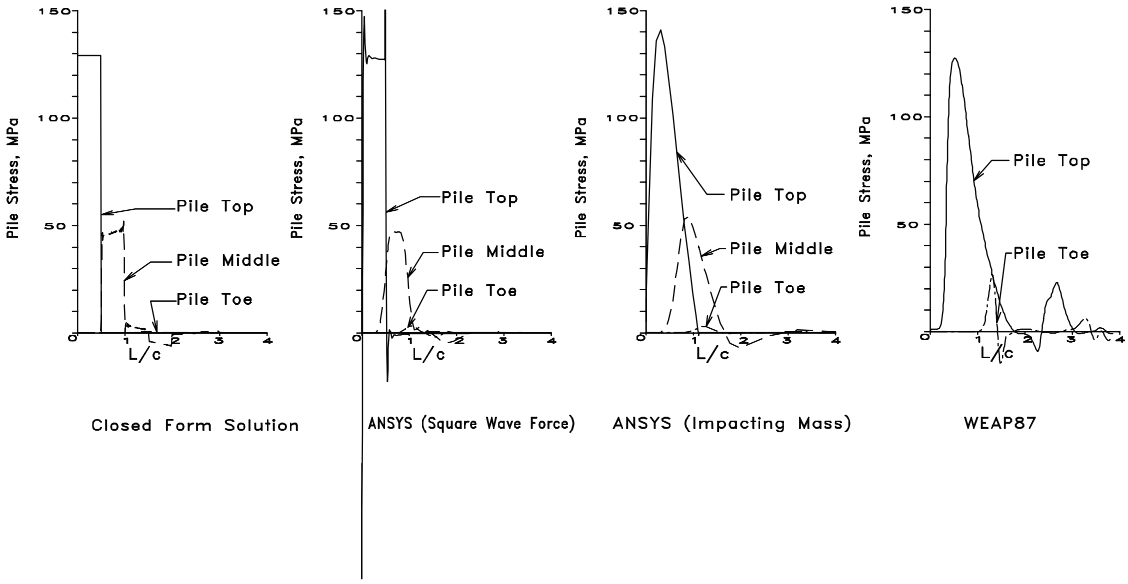


Figure 13 Damped Case, Comparison of Pile Locations, Stresses

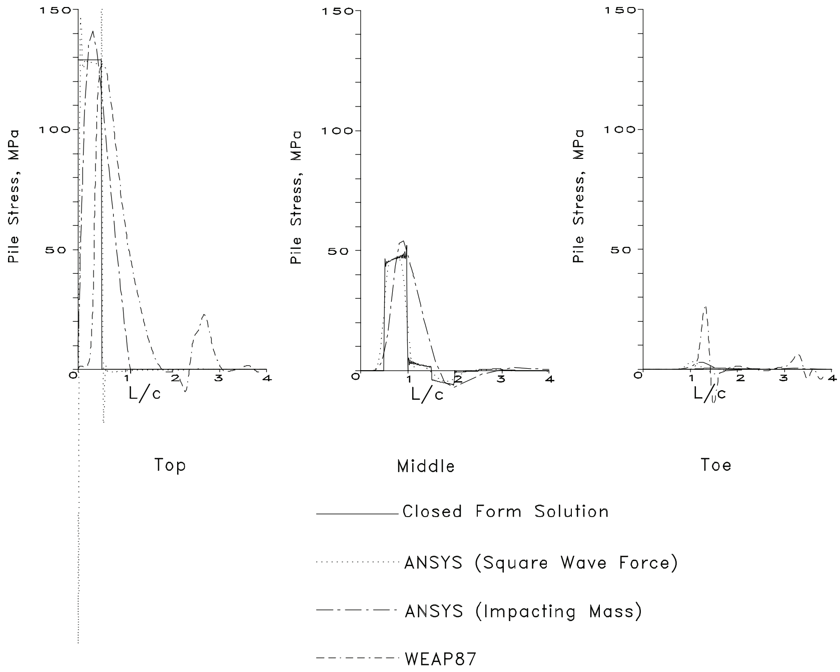


Figure 14 Damped Case, Comparison of Methods, Stresses

The closed form solution shows the difficulties that Fourier series experience with step type boundary or initial conditions; convergence at the "corners" is very difficult to achieve. This was also experienced with Laura et. al. (1974). The undamped case shows that this problem can be overcome with more realistic type loading functions.

The comparison between the closed form solution and the ANSYS run with the square wave load is not as precise as with the displacements. This is probably due to a discretization problem. However, the peak results correlate very well. ANSYS also experienced problems with the step type of loading at the corners.

The situation with the stresses from the ANSYS run with the impacting mass is very similar to that with the displacements.

The stress results from WEAP87 compares more closely with the other solutions than the displacements, although there are still oscillations at the toe.

VII. Discussions on Related Topics

Having derived the equations for the closed form solution and compared them with numerical methods, it is possible to draw some conclusions concerning them. However, before this is done, two "miscellaneous" topics need to be discussed: a) an historical perspective on the present solution and b) additional discussions on some of the aspects of the solution which relate to important issues current in the ongoing application of stress-wave theory to piles.

A. Historical Perspective of the Present Solution

Although the solution presented here is by no means the *ne plus ultra* in closed form solutions (to say nothing of numerical methods), the mathematical techniques employed here have been in existence since Isaacs (1931). This begs the obvious question: why was a solution involving Laplace transformations, Fourier series or both not used sooner? Leaving out discussions of interdisciplinary interaction (or lack of it) between civil engineers and mathematicians on this question, there are three "technical" reasons why this did not take place sooner.

The first is a limitation of the method, namely that the impulse force of the hammer be substantially complete at time $t \leq L/c$. This favours longer piles; however, the types of piles that inspired interest in this subject were generally short (10 m - 20 m long) concrete piles. So the method is not as suitable as it is for the longer piles that are now installed because they can be analysed by stress wave techniques.

Second, the computational labour for this method is considerable, as has been discussed before. This is especially true with the Fourier series.

Third – and perhaps most important – the earliest researchers did not have the understanding of the constitutive models of the soil that are necessary for successful stress wave analysis. Glanville et. al. (1938) were conscious of this problem; this is attested to in the following citation:

> *The foot resistance is, in practice, partly elastic and partly plastic. Any differences between the assumed and actual foot conditions will cause the calculated and observed values to differ most widely at the foot, and it is therefore best to assume a foot condition which corresponds most closely to that in which the foot-stress may become important, namely, severe driving against a hard stratum. Under such conditions the greater portion of the set is elastic.*
>
> *It was therefore decided to assume a purely elastic foot condition, and when applying the theory in practice to use an approximate formula equating the observed plastic plus elastic set to an equivalent purely elastic set, to which the theory would be applied directly.*
>
> *The effect of dissipation of energy due to propagation losses in the pile has not been included in the theory, since very little information concerning it exists. Dissipation will tend to reduce the stresses. The effect of skin friction will be of the same nature from the practical point of view, since it will tend to decrease the amplitude of the stress-waves as they travel along the pile. The effect of neglecting*

both propagation losses and skin friction is therefore to make the theoretical stresses higher than the actual stresses; that is, the error is on the safe side. (pp. 89-90)

Given the lack of knowledge concerning the soil response, Glanville's solution was simply to combine convenient simplifications with conservative assumptions. This certainly allowed an advance in the understanding of stress-wave phenomena in piles , but it results in a solution with a fairly large "factor of ignorance" built into it.

It is in this light that the soil modelling proposed by Smith (1960) should be considered. He states the basis for his soil model (and its limitations) as follows:

> *In order to make a pile calculation, it must be assumed that the soil will act in some particular way. When future investigators develop new facts, the mathematical method explained herein can be modified readily to take account of them, but on the basis of information presently available, the assumptions listed in what follows are recommended...Starting at 0, the pile point moves ahead a distance Q (usually assumed to be 0.1 in.) compressing the soil elastically so that at point A the ground resistance has built up to its ultimate value R_u. Plastic failure then occurs and ground resistance remains equal to R_u until the pile reaches point B. Elastic rebound equal to Q then occurs, and motion ceases at point C where all forces are zero...This conception fails to consider the element of time. Some piles penetrate the ground more rapidly than others. Obviously, the ground will offer more instantaneous resistance to rapid motion than to slow motion. We therefore introduce the additional factor of "viscous damping" which is commonly used in vibration problems (Smith (1960), p. 40).*

Smith's paper is the most cited work in the literature on pile dynamics. The soil model he proposed, which is for the most part an empirical one, has been widely discussed, criticized, modified, and applied ever since it was formulated, but at this point the basic validity of the visco-elastic-plastic model for both pile shaft and toe has been confirmed and this model is in reality the most used model for the response of the soil. (It is equally noteworthy that Smith never considered this model to be the last word on the subject, either.) It is, in a sense, like Lysmer's Analogue; a model proposed with rather limited theoretical backing but which represents a "quantum leap" in the science of which it is a part.

The present solution, in one sense, is necessary to at least bring the closed form solution up to parity with its numerical counterparts with respect to soil response. Although in this respect it represents an advance, without the inclusion of effects due to plasticity the result is not complete.

B. Pile Top Monitoring and Force-Time Characteristics

Although this thesis is devoted to the wave equation's ability to predict pile performance though analytical modelling, the most widespread use of wave mechanics is not in prediction but in analysis of driving. When reasonably reliable instrumentation was developed to monitor pile top displacements and accelerations (and through integration the velocities), interest began in the possibility of using this information to predict the capacity of the pile. This is especially valuable since soils are rather variable in nature and it is presently difficult to formulate from strictly theoretical calculations the pile capacity with both accuracy and precision. In addition to theoretical difficulties, the main problem rested in the perceived necessity of monitoring the desired parameters at several points along the pile (as Glanville et. al. (1938) did), which made the instrumentation as expensive as it was disposable. The solution to this problem is best described by Fellenius (1996):

> *Then came the second Case Seminar in Cleveland, where Dr. George Goble demonstrated that when having both gage types and placing them at the pile head, the fact that the force and velocity records have opposing trends (i.e., the strain gage and accelerometer responses on arrival of the soil reflections to the pile head) could be used to "tell it all"; because the separation of the traces actually 'reflects' the dynamic soil resistance along the pile shaft. The origin of involving both types of gages in the test was based on the idea that the strain gage would give the force and the velocity must be the force without damping, that is, the pile capacity. Well, it was not quite that simple, but the gage combination was there and its significance was quickly realized. The two independent measurements gave, qualitatively, a visual picture of the distribution of shaft resistance along the pile and a good picture of whether or not there was significant toe resistance. Records from initial driving and restriking obtained when both gage types were placed at the pile head could not give a clear indication of soil set-up, for example. (p. 9, Keynote Address)*

Put into the terms of this thesis, a semi-infinite undamped pile behaves at the pile top according to Equation (17) and anything else does not. The development of the various pile driving monitoring and data processing methods and equipment is based on the idea that the violation of this equality is actually informative, since how the stress waves are reflected is based on the parameters of the problem. An approach to this problem based on a closed form type solution is to be found in Liang and Zhou (1996).

It is interesting to note that pile monitoring such as this, which is common on pile driving job sites today, is the most important fulfilment of a comment that appeared at the end of Isaacs (1931):

> *The new method (of wave mechanics) cannot yet be taken as definitely given all that is desired...Particularly is this so in regard to the relationship between driving resistance and bearing resistance for various classes of ground, and the correlation of load tests with pile formulae...Further research may indicate an empirical solution. Present knowledge, however, if intelligently applied can be very useful, and in certain cases of ground may serve nearly as well as a static load test. (p. 323).*

VIII. CONCLUSIONS AND RECOMMENDATIONS

A. Conclusions

1. Within its stated limitations, the closed form solution presented in this thesis is a viable tool for the analysis of stress wave phenomena in piles on a theoretical basis. This is true both for the undamped and the damped cases. The method generally both correlates with and illuminates the results of numerical analysis, provided that the numerical method used can be properly compared with the theoretical basis of the closed form solution.
2. The substitution of a strain related pile toe dynamic resistance (as opposed to a velocity related one) by assuming the existence of a semi-infinite soil column under the toe shows initial success but requires further study, both theoretical and experimental, as the entire subject of pile shaft and toe response is still not adequately quantified.
3. The limitations on the Bessel function argument induced by the series substitution and the reduction of the force to a square wave pulse indicate that, although the damped solution is an adequate first approximation, the ultimate solution of this problem is either a numerical one or the numerical integration of a closed form solution.
4. The results of the ANSYS modelling shows that this program is capable of the basic modelling of wave propagation in piles.

B. Recommendations for Further Research

1. Semi-infinite pile velocity-time or displacement-time solutions for undamped piles need to be done for the case of a damped cushion both between the ram and the driving accessory and the driving and accessory and the pile. This will include the usual setup for concrete piles. Also, further solutions for cushionless hammers need to be developed, preferably in closed form.
2. The solution of the displacement and velocity characteristics of the pile top for damped piles for a variety of hammer configurations needs to be done. This will increase confidence in both the closed form and numerical solutions for the damped pile.
3. Further work on the closed form solution of the wave equation for piles should centre around the development of "semi-numerical" methods such as fast Fourier transforms and their inverses or numerical solutions of Laplace transforms for finite piles. This is necessary to extend the use of semi-infinite pile solutions (damped or undamped) to at least 2L/c, and thus make them applicable for shorter piles. Solutions should also be developed to deal with the inherent non-linearities of the problem, especially those of soil plasticity.
4. More research needs to be done concerning soil response to impact pile driving. In addition to problems which are widely understood in this field, the investigation of the strain related model, both theoretical and experimental, needs to be furthered for the sake of closed form solutions if not for numerical ones.

References

IX. REFERENCES

ABELL, M.L., and BRASELTON, J.P. (1994a) *The Maple V Handbook*. Cambridge, MA: Academic Press.

ABELL, M.L., and BRASELTON, J.P. (1994b) *Differential Equations with Maple V*. Cambridge, MA: Academic Press.

BILLINGTON, D.P. (1983) *The Tower and the Bridge*. Princeton, NJ; Princeton University Press.

BOSSARD, A., and CORTÉ, J.-F. (1983*) Battage des pieux – programme de calcul Batlab*. Paris: Laboratoire Central des Ponts et Chauseés.

CORTÉ, J.-F., and LEPERT, P. (1986) "Lateral Resistance During Driving and Dynamic Pile Testing." *Numerical Methods in Offshore Piling*, pp. 19-34. Paris: Éditions Technip.

COUTINHO, A.L.G.A, COSTA, A.M., ALVES, J.L.D., LANDAU, L., AND EBECKEN, N.F.F. (1988) "Pile Driving Simulation and Analysis by the Finite Element Method." *Proceedings of the Third International Conference on the Application of Stress-Wave Theory to Piles,* pp. 197-207. Vancouver: Bi-Tech Publishers.

DAS, B.M. (1984) *Principles of Foundation Engineering*. Boston: PWS Publishers.

DAVIS, R.O., and PHELAN, P.J. (1988) "Tests for Errors in Numerical Calculation of Pile Stress-Waves." *Proceedings of the Third International Conference on the Application of Stress-Wave Theory to Piles,* pp. 377-382. Vancouver: Bi-Tech Publishers.

DEEKS, A.J. (1992) "Numerical Analysis of Pile Driving Dynamics." Ph.D. Thesis, University of Western Australia.

ESPINOZA, D. (1991) "Application of Wave Propagation Theory in Pile Driving Analysis." Internal Report AAE-646. West Lafayette, IN: Purdue University, December.

FELLENIUS, B.H. (1996) "Reflections on Pile Dynamics." Keynote Address, Fifth International Conference on the Application of Stress-Wave Theory to Piles, Orlando, FL, 11-13 September 1996.

GLANVILLE, W.H., GRIME, G., FOX, E.N, and DAVIES, W.W (1938). "An Investigation of the Stresses in Reinforced Concrete Piles During Driving." *Department Sci. Ind. Research, British Building Research Board Technical Paper No. 20*.

GOBLE, G.G., and RAUSCHE, F. (1976) "Wave Equation Analysis of Pile Driving, WEAP Program." Federal Highway Administration Report FHWA-IP-76-14. Washington: Federal Highway Administration.

GOBLE, G.G., and RAUSCHE, F. (1986) "Wave Equation Analysis of Pile Foundations, WEAP86 Program." Federal Highway Administration Report Contract DTFH61-84-C-00100. Washington: Federal Highway Administration.

HANSEN, B., and DENVER, H. (1980) "Wave equation analysis of a pile – An analytic model." *Proceedings of the International Seminar on the Application of Stress-Wave Theory On Piles.* Rotterdam: A.A. Balkema.

HEJAZI, H.A. (1963) "The Influence of Forced Longitudinal Vibration on Rods Penetrating Soils," Ph.D. Thesis, Ohio State University, Columbus, OH. UMI ProQuest AAC 6401264.

HIRSCH, T.J., CARR, L. and LOWERY, L.L., JR. (1976) "Pile Driving Analysis – Wave Equation Users' Manual, TTI Program," Federal Highway Administration Report FHWA-IP-76-13. Washington: Federal Highway Administration.

HOLEYMAN, A.E. (1985) "Static Versus Dynamic Pile Bearing Capacity (Discussion to Session 4)" *Proceedings of the International Symposium on Penetrability and Drivability of Piles, San Francisco, 10 August 1985*. Tokyo: Japanese Society of Soil Mechanics and Foundation Engineering.

HOLEYMAN, A.E. (1988) "Modeling of Dynamic Behavior at the Pile Base." *Proceedings of the Third International Conference on the Application of Stress-Wave Theory to Piles,* pp. 174-185. Vancouver: Bi-Tech Publishers.

HOLLOWAY, D.M. (1975) *Wave Equation Analyses of Pile Driving*. Technical Report S-75-5. Vicksburg, MS: U.S. Army Engineer Waterways Experiment Station.

ISAACS, D.V. (1931) "Reinforced Concrete Pile Formulae." *Journal of the Institution of Engineers Australia*, Vol. 3, No. 9, September, pp. 305-323.

JACOBY, H.S., and DAVIS, R.P. (1941) *Foundations of Bridges and Buildings*. New York: McGraw Hill Book Company.

KREYSZIG, E. (1993) *Advanced Engineering Mathematics*. Seventh Edition. New York: John Wiley & Sons, Inc.

KÜMMEL, F. (1984) "The Kümmel Method for Calculation of Impact Forces in Piles." *Proceedings of the Second International Conference on the Application of Stress-Wave Theory On Piles.* Rotterdam: A.A. Balkema.

LAURA, P.J., REYES, J.A., and ROSSI, R.E. (1974) "Dynamic Behavior of a Cable-Payload System Suddenly Stopped at One End." *Journal of Sound and Vibration*, 34(1), pp. 81-95.

LEONARDS, G.A., DESCHAMPS, R.J., and FENG, Z. (1995) "Drivability, Load/Settlement and Bearing Capacity of Piles Installed with Vibratory Hammers: Final Report on Phase I of the Study." Submitted to the Deep Foundations Institute. West Lafayette, IN: Purdue University.

LIANG, R., and ZHOU, J. (1996) "Pile Capacity Estimation Using New HST Interpretation Method." Presented at the Fifth International Conference on the Application of Stress-Wave Theory to Piles, Orlando, FL, 11-13 September 1996.

LOGAN, D.L. (1992) *A First Course in the Finite Element Method.* Boston: PWS Publishing Company.

LOWERY, L.L, HIRSCH, T.J., EDWARDS, T.C., COYLE, H.M. and SAMSON, C.H. (1969). *Pile Driving Analysis – State of the Art.* Research Report 33-13. College Station: Texas Transportation Insititute.

LYSMER, J. (1965) *Vertical Motion of Rigid Footings*. Contract Report No. 3-115, Sponsored by Defense Atomic Support Agency NWER Subtask 13. Vicksburg, MS: U.S. Army Corps of Engineers, Waterways Experiment Station.

MCCURDY, J.C. (1993) *Eighteenth Century Solutions To The Wave Equation And The Modern Method Of Finding A Fourier Series Solution*. M.S. Thesis, Texas Woman's University. UMI ProQuest AAC 1356255.

MESECK, H. (1985) "Application of a Wave Equation Programma to Establish the Bearing Capacity of Driven Piles." *Proceedings of the International Symposium on Penetrability and Drivability of Piles, San Francisco, 10 August 1985*. Tokyo: Japanese Society of Soil Mechanics and Foundation Engineering.

OBERHETTINGER, F., and BADII, L. (1973) *Tables of Laplace Transforms*. New York: Springer-Verlag.

PARKER, E., GUAITA, P. and RENTOCCHINI, R. (1996) “Closed Form Solution to Wave Equation for Steam and Hydraulic Hammers.” Presented at the Fifth International Conference on the Application of Stress-Wave Theory to Piles, Orlando, FL, 11-13 September 1996.

PAROLA, J.F. (1970) *Mechanics of Impact Pile Driving*. Ph.D. Thesis, University Of Illinois At Urbana-Champaign. UMI ProQuest AAC 7114903.

PERRY, J.V. Jr. (1963) *Soil Motions Under Vibrating Foundations*. Ph.D. Dissertation, A&M College of Texas. UMI ProQuest AAC 6406217.

PETROVSKII, I.G. (1967) *Partial Differential Equations*. Philadelphia: W.B. Saunders Company.

PRESCOTT, J. (1924) *Applied Elasticity*. London: Longmans, Green and Company.

RABENSTEIN, A.L. (1972) *Introduction to Ordinary Differential Equations*. Second Enlarged Edition. New York: Academic Press

RANDOLPH, M.F., and SIMONS, H.A. (1986) “An Improved Soil Model for One-Dimensional Pile Driving Analysis.” *Numerical Method in Offshore Piling*, pp. 3-18. Paris: Éditions Technip.

RAUSCHE, F., GOBLE, G.G., and LIKINS, G. (1985) “Dynamic Determination of Pile Capacity.” *Journal of Geotechnical Engineering*, Vol. 111, No. 3, March, pp. 367-383. New York: American Society of Civil Engineers.

SMART, J.D. (1970) "Vibratory Pile Driving." Doctoral Thesis, University of Illinois at Urbana-Champaign. UMI ProQuest AAC 7000983.

SMITH, E.A.L. (1960) “Pile-Driving Analysis by the Wave Equation.” *Journal of the Soil Mechanics and Foundations Division*, August, pp. 35-61. New York: American Society of Civil Engineers.

STARKEY, B.J. (1954) *Laplace Transforms for Electrical Engineers*. London: Iliffe and Sons, Ltd.

TADA, H., OHSHIMA, K., KAMINAGA, K., UEKI, Y., and FUKUWAKA, M. (1985) "New Dynamic Formula Applied to Hydraulic Pile Hammer.: *Proceedings of the International Symposium on Penetrability and Drivability of Piles, San Francisco, 10 August 1985*. Tokyo: Japanese Society of Soil Mechanics and Foundation Engineering.

TIMOSHENKO, S.P., and GOODIER, J.N. (1970). *Theory of Elasticity.* Third Edition. New York: McGraw-Hill, Incorporated.

TOLSTOV, G.P. (1962) *Fourier Series*. Englewood Cliffs, NJ: Prentice-Hall, Inc..

UTO, K., FUYUKI, M. and SAKURAI, M. (1985) "An equation for the Dynamic Bearing Capacity of a Pile Based on Wave Theory." *Proceedings of the International Symposium on Penetrability and Drivability of Piles, San Francisco, 10 August 1985*. Tokyo: Japanese Society of Soil Mechanics and Foundation Engineering.

VAN KOTEN, H., MIDDENDORP, P., and VAN BREDERODE, P. (1980) "An analysis of dissapative wave propagation in a pile." *Proceedings of the International Seminar on the Application of Stress-Wave Theory On Piles.* Rotterdam: A.A. Balkema.

VAN WEELE, A.F., and KAY, S. (1984) "Analytical Results with Numerical Programs." *Proceedings of the Second International Conference on the Application of Stress-Wave Theory On Piles.* Rotterdam: A.A. Balkema.

WANG, Y.X. "Determination of Capacity of Shaft Bearing Piles Using the Wave Equation." *Proceedings of the Third International Conference on the Application of Stress-Wave Theory to Piles,* pp. 337-342. Vancouver: Bi-Tech Publishers.

WARRINGTON, D.C. (1987) "A Proposal for a Simplified Model for the Determination of Dynamic Loads and Stresses During Pile Driving." *Proceedings of the Nineteenth Annual Offshore Technology Conference*, Dallas, TX. OTC 5395.

WARRINGTON, D. C. (1996) “Development and Potential of the Wave Equation in Closed Form as Applied to Pile Dynamics.” Presented at the Fifth International Conference on the Application of Stress-Wave Theory to Piles, 11-13 September 1996, Orlando, FL.

WEBSTER, A.G. (1966) *Partial Differential Equations of Mathematical Physics*. Second Corrected Edition, pp. 173-179. New York: Dover Publications.

WHITTLE, A.J. (1993) “Assessment of An Effective Stress Analysis for Predicting the Performance of Driven Piles in Clays.” *Proceedings of the Workshop on Effects of Piles on Soil Properties, 13-15 July 1993*. Miscellaneous Paper GL-95-2. Vicksburg, MS: U.S. Army Corps of Engineers, Waterways Experiment Station.

WYLIE, C.R. (1979) *Differential Equations*. New York: McGraw-Hill Book Company.

YAMAGATA, K., and SETO, T. (1985). "Method for Preventing the Local Buckling of Hammer-Driven Steel Pile Piles." *Proceedings of the International Symposium on Penetrability and Drivability of Piles, San Francisco, 10 August 1985*. Tokyo: Japanese Society of Soil Mechanics and Foundation Engineering.

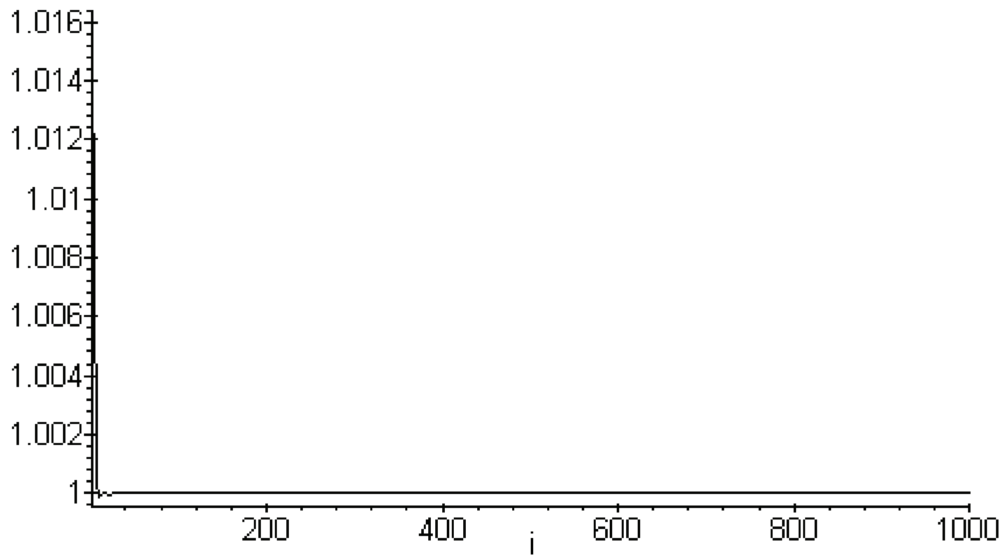
ZHOU, J. and LIANG, R.Y. (1996) “Identification of Soil-Pile Interaction Model Parameters from HST Data.” Presented at the Fifth International Conference on the Application of Stress-Wave Theory to Piles, 11-13 September 1996, Orlando, FL.

Appendices

X. APPENDICES

A. Solution of the Prescott-Laura Problem using Maple

(not included in Internet Edition except for figure below)



```
STUDENT > #Solution for Laura et. al. (1974)

STUDENT > restart;

STUDENT > with(linalg):
Warning: new definition for   norm
Warning: new definition for   trace

STUDENT > u(x,t):=X(x)*T(t);
```

$$u(x,t) := X(x)\,T(t)$$

```
STUDENT > f:=diff(u(x,t),x$2)/(X(x)*T(t))=diff(u(x,t),t$2)/(c^2*X(x)
          *T(t));
```

$$f := \frac{\frac{\partial^2}{\partial x^2} X(x)}{X(x)} = \frac{\frac{\partial^2}{\partial t^2} T(t)}{c^2\,T(t)}$$

```
STUDENT > f1:=diff(X(x),x$2)+beta^2*X(x);
```

$$f1 := \left(\frac{\partial^2}{\partial x^2} X(x)\right) + \beta^2\,X(x)$$

```
STUDENT > X(x):=rhs(dsolve(f1,X(x),X(0)=0));
```

$$X(x) := \_C1\cos(\beta\,x) + \_C2\sin(\beta\,x)$$

```
STUDENT > #First Boundary Condition

STUDENT > Bound1:=eval(subs(x=0,X(x)))=0;
```

$$Bound1 := \_C1 = 0$$

```
STUDENT > _C1:=solve(Bound1,_C1);
```

$$\_C1 := 0$$

```
STUDENT > _C2:=1;
```

$$\_C2 := 1$$

```
STUDENT > X(x):=eval(X(x));
```

$$X(x) := \sin(\beta\,x)$$

```
STUDENT > T(t):=rhs(dsolve(diff(T(t),t$2)+beta^2*c^2*T(t)=0,T(t)));
```

$$T(t) := \_C3\cos(\beta\,c\,t) + \_C4\sin(\beta\,c\,t)$$

```
STUDENT > #First Initial Condition

STUDENT > Init1:=eval(subs(t=0,T(t)))=0;
```

$$Init1 := \_C3 = 0$$

```
STUDENT > _C3:=solve(Init1,_C3);
```

$$\_C3 := 0$$

```
STUDENT > u(x,t):=eval(X(x)*T(t));
```

$$u(x,t) := \sin(\beta\, x)\, \_C4\, \sin(\beta\, c\, t)$$

```
STUDENT > #Second Boundary Condition
```

```
STUDENT > f2:=-E1*A*diff(u(x,t),x)=M*diff(u(x,t),t$2);
```

$$f2 := -E1\, A\, \cos(\beta\, x)\, \beta\, \_C4\, \sin(\beta\, c\, t) = -M\, \sin(\beta\, x)\, \_C4\, \sin(\beta\, c\, t)\, \beta^2\, c^2$$

```
STUDENT > f2:=eval(subs(x=L,f2/(-cos(beta*x)*M*beta*c^2*sin(beta*c*t
)*_C4)));
```

$$f2 := \frac{E1\, A}{M\, c^2} = \frac{\beta \sin(\beta\, L)}{\cos(\beta\, L)}$$

```
STUDENT > C=lhs(f2);
```

$$C = \frac{E1\, A}{M\, c^2}$$

```
STUDENT > v(x,t):=diff(u(x,t),t);
```

$$v(x,t) := \sin(\beta\, x)\, \_C4\, \cos(\beta\, c\, t)\, \beta\, c$$

```
STUDENT > f3:=V0=c*Sum(C4[i]*beta[i]*sin(beta[i]*x),i=1..infinity);
```

$$f3 := V0 = c\left(\sum_{i=1}^{\infty} C4_i\, \beta_i \sin(\beta_i\, x)\right)$$

```
STUDENT > f4:=f3*sin(beta[j]*x);
```

$$f4 := \sin(\beta_j\, x)\, V0 = \sin(\beta_j\, x)\, c\left(\sum_{i=1}^{\infty} C4_i\, \beta_i \sin(\beta_i\, x)\right)$$

```
STUDENT > #Case for j<>i
```

```
STUDENT > f4a:=c*C4[i]*beta[i]*expand(int(sin(beta[i]*x)*sin(beta[j]
*x),x=0..L));
```

$$f4a := c\, C4_i\, \beta_i \left(-\frac{\beta_i \cos(\beta_i\, L) \sin(\beta_j\, L)}{(\beta_i - \beta_j)(\beta_i + \beta_j)} + \frac{\beta_j \sin(\beta_i\, L) \cos(\beta_j\, L)}{(\beta_i - \beta_j)(\beta_i + \beta_j)}\right)$$

```
STUDENT > f4b:=c*C4[i]*beta[i]*(beta[i]^2*sin(beta[i]*L)*sin(beta[j]
*L)-beta[j]^2*sin(beta[i]*L)*sin(beta[j]*L))/(C*(beta[j]^2
-beta[i]^2));
```

$$f4b := \frac{c\, C4_i\, \beta_i \left(\beta_i^{\,2} \sin(\beta_i\, L) \sin(\beta_j\, L) - \beta_j^{\,2} \sin(\beta_i\, L) \sin(\beta_j\, L)\right)}{C\left(\beta_j^{\,2} - \beta_i^{\,2}\right)}$$

```
STUDENT > f4b:=simplify(f4b);
```

$$f4b := -\frac{\sin(\beta_j\, L) \sin(\beta_i\, L)\, \beta_i\, C4_i\, c}{C}$$

```
STUDENT > V0=Sum(beta[i]*C4[i]*c*sin(beta[i]*L));#Substitute this
```

```
into f4b
```

$$V0 = \sum \beta_i \, C4_i \, c \sin(\beta_i L)$$

```
STUDENT > f4c:=-V0*sin(beta[j]*L)/C;
```

$$f4c := -\frac{V0 \sin(\beta_j L)}{C}$$

```
STUDENT > #for i=j
```

```
STUDENT > f5:=factor(c*C5[j]*beta[j]*int(sin(beta[j]*x)^2,x=0..L));
```

$$f5 := \frac{1}{2} c \, C5_j \, (-\cos(\beta_j L) \sin(\beta_j L) + \beta_j L)$$

```
STUDENT > f5a:=c*C5[j]*beta[j]*(sin(beta[j]*L)^2/C+L)/2;
```

$$f5a := \frac{1}{2} c \, C5_j \, \beta_j \left(\frac{\sin(\beta_j L)^2}{C} + L \right)$$

```
STUDENT >
```

```
STUDENT > f5b:=int(lhs(f4),x=0..L);
```

$$f5b := -\frac{\cos(\beta_j L) \, V0}{\beta_j} + \frac{V0}{\beta_j}$$

```
STUDENT > f6:=f5b=f4c+f5a;
```

$$f6 := -\frac{\cos(\beta_j L) \, V0}{\beta_j} + \frac{V0}{\beta_j} = -\frac{V0 \sin(\beta_j L)}{C} + \frac{1}{2} c \, C5_j \, \beta_j \left(\frac{\sin(\beta_j L)^2}{C} + L \right)$$

```
STUDENT > factor(solve(f6,C5[j]));
```

$$2 \frac{V0 \, (-\cos(\beta_j L) \, C + C + \sin(\beta_j L) \, \beta_j)}{\beta_j^2 \, c \, (\sin(\beta_j L)^2 + L \, C)}$$

```
STUDENT > C5[j]:=simplify(2*V0*C/(beta[j]*c*(sin(beta[j]*L)*cos(beta
          [j]*L)*C+L*C*beta[j])));
```

$$C5_j := 2 \frac{V0}{\beta_j \, c \, (\cos(\beta_j L) \sin(\beta_j L) + \beta_j L)}$$

```
STUDENT > _C4:=subs(beta[j]=beta,C5[j]);
```

$$\_C4 := 2 \frac{V0}{\beta \, c \, (\cos(\beta L) \sin(\beta L) + \beta L)}$$

```
STUDENT > u1:=eval(u(x,t));
```

$$u1 := 2 \frac{\sin(\beta x) \, V0 \sin(\beta c t)}{\beta \, c \, (\cos(\beta L) \sin(\beta L) + \beta L)}$$

```
STUDENT > v1:=diff(u1,t);
```

$$v1 := 2\,\frac{\sin(\beta\,x)\,V0\,\cos(\beta\,c\,t)}{\cos(\beta\,L)\,\sin(\beta\,L)+\beta\,L}$$

```
STUDENT > sigma1:=E1*diff(u1,x);
```

$$\sigma 1 := 2\,\frac{E1\,\cos(\beta\,x)\,V0\,\sin(\beta\,c\,t)}{c\,(\cos(\beta\,L)\,\sin(\beta\,L)+\beta\,L)}$$

```
STUDENT > L:=100;M:=1000;Mrod:=100;rho:=7750;A:=Mrod/(rho*L);c:=5000
          ;E1:=c^2*rho;x:=100;
```

$$L := 100$$

$$M := 1000$$

$$Mrod := 100$$

$$\rho := 7750$$

$$A := \frac{1}{7750}$$

$$c := 5000$$

$$E1 := 193750000000$$

$$x := 100$$

```
STUDENT > #Convergence Test

STUDENT > vplot:=matrix(1000,1,[seq(0,i=1..1000)]):

STUDENT > vprime:=eval(subs(t=0,v1)/V0);
```

$$vprime := 2\,\frac{\sin(100\,\beta)}{\cos(100\,\beta)\,\sin(100\,\beta)+100\,\beta}$$

```
STUDENT > vsum:=0;
```

$$vsum := 0$$

```
STUDENT > for i from 1 by 1 while i<1001 do;

STUDENT > beta:='beta';

STUDENT > bound1:=((i-1)*Pi-Pi/2)/L;

STUDENT > bound2:=((i-1)*Pi+Pi/2)/L;

STUDENT > beta:=fsolve(f2,beta,beta=bound1..bound2);

STUDENT > eval(vprime);
```

```
STUDENT > vsum:=vsum+eval(vprime);vplot[i,1]:=vsum;

STUDENT > od:

STUDENT > i:='i';
```

$$i := i$$

```
STUDENT > plot(vplot[i,1],i=1..1000);

STUDENT > eval(vsum);
```

$$.9999999623$$

```
STUDENT >
```

B. Solution of the Ram/Cushion/Cap/Pile Top Problem using Maple

(not included in Internet Edition)

```
STUDENT > #Appendix B -- Solution for Ram/Damped Cushion/Pile Cap
          Problem for undamped semi-infinite pile

STUDENT > restart;

STUDENT > Z1=sqrt(K*M);K:=Z2^2/(Zprime^2*M);
```

$$Z1 = \sqrt{K\,M}$$

$$K := \frac{Z2^2}{Zprime^2\,M}$$

```
STUDENT > Z2=E1*A/c;
```

$$Z2 = \frac{E1\,A}{c}$$

```
STUDENT > Z1:=Z2/Zprime;
```

$$Z1 := \frac{Z2}{Zprime}$$

```
STUDENT > m:=mprime*M;
```

$$m := mprime\,M$$

```
STUDENT > C:=cprime*Z1;
```

$$C := \frac{cprime\,Z2}{Zprime}$$

```
STUDENT > f1:=M*diff(ur(t),t$2)+K*(ur(t)-ua(t))+C*(diff(ur(t),t)-dif
          f(ua(t),t))=0;
```

$$f1 := M\left(\frac{\partial^2}{\partial t^2}\mathrm{ur}(t)\right) + \frac{Z2^2\,(\mathrm{ur}(t) - \mathrm{ua}(t))}{Zprime^2\,M} + \frac{cprime\,Z2\left(\left(\frac{\partial}{\partial t}\mathrm{ur}(t)\right) - \left(\frac{\partial}{\partial t}\mathrm{ua}(t)\right)\right)}{Zprime} = 0$$

```
STUDENT > f2:=m*diff(ua(t),t$2)+Z2*diff(ua(t),t)+K*(ua(t)-ur(t))+C*(
          diff(ur(t),t)-diff(ua(t),t))=0;
```

$$f2 := mprime\,M\left(\frac{\partial^2}{\partial t^2}\mathrm{ua}(t)\right) + Z2\left(\frac{\partial}{\partial t}\mathrm{ua}(t)\right) + \frac{Z2^2\,(\mathrm{ua}(t) - \mathrm{ur}(t))}{Zprime^2\,M} + \frac{cprime\,Z2\left(\left(\frac{\partial}{\partial t}\mathrm{ur}(t)\right) - \left(\frac{\partial}{\partial t}\mathrm{ua}(t)\right)\right)}{Zprime} = 0$$

```
STUDENT > F1:=laplace(f1,t,s);
```

$$F1 := M\,((\mathrm{laplace}(\mathrm{ur}(t), t, s)\,s - \mathrm{ur}(0))\,s - \mathrm{D}(ur)(0)) + \frac{Z2^2\,(\mathrm{laplace}(\mathrm{ur}(t), t, s) - \mathrm{laplace}(\mathrm{ua}(t), t, s))}{Zprime^2\,M}$$

$$+\frac{cprime\ Z2\ (\text{laplace}(\text{ur}(t),t,s)\ s-\text{ur}(0)-\text{laplace}(\text{ua}(t),t,s)\ s+\text{ua}(0))}{Zprime}=0$$

```
STUDENT > F2:=laplace(f2,t,s);
```

$$F2 := mprime\ M\ ((\text{laplace}(\text{ua}(t),t,s)\ s-\text{ua}(0))\ s-\text{D}(ua)(0))$$
$$+Z2\ (\text{laplace}(\text{ua}(t),t,s)\ s-\text{ua}(0))+\frac{Z2^2\ (\text{laplace}(\text{ua}(t),t,s)-\text{laplace}(\text{ur}(t),t,s))}{Zprime^2\ M}$$
$$+\frac{cprime\ Z2\ (\text{laplace}(\text{ur}(t),t,s)\ s-\text{ur}(0)-\text{laplace}(\text{ua}(t),t,s)\ s+\text{ua}(0))}{Zprime}=0$$

```
STUDENT > F1a:=subs({ur(0)=0,ua(0)=0,D(ur)(0)=V0},F1);
```

$$F1a := M\ (\text{laplace}(\text{ur}(t),t,s)\ s^2-V0)+\frac{Z2^2\ (\text{laplace}(\text{ur}(t),t,s)-\text{laplace}(\text{ua}(t),t,s))}{Zprime^2\ M}$$
$$+\frac{cprime\ Z2\ (\text{laplace}(\text{ur}(t),t,s)\ s-\text{laplace}(\text{ua}(t),t,s)\ s)}{Zprime}=0$$

```
STUDENT > F2a:=subs(ur(0)=0,ua(0)=0,D(ua)(0)=0,F2);
```

$$F2a := mprime\ M\ \text{laplace}(\text{ua}(t),t,s)\ s^2+Z2\ \text{laplace}(\text{ua}(t),t,s)\ s$$
$$+\frac{Z2^2\ (\text{laplace}(\text{ua}(t),t,s)-\text{laplace}(\text{ur}(t),t,s))}{Zprime^2\ M}$$
$$+\frac{cprime\ Z2\ (\text{laplace}(\text{ur}(t),t,s)\ s-\text{laplace}(\text{ua}(t),t,s)\ s)}{Zprime}=0$$

```
STUDENT > Ura:=solve(F1a,laplace(ur(t),t,s));
```

$$Ura := -\frac{-M\ V0-\dfrac{Z2^2\ \text{laplace}(\text{ua}(t),t,s)}{Zprime^2\ M}-\dfrac{cprime\ Z2\ \text{laplace}(\text{ua}(t),t,s)\ s}{Zprime}}{M\ s^2+\dfrac{Z2^2}{Zprime^2\ M}+\dfrac{cprime\ Z2\ s}{Zprime}}$$

```
STUDENT > Urb:=solve(F2a,laplace(ur(t),t,s));
```

$$Urb := -\left(mprime\ M\ \text{laplace}(\text{ua}(t),t,s)\ s^2+Z2\ \text{laplace}(\text{ua}(t),t,s)\ s+\frac{Z2^2\ \text{laplace}(\text{ua}(t),t,s)}{Zprime^2\ M}\right.$$
$$\left.-\frac{cprime\ Z2\ \text{laplace}(\text{ua}(t),t,s)\ s}{Zprime}\right)\Bigg/\left(-\frac{Z2^2}{Zprime^2\ M}+\frac{cprime\ Z2\ s}{Zprime}\right)$$

```
STUDENT > Ua:=simplify(solve(Ura=Urb,laplace(ua(t),t,s)));
```

$$Ua := -(-Z2+cprime\ s\ Zprime\ M)\ Z2\ V0\ M\ /\ (s\ (Zprime^2\ M^3\ mprime\ s^3+M\ mprime\ s\ Z2^2$$
$$+Zprime\ M^2\ mprime\ s^2\ cprime\ Z2+Zprime^2\ M^2\ Z2\ s^2+Z2^3+Zprime\ M\ cprime\ Z2^2\ s$$
$$+Z2^2\ M\ s-cprime\ Z2\ s^2\ Zprime\ M^2))$$

```
STUDENT > Va:=s*Ua;
```

$$Va := -(-Z2+cprime\ s\ Zprime\ M)\ Z2\ V0\ M\ /\ (Zprime^2\ M^3\ mprime\ s^3+M\ mprime\ s\ Z2^2$$
$$+Zprime\ M^2\ mprime\ s^2\ cprime\ Z2+Zprime^2\ M^2\ Z2\ s^2+Z2^3+Zprime\ M\ cprime\ Z2^2\ s$$

$$+ Z2^2\, M\, s - cprime\, Z2\, s^2\, Zprime\, M^2)$$

```
STUDENT > Van:=factor(numer(Va))/(Zprime^2*M^3*mprime);Vad:=expand(d
          enom(Va)/(Zprime^2*M^3*mprime));
```

$$Van := -\frac{(-Z2 + cprime\, s\, Zprime\, M)\, Z2\, V0}{M^2\, Zprime^2\, mprime}$$

$$Vad := s^3 + \frac{s\, Z2^2}{Zprime^2\, M^2} + \frac{s^2\, cprime\, Z2}{Zprime\, M} + \frac{Z2\, s^2}{M\, mprime} + \frac{Z2^3}{Zprime^2\, M^3\, mprime} + \frac{cprime\, Z2^2\, s}{Zprime\, M^2\, mprime} + \frac{Z2^2\, s}{Zprime^2\, M^2\, mprime} - \frac{cprime\, Z2\, s^2}{Zprime\, M\, mprime}$$

```
STUDENT > Varoots:=solve(Vad=0,s);
```

$$Varoots := \%2^{1/3} + \frac{1}{9}\%3 - \frac{1}{3}\frac{Z2\,(Zprime + mprime\, cprime - cprime)}{Zprime\, M\, mprime},$$
$$-\frac{1}{2}\%2^{1/3} - \frac{1}{18}\%3 - \frac{1}{3}\frac{Z2\,(Zprime + mprime\, cprime - cprime)}{Zprime\, M\, mprime} + \frac{1}{2} I \sqrt{3}\left(\%2^{1/3} - \frac{1}{9}\%3\right),$$
$$-\frac{1}{2}\%2^{1/3} - \frac{1}{18}\%3 - \frac{1}{3}\frac{Z2\,(Zprime + mprime\, cprime - cprime)}{Zprime\, M\, mprime} - \frac{1}{2} I \sqrt{3}\left(\%2^{1/3} - \frac{1}{9}\%3\right)$$

$$\%1 := mprime\, cprime\, Zprime$$

$$\%2 := -\frac{1}{54} Z2^3\,(18\, Zprime\, mprime^2 - 9\, mprime^3\, cprime - 9\, mprime\, Zprime + 9\, mprime\, cprime - 3\, mprime\, cprime\, Zprime^2 - 3\, cprime^2\, Zprime\, mprime^2 - 3\, cprime^2\, Zprime\, mprime + 2\, Zprime^3 - 6\, cprime\, Zprime^2 + 6\, cprime^2\, Zprime + 2\, mprime^3\, cprime^3 - 6\, mprime^2\, cprime^3 + 6\, mprime\, cprime^3 - 2\, cprime^3) \Big/ (Zprime^3\, M^3\, mprime^3) + \frac{1}{18} Z2^3\,(12\, mprime^4 - 24\, mprime^3\, cprime\, Zprime + 66\, mprime^2\, cprime\, Zprime + 6\, mprime^2\, cprime^2\, Zprime^2 - 24\, mprime\, cprime\, Zprime^3 + 18\, mprime\, cprime^2\, Zprime^2 + 6\, mprime^3\, cprime^3\, Zprime - 30\, mprime^2\, cprime^3\, Zprime + 42\, mprime\, cprime^3\, Zprime - 3\, mprime^2\, cprime^4\, Zprime^2 + 6\, mprime\, cprime^3\, Zprime^3 + 6\, mprime\, cprime^4\, Zprime^2 + 24\, mprime^2\, Zprime^2 - 3\, mprime^4\, cprime^2 - 42\, Zprime^3\, cprime + 48\, Zprime^2\, cprime^2 - 18\, cprime^3\, Zprime - 3\, Zprime^2\, cprime^4 - 3\, Zprime^4\, cprime^2 + 6\, Zprime^3\, cprime^3 + 36\, mprime^3 + 36\, mprime^2 + 6\, mprime^2\, cprime^2 - 3\, cprime^2 + 96\,\%1 + 12\, Zprime^4 - 3\, Zprime^2 + 6\, cprime\, Zprime - 60\, Zprime^2\, mprime + 12\, mprime)^{1/2} \Big/ (mprime^2\, Zprime^3\, M^3)$$

$$\%3 := Z2^2\,(-3\, mprime^2 - 3\, mprime - \%1 + Zprime^2 - 2\, cprime\, Zprime + mprime^2\, cprime^2 - 2\, mprime\, cprime^2 + cprime^2) \Big/ (Zprime^2\, M^2\, mprime^2\, \%2^{1/3})$$

```
STUDENT > alpha1=factor(Varoots[1]);
```

$$\alpha 1 = \frac{1}{9}(9\, Zprime^2\, M^2\, mprime^2\, \%1^{2/3} - 3\, Z2^2\, mprime^2 - 3\, Z2^2\, mprime$$
$$- Z2^2\, mprime\, cprime\, Zprime + Z2^2\, Zprime^2 - 2\, Z2^2\, cprime\, Zprime + Z2^2\, mprime^2\, cprime^2$$
$$- 2\, Z2^2\, mprime\, cprime^2 + Z2^2\, cprime^2 - 3\, Z2\, Zprime^2\, M\, mprime\, \%1^{1/3}$$
$$- 3\, Z2\, Zprime\, M\, mprime^2\, \%1^{1/3}\, cprime + 3\, Z2\, Zprime\, M\, mprime\, \%1^{1/3}\, cprime) \Big/ ($$
$$Zprime^2\, M^2\, mprime^2\, \%1^{1/3})$$

$$\%1 := -\frac{1}{54} Z2^3\, (18\, Zprime\, mprime^2 - 9\, mprime^3\, cprime - 9\, mprime\, Zprime$$
$$+ 9\, mprime\, cprime - 3\, mprime\, cprime\, Zprime^2 - 3\, cprime^2\, Zprime\, mprime^2$$
$$- 3\, cprime^2\, Zprime\, mprime + 2\, Zprime^3 - 6\, cprime\, Zprime^2 + 6\, cprime^2\, Zprime$$
$$+ 2\, mprime^3\, cprime^3 - 6\, mprime^2\, cprime^3 + 6\, mprime\, cprime^3 - 2\, cprime^3 - 3\, (12\, mprime^4$$
$$- 24\, mprime^3\, cprime\, Zprime + 66\, mprime^2\, cprime\, Zprime + 6\, mprime^2\, cprime^2\, Zprime^2$$
$$- 24\, mprime\, cprime\, Zprime^3 + 18\, mprime\, cprime^2\, Zprime^2 + 6\, mprime^3\, cprime^3\, Zprime$$
$$- 30\, mprime^2\, cprime^3\, Zprime + 42\, mprime\, cprime^3\, Zprime - 3\, mprime^2\, cprime^4\, Zprime^2$$
$$+ 6\, mprime\, cprime^3\, Zprime^3 + 6\, mprime\, cprime^4\, Zprime^2 + 24\, mprime^2\, Zprime^2$$
$$- 3\, mprime^4\, cprime^2 - 42\, Zprime^3\, cprime + 48\, Zprime^2\, cprime^2 - 18\, cprime^3\, Zprime$$
$$- 3\, Zprime^2\, cprime^4 - 3\, Zprime^4\, cprime^2 + 6\, Zprime^3\, cprime^3 + 36\, mprime^3 + 36\, mprime^2$$
$$+ 6\, mprime^2\, cprime^2 - 3\, cprime^2 + 96\, mprime\, cprime\, Zprime + 12\, Zprime^4 - 3\, Zprime^2$$
$$+ 6\, cprime\, Zprime - 60\, Zprime^2\, mprime + 12\, mprime)^{1/2}\, mprime) \Big/ (Zprime^3\, M^3\, mprime^3)$$

```
STUDENT > alpha2=sum(op(n,Varoots[2]),n=1..3);
```

$$\alpha 2 = -\frac{1}{2}\Big(-\frac{1}{54} Z2^3\, (18\, Zprime\, mprime^2 - 9\, mprime^3\, cprime - 9\, mprime\, Zprime$$
$$+ 9\, mprime\, cprime - 3\, mprime\, cprime\, Zprime^2 - 3\, cprime^2\, Zprime\, mprime^2$$
$$- 3\, cprime^2\, Zprime\, mprime + 2\, Zprime^3 - 6\, cprime\, Zprime^2 + 6\, cprime^2\, Zprime$$
$$+ 2\, mprime^3\, cprime^3 - 6\, mprime^2\, cprime^3 + 6\, mprime\, cprime^3 - 2\, cprime^3) \Big/ (Zprime^3\, M^3$$
$$mprime^3) + \frac{1}{18} Z2^3\, (12\, mprime^4 - 24\, mprime^3\, cprime\, Zprime + 66\, mprime^2\, cprime\, Zprime$$
$$+ 6\, mprime^2\, cprime^2\, Zprime^2 - 24\, mprime\, cprime\, Zprime^3 + 18\, mprime\, cprime^2\, Zprime^2$$
$$+ 6\, mprime^3\, cprime^3\, Zprime - 30\, mprime^2\, cprime^3\, Zprime + 42\, mprime\, cprime^3\, Zprime$$
$$- 3\, mprime^2\, cprime^4\, Zprime^2 + 6\, mprime\, cprime^3\, Zprime^3 + 6\, mprime\, cprime^4\, Zprime^2$$
$$+ 24\, mprime^2\, Zprime^2 - 3\, mprime^4\, cprime^2 - 42\, Zprime^3\, cprime + 48\, Zprime^2\, cprime^2$$
$$- 18\, cprime^3\, Zprime - 3\, Zprime^2\, cprime^4 - 3\, Zprime^4\, cprime^2 + 6\, Zprime^3\, cprime^3$$
$$+ 36\, mprime^3 + 36\, mprime^2 + 6\, mprime^2\, cprime^2 - 3\, cprime^2 + 96\, mprime\, cprime\, Zprime$$

$$+ 12\, Zprime^4 - 3\, Zprime^2 + 6\, cprime\, Zprime - 60\, Zprime^2\, mprime + 12\, mprime)^{1/2} \Big/ ($$

$$mprime^2\, Zprime^3\, M^3)\Bigg)^{1/3} - \frac{1}{18} Z2^2\, (-3\, mprime^2 - 3\, mprime - mprime\, cprime\, Zprime$$

$$+ Zprime^2 - 2\, cprime\, Zprime + mprime^2\, cprime^2 - 2\, mprime\, cprime^2 + cprime^2) \Big/ \Bigg(Zprime^2$$

$$M^2\, mprime^2 \Bigg(-\frac{1}{54} Z2^3\, (18\, Zprime\, mprime^2 - 9\, mprime^3\, cprime - 9\, mprime\, Zprime$$

$$+ 9\, mprime\, cprime - 3\, mprime\, cprime\, Zprime^2 - 3\, cprime^2\, Zprime\, mprime^2$$

$$- 3\, cprime^2\, Zprime\, mprime + 2\, Zprime^3 - 6\, cprime\, Zprime^2 + 6\, cprime^2\, Zprime$$

$$+ 2\, mprime^3\, cprime^3 - 6\, mprime^2\, cprime^3 + 6\, mprime\, cprime^3 - 2\, cprime^3) \Big/ (Zprime^3\, M^3$$

$$mprime^3) + \frac{1}{18} Z2^3\, (12\, mprime^4 - 24\, mprime^3\, cprime\, Zprime + 66\, mprime^2\, cprime\, Zprime$$

$$+ 6\, mprime^2\, cprime^2\, Zprime^2 - 24\, mprime\, cprime\, Zprime^3 + 18\, mprime\, cprime^2\, Zprime^2$$

$$+ 6\, mprime^3\, cprime^3\, Zprime - 30\, mprime^2\, cprime^3\, Zprime + 42\, mprime\, cprime^3\, Zprime$$

$$- 3\, mprime^2\, cprime^4\, Zprime^2 + 6\, mprime\, cprime^3\, Zprime^3 + 6\, mprime\, cprime^4\, Zprime^2$$

$$+ 24\, mprime^2\, Zprime^2 - 3\, mprime^4\, cprime^2 - 42\, Zprime^3\, cprime + 48\, Zprime^2\, cprime^2$$

$$- 18\, cprime^3\, Zprime - 3\, Zprime^2\, cprime^4 - 3\, Zprime^4\, cprime^2 + 6\, Zprime^3\, cprime^3$$

$$+ 36\, mprime^3 + 36\, mprime^2 + 6\, mprime^2\, cprime^2 - 3\, cprime^2 + 96\, mprime\, cprime\, Zprime$$

$$+ 12\, Zprime^4 - 3\, Zprime^2 + 6\, cprime\, Zprime - 60\, Zprime^2\, mprime + 12\, mprime)^{1/2} \Big/ ($$

$$mprime^2\, Zprime^3\, M^3)\Bigg)^{1/3}\Bigg) - \frac{1}{3} \frac{Z2\, (Zprime + mprime\, cprime - cprime)}{Zprime\, M\, mprime}$$

```
STUDENT > alpha3=(op(4,Varoots[2]))/I;
```

$$\alpha 3 = \frac{1}{2}\sqrt{3}\, \Bigg(\Bigg(\Bigg(-\frac{1}{54} Z2^3\, (18\, Zprime\, mprime^2 - 9\, mprime^3\, cprime - 9\, mprime\, Zprime$$

$$+ 9\, mprime\, cprime - 3\, mprime\, cprime\, Zprime^2 - 3\, cprime^2\, Zprime\, mprime^2$$

$$- 3\, cprime^2\, Zprime\, mprime + 2\, Zprime^3 - 6\, cprime\, Zprime^2 + 6\, cprime^2\, Zprime$$

$$+ 2\, mprime^3\, cprime^3 - 6\, mprime^2\, cprime^3 + 6\, mprime\, cprime^3 - 2\, cprime^3) \Big/ (Zprime^3\, M^3$$

$$mprime^3) + \frac{1}{18} Z2^3\, (12\, mprime^4 - 24\, mprime^3\, cprime\, Zprime + 66\, mprime^2\, cprime\, Zprime$$

$$+ 6\, mprime^2\, cprime^2\, Zprime^2 - 24\, mprime\, cprime\, Zprime^3 + 18\, mprime\, cprime^2\, Zprime^2$$

$$+ 6\, mprime^3\, cprime^3\, Zprime - 30\, mprime^2\, cprime^3\, Zprime + 42\, mprime\, cprime^3\, Zprime$$

$$- 3\, mprime^2\, cprime^4\, Zprime^2 + 6\, mprime\, cprime^3\, Zprime^3 + 6\, mprime\, cprime^4\, Zprime^2$$

$$+ 24\, mprime^2\, Zprime^2 - 3\, mprime^4\, cprime^2 - 42\, Zprime^3\, cprime + 48\, Zprime^2\, cprime^2$$

$$- 18\, cprime^3\, Zprime - 3\, Zprime^2\, cprime^4 - 3\, Zprime^4\, cprime^2 + 6\, Zprime^3\, cprime^3$$

$$+ 36\,mprime^3 + 36\,mprime^2 + 6\,mprime^2\,cprime^2 - 3\,cprime^2 + 96\,mprime\,cprime\,Zprime$$
$$+ 12\,Zprime^4 - 3\,Zprime^2 + 6\,cprime\,Zprime - 60\,Zprime^2\,mprime + 12\,mprime)^{1/2} \Big/ ($$
$$mprime^2\,Zprime^3\,M^3)\Big)^{1/3} - \frac{1}{9} Z2^2\,(-3\,mprime^2 - 3\,mprime - mprime\,cprime\,Zprime$$
$$+ Zprime^2 - 2\,cprime\,Zprime + mprime^2\,cprime^2 - 2\,mprime\,cprime^2 + cprime^2) \Big/ \Big(Zprime^2$$
$$M^2\,mprime^2 \Big(-\frac{1}{54} Z2^3\,(18\,Zprime\,mprime^2 - 9\,mprime^3\,cprime - 9\,mprime\,Zprime$$
$$+ 9\,mprime\,cprime - 3\,mprime\,cprime\,Zprime^2 - 3\,cprime^2\,Zprime\,mprime^2$$
$$- 3\,cprime^2\,Zprime\,mprime + 2\,Zprime^3 - 6\,cprime\,Zprime^2 + 6\,cprime^2\,Zprime$$
$$+ 2\,mprime^3\,cprime^3 - 6\,mprime^2\,cprime^3 + 6\,mprime\,cprime^3 - 2\,cprime^3) \Big/ (Zprime^3\,M^3$$
$$mprime^3) + \frac{1}{18} Z2^3\,(12\,mprime^4 - 24\,mprime^3\,cprime\,Zprime + 66\,mprime^2\,cprime\,Zprime$$
$$+ 6\,mprime^2\,cprime^2\,Zprime^2 - 24\,mprime\,cprime\,Zprime^3 + 18\,mprime\,cprime^2\,Zprime^2$$
$$+ 6\,mprime^3\,cprime^3\,Zprime - 30\,mprime^2\,cprime^3\,Zprime + 42\,mprime\,cprime^3\,Zprime$$
$$- 3\,mprime^2\,cprime^4\,Zprime^2 + 6\,mprime\,cprime^3\,Zprime^3 + 6\,mprime\,cprime^4\,Zprime^2$$
$$+ 24\,mprime^2\,Zprime^2 - 3\,mprime^4\,cprime^2 - 42\,Zprime^3\,cprime + 48\,Zprime^2\,cprime^2$$
$$- 18\,cprime^3\,Zprime - 3\,Zprime^2\,cprime^4 - 3\,Zprime^4\,cprime^2 + 6\,Zprime^3\,cprime^3$$
$$+ 36\,mprime^3 + 36\,mprime^2 + 6\,mprime^2\,cprime^2 - 3\,cprime^2 + 96\,mprime\,cprime\,Zprime$$
$$+ 12\,Zprime^4 - 3\,Zprime^2 + 6\,cprime\,Zprime - 60\,Zprime^2\,mprime + 12\,mprime)^{1/2} \Big/ ($$
$$mprime^2\,Zprime^3\,M^3)\Big)^{1/3}\Big)\Big)$$

```
STUDENT > beta1:=alpha1;beta2:=alpha2+I*alpha3;beta3:=alpha2-I*alpha
          3;
```

$$\beta 1 := \alpha 1$$
$$\beta 2 := \alpha 2 + I\,\alpha 3$$
$$\beta 3 := \alpha 2 - I\,\alpha 3$$

```
STUDENT > Va1n:=Van*exp(s*t);Va1d:=(s-beta1)*(s-beta2)*(s-beta3);Var
          es:=Va1n/diff(Va1d,s);
```

$$Va1n := -\frac{(-Z2 + cprime\,s\,Zprime\,M)\,Z2\,V0\,e^{(s\,t)}}{M^2\,Zprime^2\,mprime}$$
$$Va1d := (s - \alpha 1)\,(s - \alpha 2 - I\,\alpha 3)\,(s - \alpha 2 + I\,\alpha 3)$$
$$Vares := -(-Z2 + cprime\,s\,Zprime\,M)\,Z2\,V0\,e^{(s\,t)} \Big/ (M^2\,Zprime^2\,mprime$$
$$((s - \alpha 2 - I\,\alpha 3)\,(s - \alpha 2 + I\,\alpha 3) + (s - \alpha 1)\,(s - \alpha 2 + I\,\alpha 3) + (s - \alpha 1)\,(s - \alpha 2 - I\,\alpha 3)))$$

```
STUDENT > v0:=simplify(simplify(evalc(subs(s=beta1,Vares)))+simplify
```

```
(evalc(subs(s=beta2,Vares))+evalc(subs(s=beta3,Vares)));
```

$$v0 := Z2\ V0\ (e^{(\alpha 1\ t)}\ \alpha 3\ Z2 - e^{(\alpha 1\ t)}\ \alpha 3\ cprime\ \alpha 1\ Zprime\ M + e^{(t\ \alpha 2)}\ cprime\ \alpha 3 \cos(t\ \alpha 3)\ Zprime\ M\ \alpha 1 - e^{(t\ \alpha 2)} \sin(t\ \alpha 3)\ Z2\ \alpha 1 + e^{(t\ \alpha 2)} \sin(t\ \alpha 3)\ Z2\ \alpha 2 + e^{(t\ \alpha 2)} \sin(t\ \alpha 3)\ cprime\ Zprime\ M\ \alpha 2\ \alpha 1 - e^{(t\ \alpha 2)} \sin(t\ \alpha 3)\ cprime\ Zprime\ M\ \alpha 2^2 - e^{(t\ \alpha 2)}\ \alpha 3 \cos(t\ \alpha 3)\ Z2 - e^{(t\ \alpha 2)}\ cprime\ \alpha 3^2 \sin(t\ \alpha 3)\ Zprime\ M) \Big/ ((\alpha 1^2 - 2\ \alpha 1\ \alpha 2 + \alpha 2^2 + \alpha 3^2)\ mprime\ Zprime^2\ M^2\ \alpha 3)$$

```
STUDENT > gamma1=V0*Z2/(mprime*(Zprime*M)^2*alpha3*(alpha1^2-2*alpha
          1*alpha2+alpha2^2+alpha3^2));gamma2=alpha3*(Z2-cprime*Zpri
          me*M*alpha1);gamma3=Z2*(alpha2-alpha1)+M*Zprime*(cprime*(a
          lpha1*alpha2-alpha2^2-alpha3^2));v1:=gamma1*(gamma2*exp(al
          pha1*t)+exp(alpha2*t)*(gamma3*sin(alpha2*t)-gamma2*cos(alp
          ha2*t)));
```

$$\gamma 1 = \frac{V0\ Z2}{(\alpha 1^2 - 2\ \alpha 1\ \alpha 2 + \alpha 2^2 + \alpha 3^2)\ mprime\ Zprime^2\ M^2\ \alpha 3}$$

$$\gamma 2 = \alpha 3\ (Z2 - cprime\ \alpha 1\ Zprime\ M)$$

$$\gamma 3 = Z2\ (\alpha 2 - \alpha 1) + M\ Zprime\ cprime\ (\alpha 1\ \alpha 2 - \alpha 2^2 - \alpha 3^2)$$

$$v1 := \gamma 1\ (\gamma 2\ e^{(\alpha 1\ t)} + e^{(t\ \alpha 2)}\ (\gamma 3 \sin(t\ \alpha 2) - \gamma 2 \cos(t\ \alpha 2)))$$

```
STUDENT > impulse:=simplify(int(v1*Z2,t=0..L/c));
```

$$impulse := \frac{1}{2}\gamma 1\ Z2 \left(2\ e^{\left(\frac{\alpha 1\ L}{c}\right)}\ \gamma 2\ \alpha 2 - e^{\left(\frac{L\ \alpha 2}{c}\right)}\ \alpha 1\ \gamma 3 \cos\left(\frac{L\ \alpha 2}{c}\right) + e^{\left(\frac{L\ \alpha 2}{c}\right)}\ \alpha 1\ \gamma 3 \sin\left(\frac{L\ \alpha 2}{c}\right) - e^{\left(\frac{L\ \alpha 2}{c}\right)}\ \alpha 1\ \gamma 2 \cos\left(\frac{L\ \alpha 2}{c}\right) - e^{\left(\frac{L\ \alpha 2}{c}\right)}\ \alpha 1\ \gamma 2 \sin\left(\frac{L\ \alpha 2}{c}\right) - 2\ \gamma 2\ \alpha 2 + \alpha 1\ \gamma 2 + \alpha 1\ \gamma 3 \right) \Big/ (\alpha 1\ \alpha 2)$$

```
STUDENT >
```

C. Symbolic Solution of the Damped Wave Equation with pile top step loading using Maple

(not included in Internet Edition)

```
STUDENT > #Solution of Wave Equation for Damped Case Using Step
          Loading at Pile Top
```

```
STUDENT > restart;
```

```
STUDENT > fz:=Z2=E1*A2/c;E1:=solve(fz,E1);
```

$$fz := Z2 = \frac{E1\,A2}{c}$$

$$E1 := \frac{Z2\,c}{A2}$$

```
STUDENT > f:=c^2*diff(u(x,t),x$2)=diff(u(x,t),t$2)+a*u(x,t)+2*b*diff
          (u(x,t),t);
```

$$f := c^2\left(\frac{\partial^2}{\partial x^2}\mathrm{u}(x,t)\right) = \left(\frac{\partial^2}{\partial t^2}\mathrm{u}(x,t)\right) + a\,\mathrm{u}(x,t) + 2\,b\left(\frac{\partial}{\partial t}\mathrm{u}(x,t)\right)$$

```
STUDENT > G1:=laplace(f,t,s);#Solution of Wave Equation for
          Semi-Infinite Pile and t<L/case
```

$$G1 := c^2\left(\frac{\partial^2}{\partial x^2}\mathrm{laplace}(\mathrm{u}(x,t),t,s)\right) = (\mathrm{laplace}(\mathrm{u}(x,t),t,s)\,s - \mathrm{u}(x,0))\,s - \left(\frac{\partial}{\partial t}\mathrm{u}(x,0)\right)$$
$$+ a\,\mathrm{laplace}(\mathrm{u}(x,t),t,s) + 2\,b\,(\mathrm{laplace}(\mathrm{u}(x,t),t,s)\,s - \mathrm{u}(x,0))$$

```
STUDENT > G1:=subs(u(x,0)=0,laplace(u(x,t),t,s)=U(x),G1);
```

$$G1 := c^2\left(\frac{\partial^2}{\partial x^2}\mathrm{U}(x)\right) = \mathrm{U}(x)\,s^2 - \left(\frac{\partial}{\partial t}0\right) + a\,\mathrm{U}(x) + 2\,b\,\mathrm{U}(x)\,s$$

```
STUDENT > g1:=dsolve(G1,U(x));
```

$$g1 := \mathrm{U}(x) = \frac{\frac{\partial}{\partial t}0}{s^2 + a + 2\,s\,b} + \_C1\,e^{\left(\frac{\sqrt{s^2+a+2\,s\,b}\,x}{c}\right)} + \_C2\,e^{\left(-\frac{\sqrt{s^2+a+2\,s\,b}\,x}{c}\right)}$$

```
STUDENT > U1(x):=op(3,rhs(g1));#Set Initial Velocity to Zero; Set
          _C1=0 to avoid unbounded condition
```

$$\mathrm{U1}(x) := \_C2\,e^{\left(-\frac{\sqrt{s^2+a+2\,s\,b}\,x}{c}\right)}$$

```
STUDENT > g3:=-Z2*c*diff(u(x,t),x)=F0;
```

$$g3 := -Z2\,c\left(\frac{\partial}{\partial x}\mathrm{u}(x,t)\right) = F0$$

```
STUDENT > G3:=laplace(g3,t,s);
```

$$G3 := -Z2\,c\left(\frac{\partial}{\partial x}\mathrm{laplace}(\mathrm{u}(x,t),t,s)\right) = \frac{F0}{s}$$

```
STUDENT > G3:=eval(subs(laplace(u(x,t),t,s)=U1(x),G3));
```

$$G3 := Z2\_C2\sqrt{s^2+a+2sb}\,e^{\left(-\frac{\sqrt{s^2+a+2sb}\,x}{c}\right)} = \frac{F0}{s}$$

```
STUDENT > g2a:=eval(subs(x=0,G3));
```

$$g2a := Z2\_C2\sqrt{s^2+a+2sb} = \frac{F0}{s}$$

```
STUDENT > _C2:=solve(g2a,_C2);
```

$$\_C2 := \frac{F0}{s\,Z2\sqrt{s^2+a+2sb}}$$

```
STUDENT > U2(x):=simplify(eval(U1(x)));
```

$$\mathrm{U2}(x) := \frac{F0\,e^{\left(-\frac{\sqrt{s^2+a+2sb}\,x}{c}\right)}}{s\,Z2\sqrt{s^2+a+2sb}}$$

```
STUDENT > g4:=s^2+a+2*b*s;F(s):=exp(-sqrt(g4)*x/c)/sqrt(g4);G(s):=rh
          s(g2a)/Z2;g5:=solve(g4=0,s);
```

$$g4 := s^2+a+2sb$$

$$\mathrm{F}(s) := \frac{e^{\left(-\frac{\sqrt{s^2+a+2sb}\,x}{c}\right)}}{\sqrt{s^2+a+2sb}}$$

$$\mathrm{G}(s) := \frac{F0}{s\,Z2}$$

$$g5 := -b+\sqrt{b^2-a},\ -b-\sqrt{b^2-a}$$

```
STUDENT > fconv(t):=exp(-(ah*t+bh*t)/2)*BesselI(0,(ah-bh)*sqrt(t^2-c
          h^2)/2);ch:=x/c;ah:=-g5[1];bh:=-g5[2];radsimp(fconv(t));d=
          (b^2-a)*(L/c)^2;bessarg:=d*((t*c/L)^2-(x/L)^2);gconv(t):=F
          0/Z1;intconv(t):=Int(radsimp(subs(ch=x/c,ah=-g5[1],bh=-g5[
          2],fconv(t)))*subs(t=t-tau,gconv(t)),tau=x/c..t);
```

$$\mathrm{fconv}(t) := e^{(-1/2\,(b-\sqrt{b^2-a})\,t-1/2\,(b+\sqrt{b^2-a})\,t)}\,\mathrm{BesselI}\left(0,\sqrt{b^2-a}\sqrt{t^2-\frac{x^2}{c^2}}\right)$$

$$ch := \frac{x}{c}$$

$$ah := b-\sqrt{b^2-a}$$

$$bh := b+\sqrt{b^2-a}$$

$$e^{(-t\,b)}\,\mathrm{BesselI}\left(0,\sqrt{b^2-a}\sqrt{\frac{t^2c^2-x^2}{c^2}}\right)$$

$$d = \frac{(b^2 - a) L^2}{c^2}$$

$$bessarg := d \left(\frac{t^2 c^2}{L^2} - \frac{x^2}{L^2} \right)$$

$$\mathrm{gconv}(t) := \frac{F0}{Z1}$$

$$\mathrm{intconv}(t) := \int_{\frac{x}{c}}^{t} \frac{e^{(-t\,b)} \, \mathrm{BesselI}\left(0, \sqrt{b^2 - a} \sqrt{\frac{t^2 c^2 - x^2}{c^2}}\right) F0}{Z1} \, d\tau$$

```
STUDENT > fconv:=exp(-b*u)*sum((subs(t=u,bessarg)/4)^n/n!^2,n=0..2);
```

$$fconv := e^{(-b\,u)} \left(1 + \frac{1}{4} d \left(\frac{u^2 c^2}{L^2} - \frac{x^2}{L^2} \right) + \frac{1}{64} d^2 \left(\frac{u^2 c^2}{L^2} - \frac{x^2}{L^2} \right)^2 \right)$$

```
STUDENT > gconv:=F0/Z2;
```

$$gconv := \frac{F0}{Z2}$$

```
STUDENT > intconv:=fconv*gconv;
```

$$intconv := \frac{e^{(-b\,u)} \left(1 + \frac{1}{4} d \left(\frac{u^2 c^2}{L^2} - \frac{x^2}{L^2} \right) + \frac{1}{64} d^2 \left(\frac{u^2 c^2}{L^2} - \frac{x^2}{L^2} \right)^2 \right) F0}{Z2}$$

```
STUDENT > u1:=combine(int(intconv,u=x/c..t),exp);#Displacement Due
          to Constant Pile Head Velocity for t<delta
```

$$u1 := \left(\frac{1}{4} d\, x^2 b^4 L^2 - \frac{1}{64} d^2 x^4 b^4 - \frac{1}{2} d\, c^2 b^2 L^2 - \frac{1}{4} d\, c^2 b^4 L^2 t^2 - \frac{3}{16} d^2 c^4 b^2 t^2 + \frac{1}{16} d^2 c^2 x^2 b^2 - \frac{1}{2} d\, c^2 b^3 L^2 t - \frac{3}{8} d^2 c^4 t\, b - \frac{1}{64} d^2 c^4 b^4 t^4 - \frac{1}{16} d^2 c^4 b^3 t^3 + \frac{1}{32} d^2 c^2 x^2 b^4 t^2 - b^4 L^4 - \frac{3}{8} d^2 c^4 + \frac{1}{16} d^2 c^2 x^2 b^3 t \right) F0\, e^{(-t\,b)} \Big/ (b^5 L^4 Z2)$$
$$+ \frac{\left(\frac{1}{2} d\, c^2 b^2 L^2 + b^4 L^4 + \frac{3}{8} d^2 b\, c^3 x + \frac{3}{8} d^2 c^4 + \frac{1}{8} d^2 c^2 x^2 b^2 + \frac{1}{2} d\, b^3 L^2 c\, x \right) F0\, e^{\left(-\frac{b\,x}{c}\right)}}{b^5 L^4 Z2}$$

```
STUDENT > v1:=factor(diff(u1,t));sigma1:=E1*diff(u1,x);
```

$$v1 := \frac{1}{64} \frac{F0\, e^{(-t\,b)} (8 L^2 + d\, c^2 t^2 - d\, x^2)^2}{L^4 Z2}$$

$$\sigma1 := Z2\, c \left(\frac{\left(\frac{1}{2} d\, x\, b^4 L^2 - \frac{1}{16} d^2 x^3 b^4 + \frac{1}{8} d^2 c^2 x\, b^2 + \frac{1}{16} d^2 c^2 x\, b^4 t^2 + \frac{1}{8} d^2 c^2 x\, b^3 t\right) F0\, e^{(-t\,b)}}{b^5 L^4 Z2} \right.$$

$$+ \frac{\left(\frac{3}{8} d^2 b\, c^3 + \frac{1}{4} d^2 c^2 x\, b^2 + \frac{1}{2} d\, b^3 L^2 c\right) F0\, e^{\left(-\frac{b\,x}{c}\right)}}{b^5 L^4 Z2}$$

$$\left. - \frac{\left(\frac{1}{2} d\, c^2 b^2 L^2 + b^4 L^4 + \frac{3}{8} d^2 b\, c^3 x + \frac{3}{8} d^2 c^4 + \frac{1}{8} d^2 c^2 x^2 b^2 + \frac{1}{2} d\, b^3 L^2 c\, x\right) F0\, e^{\left(-\frac{b\,x}{c}\right)}}{b^4 L^4 Z2\, c} \right) / A2$$

```
STUDENT > ufinal1:=subs(t=delta,u1);vfinal1:=subs(t=delta,v1);#Times
          and Velocities at t=delta
```

$$ufinal1 := \left(\frac{1}{4} d\, x^2 b^4 L^2 - \frac{1}{64} d^2 x^4 b^4 - \frac{1}{2} d\, c^2 b^2 L^2 - \frac{1}{4} d\, c^2 b^4 L^2 \delta^2 - \frac{3}{16} d^2 c^4 b^2 \delta^2 \right.$$

$$+ \frac{1}{16} d^2 c^2 x^2 b^2 - \frac{1}{2} d\, c^2 b^3 L^2 \delta - \frac{3}{8} d^2 c^4 \delta\, b - \frac{1}{64} d^2 c^4 b^4 \delta^4 - \frac{1}{16} d^2 c^4 b^3 \delta^3 + \frac{1}{32} d^2 c^2 x^2 b^4 \delta^2$$

$$\left. - b^4 L^4 - \frac{3}{8} d^2 c^4 + \frac{1}{16} d^2 c^2 x^2 b^3 \delta \right) F0\, e^{(-\delta\, b)} \Big/ (b^5 L^4 Z2)$$

$$+ \frac{\left(\frac{1}{2} d\, c^2 b^2 L^2 + b^4 L^4 + \frac{3}{8} d^2 b\, c^3 x + \frac{3}{8} d^2 c^4 + \frac{1}{8} d^2 c^2 x^2 b^2 + \frac{1}{2} d\, b^3 L^2 c\, x\right) F0\, e^{\left(-\frac{b\,x}{c}\right)}}{b^5 L^4 Z2}$$

$$vfinal1 := \frac{1}{64} \frac{F0\, e^{(-\delta\, b)} (8 L^2 + d\, c^2 \delta^2 - d\, x^2)^2}{L^4 Z2}$$

```
STUDENT > #Damped Pile Wave Equation with Fourier Series for t>L/c
```

```
STUDENT > u:=exp(beta*tprime*c/L+I*lambda*x/L);
```

$$u := e^{\left(\frac{\beta\, tprime\, c}{L} + \frac{I\, \lambda\, x}{L}\right)}$$

```
STUDENT > f:=c^2*diff(u,x$2)=a*u+2*b*diff(u,tprime)+diff(u,tprime$2)
          ;
```

$$f := -\frac{c^2 \lambda^2 \%1}{L^2} = a\, \%1 + 2 \frac{b\, \beta\, c\, \%1}{L} + \frac{\beta^2 c^2 \%1}{L^2}$$

$$\%1 := e^{\left(\frac{\beta\, tprime\, c}{L} + \frac{I\, \lambda\, x}{L}\right)}$$

```
STUDENT > beta1:=solve(f,beta);omega:=radsimp(op(2,expand(beta1[1]))
          );alpha:=op(1,expand(beta1[1]));
```

$$\beta 1 := \frac{1}{2}\frac{-2\,b\,c\,L + 2\sqrt{c^2\,b^2\,L^2 - c^4\,\lambda^2 - c^2\,a\,L^2}}{c^2}, \frac{1}{2}\frac{-2\,b\,c\,L - 2\sqrt{c^2\,b^2\,L^2 - c^4\,\lambda^2 - c^2\,a\,L^2}}{c^2}$$

$$\omega := \frac{\sqrt{b^2\,L^2 - c^2\,\lambda^2 - a\,L^2}}{c}$$

$$\alpha := -\frac{b\,L}{c}$$

```
STUDENT > u:=subs(beta=beta1[1],u)+subs(beta=beta1[2],u);
```

$$u := e^{\left(1/2\frac{(-2\,b\,c\,L + 2\sqrt{c^2\,b^2\,L^2 - c^4\,\lambda^2 - c^2\,a\,L^2})\,tprime}{c\,L} + \frac{I\,\lambda\,x}{L}\right)} + e^{\left(1/2\frac{(-2\,b\,c\,L - 2\sqrt{c^2\,b^2\,L^2 - c^4\,\lambda^2 - c^2\,a\,L^2})\,tprime}{c\,L} + \frac{I\,\lambda\,x}{L}\right)}$$

```
STUDENT > convert(u,trig);ux:=C3*cos(lambda*x/L)+C4*sin(lambda*x/L);
          ut:=exp(alpha*tprime*c/L)*(C1*cosh(omega*tprime*c/L)-C2*si
          nh(omega*tprime*c/L));u:=ux*ut;
```

$$\left(\cosh\left(\frac{1}{2}\frac{(-2\,b\,c\,L + 2\,\%1)\,tprime}{c\,L}\right) + \sinh\left(\frac{1}{2}\frac{(-2\,b\,c\,L + 2\,\%1)\,tprime}{c\,L}\right)\right)$$
$$\left(\cos\left(\frac{\lambda\,x}{L}\right) + I\sin\left(\frac{\lambda\,x}{L}\right)\right) +$$
$$\left(\cosh\left(\frac{1}{2}\frac{(-2\,b\,c\,L - 2\,\%1)\,tprime}{c\,L}\right) + \sinh\left(\frac{1}{2}\frac{(-2\,b\,c\,L - 2\,\%1)\,tprime}{c\,L}\right)\right)$$
$$\left(\cos\left(\frac{\lambda\,x}{L}\right) + I\sin\left(\frac{\lambda\,x}{L}\right)\right)$$
$$\%1 := \sqrt{c^2\,b^2\,L^2 - c^4\,\lambda^2 - c^2\,a\,L^2}$$

$$ux := C3\cos\left(\frac{\lambda\,x}{L}\right) + C4\sin\left(\frac{\lambda\,x}{L}\right)$$

$$ut := e^{(-b\,tprime)}\left(C1\cosh\left(\frac{\sqrt{b^2\,L^2 - c^2\,\lambda^2 - a\,L^2}\;tprime}{L}\right) - C2\sinh\left(\frac{\sqrt{b^2\,L^2 - c^2\,\lambda^2 - a\,L^2}\;tprime}{L}\right)\right)$$

$$u := \left(C3\cos\left(\frac{\lambda\,x}{L}\right) + C4\sin\left(\frac{\lambda\,x}{L}\right)\right)e^{(-b\,tprime)}$$
$$\left(C1\cosh\left(\frac{\sqrt{b^2\,L^2 - c^2\,\lambda^2 - a\,L^2}\;tprime}{L}\right) - C2\sinh\left(\frac{\sqrt{b^2\,L^2 - c^2\,\lambda^2 - a\,L^2}\;tprime}{L}\right)\right)$$

```
STUDENT > p2:=sqrt(A/rg);
```

$$p2 := \sqrt{\frac{A}{rg}}$$

```
STUDENT > mu2:=sqrt(Gs*rho0);b=radsimp(p2*mu2/(2*rho*A));
```

$$\mu 2 := \sqrt{Gs\,\rho 0}$$

$$b = \frac{1}{2} \frac{\sqrt{\rho 0}\,\sqrt{Gs}}{\sqrt{A}\,\sqrt{rg}\,\rho}$$

```
STUDENT > K2:=Pi*Gs/p2;a=p2*K2/(rho*A);
```

$$K2 := \frac{\pi\, Gs}{\sqrt{\frac{A}{rg}}}$$

$$a = \frac{\pi\, Gs}{\rho\, A}$$

```
STUDENT > #Boundary Condition for x=0
```

```
STUDENT > f2:=E1*A*diff(u,x)=0;
```

$$f2 := Z2\, c\, A \left(-\frac{C3 \sin\left(\frac{\lambda x}{L}\right) \lambda}{L} + \frac{C4 \cos\left(\frac{\lambda x}{L}\right) \lambda}{L} \right) e^{(-b\, tprime)} \left(C1 \cosh\left(\frac{\sqrt{b^2 L^2 - c^2 \lambda^2 - a L^2}\; tprime}{L} \right) - C2 \sinh\left(\frac{\sqrt{b^2 L^2 - c^2 \lambda^2 - a L^2}\; tprime}{L} \right) \right) / A2 = 0$$

```
STUDENT > Bound1:=eval(subs(x=0,f2));
```

$$Bound1 := Z2\, c\, A\, C4\, \lambda\, e^{(-b\, tprime)} \left(C1 \cosh\left(\frac{\sqrt{b^2 L^2 - c^2 \lambda^2 - a L^2}\; tprime}{L} \right) - C2 \sinh\left(\frac{\sqrt{b^2 L^2 - c^2 \lambda^2 - a L^2}\; tprime}{L} \right) \right) / (A2\, L) = 0$$

```
STUDENT > C4:=0;
```

$$C4 := 0$$

```
STUDENT > #Boundary Condition @ x=L
```

```
STUDENT > K3=4*Gs*r0/((1-nu)*A3);Es=100*Gs/(85*(1-nu)^2);
```

$$K3 = 4 \frac{Gs\, r0}{(1-\nu)\, A3}$$

$$Es = \frac{20}{17} \frac{Gs}{(1-\nu)^2}$$

```
STUDENT > f3:=factor((A2*E1-A3*Es)*diff(u,x)+K3*A3*u=0);
```

$$f3 := -C3\, e^{(-b\, tprime)} \left(-C1 \cosh\left(\frac{\sqrt{b^2 L^2 - c^2 \lambda^2 - a L^2}\; tprime}{L} \right) + C2 \sinh\left(\frac{\sqrt{b^2 L^2 - c^2 \lambda^2 - a L^2}\; tprime}{L} \right) \right) \left(-\sin\left(\frac{\lambda x}{L}\right) \lambda\, Z2\, c + \sin\left(\frac{\lambda x}{L}\right) \lambda\, A3\, Es + K3\, A3 \cos\left(\frac{\lambda x}{L}\right) L \right) / L = 0$$

```
STUDENT > Bound2:=factor(subs(x=L,f3));
```

$$Bound2 := -\,C3\,e^{(-b\,tprime)}\left(-C1\cosh\left(\frac{\sqrt{b^2 L^2 - c^2\lambda^2 - a L^2}\;tprime}{L}\right) + C2\sinh\left(\frac{\sqrt{b^2 L^2 - c^2\lambda^2 - a L^2}\;tprime}{L}\right)\right)(-\sin(\lambda)\,\lambda\,Z2\,c + \sin(\lambda)\,\lambda\,A3\,Es + K3\,A3\cos(\lambda)\,L)\,/\,L = 0$$

```
STUDENT > Bound3:=expand(L*eval(subs(C3=1,tprime=0,C1=1,Bound2)));
```

$$Bound3 := -\sin(\lambda)\,\lambda\,Z2\,c + \sin(\lambda)\,\lambda\,A3\,Es + K3\,A3\cos(\lambda)\,L = 0$$

```
STUDENT > Bound3a:=(sin(lambda)/cos(lambda))=(1/lambda)*K3*A3*L/(-Es
          *A3+Z2*c);
```

$$Bound3a := \frac{\sin(\lambda)}{\cos(\lambda)} = \frac{K3\,A3\,L}{\lambda\,(Z2\,c - A3\,Es)}$$

```
STUDENT > u2:=exp(-b*tprime)*cos(lambda*x/L)*(C1*cos(sqrt(lambda^2-d
          )*c*tprime/L)+C2*sin(sqrt(lambda^2-d)*c*tprime/L));v2:=dif
          f(u2,tprime);sigma2:=E1*diff(u2,x);
```

$$u2 := e^{(-b\,tprime)}\cos\left(\frac{\lambda\,x}{L}\right)\left(C1\cos\left(\frac{\sqrt{\lambda^2 - d}\;c\;tprime}{L}\right) + C2\sin\left(\frac{\sqrt{\lambda^2 - d}\;c\;tprime}{L}\right)\right)$$

$$v2 := -b\,e^{(-b\,tprime)}\cos\left(\frac{\lambda\,x}{L}\right)(C1\cos(\%1) + C2\sin(\%1)) + e^{(-b\,tprime)}\cos\left(\frac{\lambda\,x}{L}\right)\left(-\frac{C1\sin(\%1)\sqrt{\lambda^2 - d}\;c}{L} + \frac{C2\cos(\%1)\sqrt{\lambda^2 - d}\;c}{L}\right)$$

$$\%1 := \frac{\sqrt{\lambda^2 - d}\;c\;tprime}{L}$$

$$\sigma2 := -\frac{Z2\,c\,e^{(-b\,tprime)}\sin\left(\frac{\lambda\,x}{L}\right)\lambda\left(C1\cos\left(\frac{\sqrt{\lambda^2 - d}\;c\;tprime}{L}\right) + C2\sin\left(\frac{\sqrt{\lambda^2 - d}\;c\;tprime}{L}\right)\right)}{A2\,L}$$

```
STUDENT > u20:=eval(subs(tprime=0,u2));v20:=simplify(eval(subs(tprim
          e=0,v2)));
```

$$u20 := \cos\left(\frac{\lambda\,x}{L}\right)C1$$

$$v20 := \frac{\cos\left(\frac{\lambda\,x}{L}\right)(-b\,C1\,L + C2\sqrt{\lambda^2 - d}\;c)}{L}$$

```
STUDENT > fournumer:=int(cos(lambda*x/L)^2,x=0..L);
```

$$fournumer := \frac{1}{2}\,\frac{L\,(\cos(\lambda)\sin(\lambda) + \lambda)}{\lambda}$$

```
STUDENT > C1:=convert((int(ufinal1*cos(lambda*x/L),x=0..delta*c)),ex
          psincos)/fournumer;
```

$$C1 := 2\left(\frac{1}{8} F0\, e^{(-\delta b)} (-8\, \%1\, L^7 \lambda^4 b^5 + 2\, c^5 d^2 \lambda^5 b^2 L^2\, \%2\, \delta^2 + 3\, d^2 b^5 \lambda\, \%2\, L^6 \delta c\right.$$

$$- 5\, c^6 d^2 \lambda^6\, \%1\, \delta L + 7\, c^5 d^2 \lambda^5 b L^2\, \%2\, \delta - 8\, c^4 L^3 \lambda^8 b\, \%1 + c^3 d^2 \delta^2 \lambda^3 b^4\, \%2\, L^4$$

$$- 12\, c^4 L^3 d \lambda^6 b\, \%1 - 16\, c^2 L^5 \lambda^6 b^3\, \%1 + 4\, L^6 d \lambda^3 b^5\, \%2\, \delta c + 8\, L^4 d \lambda^5 b^3\, \%2\, \delta c^3$$

$$+ 4\, L^2 d \lambda^7 b\, \%2\, \delta c^5 - 4\, c^6 L d \lambda^8\, \%1\, \delta - 8\, c^4 L^3 d \lambda^6 b^2\, \%1\, \delta - 8\, c^5 d^2 \lambda^5 L^2\, \%2$$

$$- 8\, c^5 L^2 \lambda^9\, \%2 - 16\, c^3 L^4 \lambda^7 b^2\, \%2 + c^7 d^2 \lambda^7\, \%2\, \delta^2 - 4\, L^7 d \lambda^2 b^5\, \%1 - 3\, d^2 b^5\, \%1\, L^7$$

$$- 10\, d^2 b^3\, \%1\, L^5 c^2 \lambda^2 - 16\, L^5 d \lambda^4 b^3\, \%1\, c^2 + c^6 d^2 \lambda^6 b\, \%1\, \delta^2 L - 6\, \%1\, d^2 c^4 \delta \lambda^4 b^2 L^3$$

$$- 8\, c^5 L^2 d \lambda^7\, \%2 - d^2 c^2 \delta \lambda^2 b^4\, \%1\, L^5 - 8\, c^3 L^4 d \lambda^5 b^2\, \%2 - 8\, c L^6 \lambda^5 b^4\, \%2$$

$$+ 2\, d^2 b^3 \lambda^4\, \%1\, L^3 \delta^2 c^4 + d^2 b^5 \lambda^2\, \%1\, L^5 \delta^2 c^2 + 10\, d^2 b^3 \lambda^3\, \%2\, L^4 \delta c^3 - 15\, d^2 b\, \%1\, L^3 c^4 \lambda^4$$

$$- 4\, \%1\, d c^2 \delta L^5 \lambda^4 b^4) \Big/ (\lambda^5 Z2 (b^2 L^2 + c^2 \lambda^2)^3)$$

$$\left. + \frac{F0\, c L^2 (2\, b^2 L^2 c^2 \lambda^2 + c^4 \lambda^4 + b^4 L^4 + d c^2 b^2 L^2 + c^4 d \lambda^2 + d^2 c^4)}{Z2 (b^2 L^2 + c^2 \lambda^2)^3}\right) \lambda / (L$$

$$(\cos(\lambda) \sin(\lambda) + \lambda))$$

$$\%1 := \sin\left(\frac{\lambda \delta c}{L}\right)$$

$$\%2 := \cos\left(\frac{\lambda \delta c}{L}\right)$$

```
STUDENT > C2:=factor((int(vfinal1*cos(lambda*x/L),x=0..delta*c)/four
          numer+b*C1n)/(sqrt(lambda^2-d)*c/L));
```

$$C2 := -\frac{1}{4}\left(-3\, F0\, e^{(-\delta b)} d^2 L^2\, \%1 + 4\, F0\, e^{(-\delta b)} L d \lambda^3 \cos\left(\frac{\lambda \delta c}{L}\right) \delta c - 8\, F0\, e^{(-\delta b)} L^2\, \%1\, \lambda^4\right.$$

$$+ F0\, e^{(-\delta b)} d^2 c^2 \delta^2 \lambda^2\, \%1 + 3\, F0\, e^{(-\delta b)} d^2 \lambda \cos\left(\frac{\lambda \delta c}{L}\right) L \delta c - 4\, F0\, e^{(-\delta b)} L^2 d \lambda^2\, \%1$$

$$\left. - 4\, b\, C1n\, L^2 \lambda^4 Z2 \cos(\lambda) \sin(\lambda) - 4\, b\, C1n\, L^2 \lambda^5 Z2\right) \Big/ (L \lambda^4 Z2 (\cos(\lambda) \sin(\lambda) + \lambda)$$

$$\sqrt{\lambda^2 - d}\; c)$$

$$\%1 := \sin\left(\frac{\lambda \delta c}{L}\right)$$

```
STUDENT > C1:='C1';C2:='C2';u3:=subs(tprime=t-delta,u2);v3:=subs(tpr
          ime=t-delta,v2);sigma3:=subs(tprime=t-delta,sigma2);
```

$$C1 := C1$$

$$C2 := C2$$

$$u3 := e^{(-b (t - \delta))} \cos\left(\frac{\lambda x}{L}\right)\left(C1 \cos\left(\frac{\sqrt{\lambda^2 - d}\; c (t - \delta)}{L}\right) + C2 \sin\left(\frac{\sqrt{\lambda^2 - d}\; c (t - \delta)}{L}\right)\right)$$

$$v3 := -b\, e^{(-b\,(t-\delta))} \cos\left(\frac{\lambda\, x}{L}\right) (C1 \cos(\%1) + C2 \sin(\%1))$$
$$+ e^{(-b\,(t-\delta))} \cos\left(\frac{\lambda\, x}{L}\right)\left(-\frac{C1 \sin(\%1)\sqrt{\lambda^2 - d\, c}}{L} + \frac{C2 \cos(\%1)\sqrt{\lambda^2 - d\, c}}{L}\right)$$
$$\%1 := \frac{\sqrt{\lambda^2 - d\, c}\,(t-\delta)}{L}$$
$$\sigma 3 := -\frac{Z2\, c\, e^{(-b\,(t-\delta))} \sin\left(\frac{\lambda\, x}{L}\right) \lambda \left(C1 \cos\left(\frac{\sqrt{\lambda^2 - d\, c}\,(t-\delta)}{L}\right) + C2 \sin\left(\frac{\sqrt{\lambda^2 - d\, c}\,(t-\delta)}{L}\right)\right)}{A2\, L}$$

STUDENT >

D. Numerical Solution Of The Damped Wave Equation With Pile Top Step Loading Using Maple

(not included in Internet Edition)

```
STUDENT > #Solution for Ram/Damped Cushion/Pile Cap Problem --
          Numerical Results

STUDENT > restart;interface(prettyprint=0);

STUDENT > #Setup of Constants -- SI Units Used

STUDENT > #Number of Terms in Fourier Series

STUDENT > nterms:=65;lambda2:=array(1..nterms);
nterms := 65
lambda2 := array(1 .. 65,[])

STUDENT > #Constants of Hammer

STUDENT > M:=15000.;stroke:=1.5;efficiency:=0.8;gravity:=9.8;m:=3000
          .;
M := 15000.
stroke := 1.5
efficiency := .8
gravity := 9.8
m := 3000.

STUDENT > V0:=sqrt(2*stroke*efficiency*gravity);
V0 := 4.849742261

STUDENT > k1:=2450000000.;C:=0;
k1 := 2450000000.
C := 0

STUDENT > Z1:=sqrt(k1*M);
Z1 := 6062177.826

STUDENT > #Constants of Pile

STUDENT > ro:=0.5;ri:=0.46;
ro := .5
ri := .46

STUDENT > A2:=evalf(Pi*(ro^2-ri^2));A3:=A2;
A2 := .1206371579
```

```
A3 := .1206371579

STUDENT > L:=50.;
L := 50.

STUDENT > E2:=210000000000.;
E2 := 210000000000.

STUDENT > rho:=7800.;
rho := 7800.

STUDENT > c:=sqrt(E2/rho);
c := 5188.745216

STUDENT > Z2:=E2*A2/c;
Z2 := 4882452.714

STUDENT > Zprime:=Z2/Z1;
Zprime := .8053958254

STUDENT > piletime:=L/c;
piletime := .9636241118e-2

STUDENT > Mp:=rho*A2*L;
Mp := 47048.49158

STUDENT > #Soil (Pile Toe) Properties

STUDENT > nu:=0.25;
nu := .25

STUDENT > G3:=11040000.;rho3:=1600.;
G3 := 11040000.
rho3 := 1600.

STUDENT > k3:=4*G3*ro/((1-nu)*A3);E3:=100*G3/(85*(1-nu)^2);
k3 := 244037579.4
E3 := 23090196.08

STUDENT > p2:=evalf(Pi*2*ro);rg:=A2/p2^2;
```

```
p2 := 3.141592654
rg := .1222309963e-1
```

```
STUDENT > mu2:=sqrt(G3*rho3);b:=radsimp(mu2/(2*rho*sqrt(A2*rg)));
mu2 := 132905.9818
b := 221.8649537
```

```
STUDENT > K2:=evalf(Pi*G3/p2);a:=K2/(rho*sqrt(A2*rg));
K2 := 11040000.00
a := 36858.97435
```

```
STUDENT > d:=(b^2-a)*(L/c)^2;
d := 1.148186310
```

```
STUDENT > mprime:=m/M;
mprime := .2000000000
```

```
STUDENT > cprime:=C/Z1;
cprime := 0
```

```
STUDENT > F0:=15.566e6;delta:=4.6735e-3;switchpoint:=ceil(100*delta/
          piletime);
F0 := .15566e8
delta := .46735e-2
switchpoint := 49
```

```
STUDENT > #Solution of Wave Equation for Damped Case Using Step
          Loading at Pile Top
```

```
STUDENT > bessarg:=d*((u*c/L)^2-(x/L)^2);fconv:=exp(-b*u)*sum((bessa
          rg/4)^n/n!^2,n=0..2);
bessarg := 12365.08334*u^2-.4592745240e-3*x^2
fconv := exp(-221.8649537*u)*(1.+3091.270835*u^2-.1148186310e-3*x^2+.2500000000*(3091.27083
2-.1148186310e-3*x^2)^2)
```

```
STUDENT > gconv:=F0/Z2;
gconv := 3.188151716
```

```
STUDENT > intconv:=evalf(fconv*gconv);
intconv := 3.188151716*exp(-221.8649537*u)*(1.+3091.270835*u^2-.1148186310e-3*x^2+.25000000
3091.270835*u^2-.1148186310e-3*x^2)^2)
```

```
STUDENT > u1:=combine(int(intconv,u=x/c..t),exp);#Displacement Due
          to Constant Pile Head Velocity for t<delta
u1 := -.1651466297e-1*exp(-221.8649537*t)+.1753533975e-5*exp(-221.8649537*t)*x^2-52.7897948
exp(-221.8649537*t)*t^2-.4736036188e-10*exp(-221.8649537*t)*x^4-618.9216607*exp(-221.864953
*t^3+.2298851882e-4*exp(-221.8649537*t)*x^2*t-.4758732194*exp(-221.8649537*t)*t-34329.25640
(-221.8649537*t)*t^4+.2550173332e-2*exp(-221.8649537*t)*x^2*t^2+.1651466297e-1*exp(-.427588
e-1*x)+.3272639835e-28*exp(-.4275888380e-1*x)*x^4+.2072298362e-6*exp(-.4275888380e-1*x)*x^2
3682902692e-17*exp(-.4275888380e-1*x)*x^3+.9171258169e-4*exp(-.4275888380e-1*x)*x

STUDENT > v1:=diff(u1,t);sigma1:=-E2*diff(u1,x);
v1 := 3.188151716*exp(-221.8649537*t)-.3660592154e-3*exp(-221.8649537*t)*x^2+9855.440418*ex
221.8649537*t)*t^2+.1050760450e-7*exp(-221.8649537*t)*x^4+7616458.882*exp(-221.8649537*t)*t
5657940882*exp(-221.8649537*t)*x^2*t^2
sigma1 := -736484.2695*exp(-221.8649537*t)*x+39.78270398*exp(-221.8649537*t)*x^3-9655177.90
exp(-221.8649537*t)*x*t-1071072799.*exp(-221.8649537*t)*x*t^2+129031554.4*exp(-.4275888380e
)+.2938622954e-18*exp(-.4275888380e-1*x)*x^4+.3307012972e-7*exp(-.4275888380e-1*x)*x^3+1860
792460*exp(-.4275888380e-1*x)*x^2+736484.2697*exp(-.4275888380e-1*x)*x

STUDENT > ufinal1:=subs(t=delta,u1);vfinal1:=subs(t=delta,v1);#Times
          and Velocities at t=delta
ufinal1 := -.1997122462e-1*exp(-1.036885861)+.1916670690e-5*exp(-1.036885861)*x^2-.47360361
10*exp(-1.036885861)*x^4+.1651466297e-1*exp(-.4275888380e-1*x)+.3272639835e-28*exp(-.427588
e-1*x)*x^4+.2072298362e-6*exp(-.4275888380e-1*x)*x^2+.3682902692e-17*exp(-.4275888380e-1*x)
+.9171258169e-4*exp(-.4275888380e-1*x)*x
vfinal1 := 3.407043800*exp(-1.036885861)-.3784170648e-3*exp(-1.036885861)*x^2+.1050760450e-
exp(-1.036885861)*x^4

STUDENT > #Damped Pile Wave Equation with Fourier Series for t>L/c

STUDENT > omega:=sqrt(lambda^2-d);alpha:=-b*L/c;evalf(omega);evalf(a
          lpha);lambda0:=evalf(solve(omega=0,lambda));
omega := (lambda^2-1.148186310)^(1/2)
alpha := -2.137944190
(lambda^2-1.148186310)^(1/2)
-2.137944190
lambda0 := 1.071534559, -1.071534559

STUDENT > Bound3a:=(sin(lambda)/cos(lambda))=(1/lambda)*k3*A3*L/(-E3
          *A3+Z2*c);
Bound3a := sin(lambda)/cos(lambda) = .5811057503e-1/lambda

STUDENT > u2:=exp(-b*tprime)*cos(lambda*x/L)*(C1*cos(omega*c*tprime/
          L)+C2*sin(omega*c*tprime/L));v2:=diff(u2,tprime);sigma2:=-
          E2*diff(u2,x);
u2 := exp(-221.8649537*tprime)*cos(.2000000000e-1*lambda*x)*(C1*cos(103.7749043*(lambda^2-1
```

```
148186310)^(1/2)*tprime)+C2*sin(103.7749043*(lambda^2-1.148186310)^(1/2)*tprime))
v2 := -221.8649537*exp(-221.8649537*tprime)*cos(.2000000000e-1*lambda*x)*(C1*cos(103.774904
lambda^2-1.148186310)^(1/2)*tprime)+C2*sin(103.7749043*(lambda^2-1.148186310)^(1/2)*tprime)
exp(-221.8649537*tprime)*cos(.2000000000e-1*lambda*x)*(-103.7749043*C1*sin(103.7749043*(lam
2-1.148186310)^(1/2)*tprime)*(lambda^2-1.148186310)^(1/2)+103.7749043*C2*cos(103.7749043*(
lambda^2-1.148186310)^(1/2)*tprime)*(lambda^2-1.148186310)^(1/2))
sigma2 := 4200000000.*exp(-221.8649537*tprime)*sin(.2000000000e-1*lambda*x)*lambda*(C1*cos(
7749043*(lambda^2-1.148186310)^(1/2)*tprime)+C2*sin(103.7749043*(lambda^2-1.148186310)^(1/2
tprime))
```

STUDENT > `fournumer:=int(cos(lambda*x/L)^2,x=0..L);`

```
fournumer := (25.00000000*cos(1.000000000*lambda)*sin(1.000000000*lambda)+25.00000000*lambd
lambda
```

STUDENT > `C1:=(int(ufinal1*cos(lambda*x/L),x=0..delta*c))/fournumer;`

```
C1 := ((-.4016217248e19*sin(.4849920154*lambda)*lambda^12+.8939020775e21*lambda^3*cos(.
4849920154*lambda)-.3124281440e20*lambda^11*cos(.4849920154*lambda)-.2154527460e21*cos(.
4849920154*lambda)*lambda^9-1038571099.*sin(.4849920154*lambda)*lambda^14-.1662083735e19*la
^13*cos(.4849920154*lambda)-.6112675213e21*lambda^7*cos(.4849920154*lambda)-.3729606595e21*
lambda^5*cos(.4849920154*lambda)-.1783006701e22*sin(.4849920154*lambda)*lambda^2+.371886798
*lambda*cos(.4849920154*lambda)-.3435643728e22*sin(.4849920154*lambda)*lambda^4-.218530185
sin(.4849920154*lambda)*lambda^6-.6126721000e21*sin(.4849920154*lambda)*lambda^8-.766789527
*sin(.4849920154*lambda)-.8007586188e20*sin(.4849920154*lambda)*lambda^10)/(.6088547593e22*
lambda^5+.6660256902e22*lambda^7+.2914259688e22*lambda^9+.6375812267e21*lambda^11+.69744954
20*lambda^13+.3051757813e19*lambda^15)+.5000000000e-1*(.4375285203e84*lambda^4+.1431617840e
lambda^2+.3072179866e82*lambda^8+.5969677965e83*lambda^6+.1762432670e85)/(.4570805360e17+.
1000000000e17*lambda^2)^5)/(25.00000000*cos(1.000000000*lambda)*sin(1.000000000*lambda)+25.
00000000*lambda)*lambda
```

STUDENT > `C2:=(int(vfinal1*cos(lambda*x/L),x=0..delta*c)/fournumer+b`
`*C1)/(sqrt(lambda^2-d)*c/L);`

```
C2 := .9636241120e-2*((56.51909208*sin(.4849920154*lambda)*lambda^4+27.94159372*sin(.484992
*lambda)-13.55144985*lambda*cos(.4849920154*lambda)+30.25644222*sin(.4849920154*lambda)*lam
2-15.73664450*lambda^3*cos(.4849920154*lambda))/lambda^4/(25.00000000*cos(1.000000000*lambd
sin(1.000000000*lambda)+25.00000000*lambda)+221.8649537*((-.4016217248e19*sin(.4849920154*
lambda)*lambda^12+.8939020775e21*lambda^3*cos(.4849920154*lambda)-.3124281440e20*lambda^11*
.4849920154*lambda)-.2154527460e21*cos(.4849920154*lambda)*lambda^9-1038571099.*sin(.484992
*lambda)*lambda^14-.1662083735e19*lambda^13*cos(.4849920154*lambda)-.6112675213e21*lambda^7
(.4849920154*lambda)-.3729606595e21*lambda^5*cos(.4849920154*lambda)-.1783006701e22*sin(.
4849920154*lambda)*lambda^2+.3718867983e21*lambda*cos(.4849920154*lambda)-.3435643728e22*si
4849920154*lambda)*lambda^4-.2185301853e22*sin(.4849920154*lambda)*lambda^6-.6126721000e21*
.4849920154*lambda)*lambda^8-.7667895270e21*sin(.4849920154*lambda)-.8007586188e20*sin(.
4849920154*lambda)*lambda^10)/(.6088547593e22*lambda^5+.6660256902e22*lambda^7+.2914259688e
lambda^9+.6375812267e21*lambda^11+.6974495485e20*lambda^13+.3051757813e19*lambda^15)+.
5000000000e-1*(.4375285203e84*lambda^4+.1431617840e85*lambda^2+.3072179866e82*lambda^8+.
5969677965e83*lambda^6+.1762432670e85)/(.4570805360e17+.1000000000e17*lambda^2)^5)/(25.0000
*cos(1.000000000*lambda)*sin(1.000000000*lambda)+25.00000000*lambda)*lambda)/(lambda^2-1.
148186310)^(1/2)
```

STUDENT > `for i from 1 by 1 while i<nterms+1 do;`

```
STUDENT > lambda:='lambda';

STUDENT > bound1:=((i-1)*Pi-Pi/2);

STUDENT > bound2:=((i-1)*Pi+Pi/2);

STUDENT > lambda2[i]:=fsolve(Bound3a,lambda,lambda=bound1..bound2);

STUDENT > od:

STUDENT > n:='n';print (lambda2);
n := n
VECTOR([.2387513611, 3.159980119, 6.292420057, 9.430939578, 12.57099317, 15.71166181, 18.
85263827, 21.99379070, 25.13505316, 28.27638897, 31.41777614, 34.55920067, 37.70065321, 40.
84212731, 43.98361834, 47.12512292, 50.26663850, 53.40816316, 56.54969537, 59.69123394, 62.
83277792, 65.97432653, 69.11587915, 72.25743525, 75.39899439, 78.54055622, 81.68212042, 84.
82368672, 87.96525491, 91.10682478, 94.24839618, 97.38996894, 100.5315429, 103.6731181, 106
8146943, 109.9562714, 113.0978493, 116.2394281, 119.3810076, 122.5225878, 125.6641686, 128.
8057499, 131.9473319, 135.0889143, 138.2304971, 141.3720805, 144.5136642, 147.6552483, 150.
7968327, 153.9384175, 157.0800026, 160.2215880, 163.3631737, 166.5047596, 169.6463458, 172.
7879323, 175.9295189, 179.0711058, 182.2126928, 185.3542801, 188.4958675, 191.6374551, 194.
7790429, 197.9206308, 201.0622188])

STUDENT > xlate:=sum(evalf(subs(lambda=lambda2[n],tprime=t-delta,u2)
          ),n=1..nterms):

STUDENT > stlate:=sum(evalf(subs(lambda=lambda2[n],tprime=t-delta,si
          gma2)),n=1..nterms):

STUDENT > x:='x';
x := x

STUDENT > writeto(dsoltop);

STUDENT > for i from 0 by 1 to switchpoint do

STUDENT > t:=i*piletime/100:

STUDENT > print(evalf(i/100),evalf(1000*subs(x=0,u1*Heaviside(t-x/c)
          )));
```

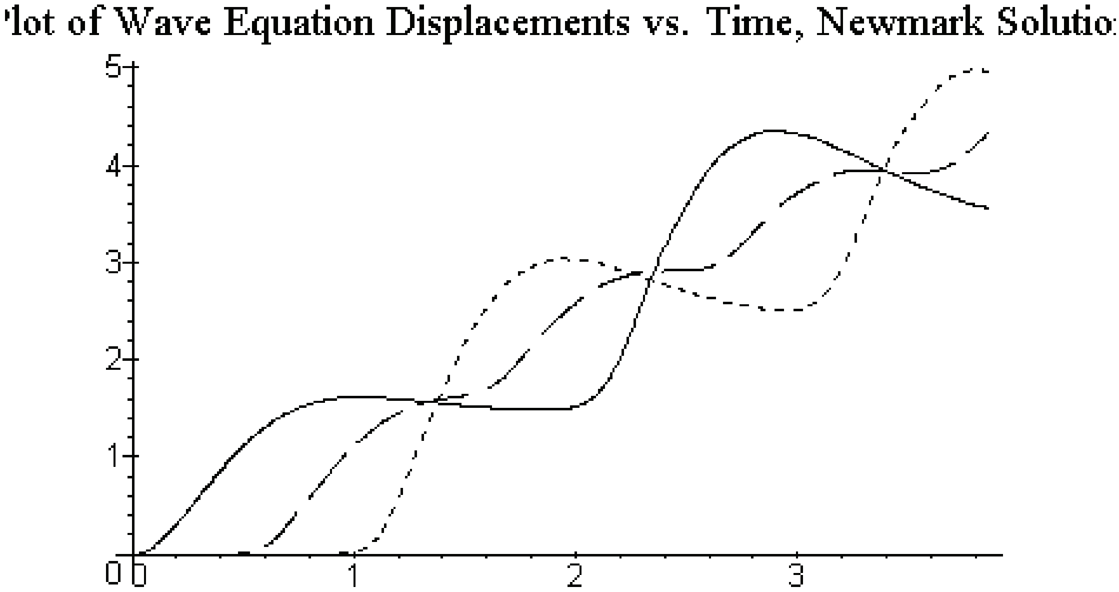
```
STUDENT > od:

STUDENT > for i from switchpoint+1 by 1 to 400 do

STUDENT > t:=i*piletime/100:

STUDENT > print(evalf(i/100),evalf(1000*subs(x=0,xlate)));

STUDENT > od:

STUDENT > writeto(dsolmid);

STUDENT > for i from 0 by 1 to switchpoint do

STUDENT > t:=i*piletime/100:

STUDENT > print(evalf(i/100),evalf(1000*subs(x=L/2,u1*Heaviside(t-x/
          c))));

STUDENT > od:

STUDENT > for i from switchpoint+1 by 1 to 400 do

STUDENT > t:=i*piletime/100:

STUDENT > print(evalf(i/100),evalf(1000*subs(x=L/2,xlate)));

STUDENT > od:

STUDENT > writeto(dsoltoe);

STUDENT > for i from 0 by 1 to switchpoint do

STUDENT > t:=i*piletime/100:

STUDENT > print(evalf(i/100),evalf(1000*subs(x=L,u1*Heaviside(t-x/c)
          )));
```

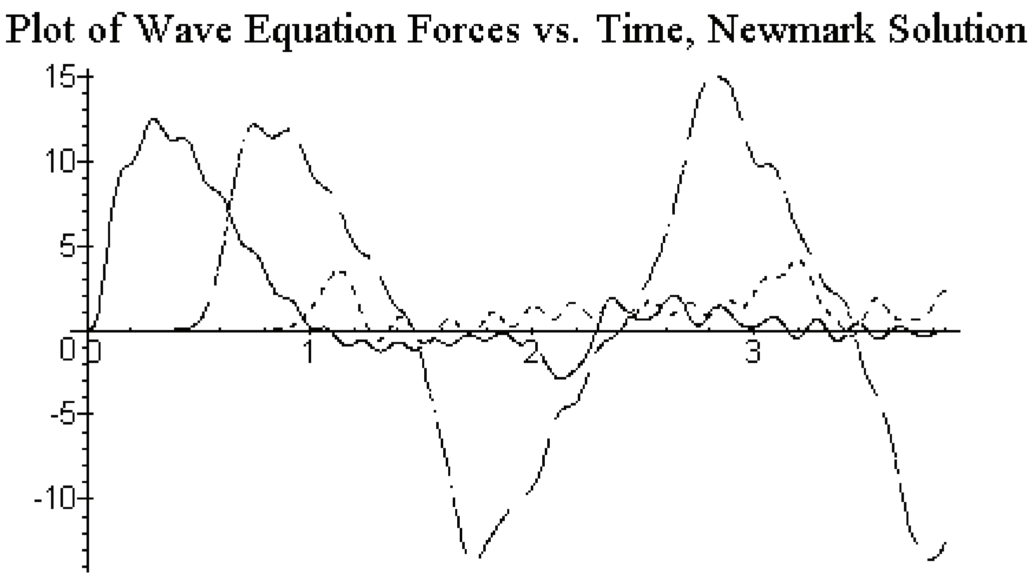
```
STUDENT > od:

STUDENT > for i from switchpoint+1 by 1 to 400 do

STUDENT > t:=i*piletime/100:

STUDENT > print(evalf(i/100),evalf(1000*subs(x=L,xlate)));

STUDENT > od:

STUDENT > writeto(ssoltop);

STUDENT > for i from 0 by 1 to switchpoint do

STUDENT > t:=i*piletime/100:

STUDENT > print(evalf(i/100),evalf(1e-7*subs(x=0,sigma1*Heaviside(t-
          x/c))));

STUDENT > od:

STUDENT > for i from switchpoint+1 by 1 to 400 do

STUDENT > t:=i*piletime/100:

STUDENT > print(evalf(i/100),evalf(1e-7*subs(x=0,stlate)));

STUDENT > od:

STUDENT > writeto(ssolmid);

STUDENT > for i from 0 by 1 to switchpoint do

STUDENT > t:=i*piletime/100:
```

```
STUDENT > print(evalf(i/100),evalf(1e-7*subs(x=L/2,sigma1*Heaviside(
          t-x/c))));

STUDENT > od:

STUDENT > for i from switchpoint+1 by 1 to 400 do

STUDENT > t:=i*piletime/100:

STUDENT > print(evalf(i/100),evalf(1e-7*subs(x=L/2,stlate)));

STUDENT > od:

STUDENT > writeto(ssoltoe);

STUDENT > for i from 0 by 1 to switchpoint do

STUDENT > t:=i*piletime/100:

STUDENT > print(evalf(i/100),evalf(1e-7*subs(x=L,sigma1*Heaviside(t-
          x/c))));

STUDENT > od:

STUDENT > for i from switchpoint+1 by 1 to 400 do

STUDENT > t:=i*piletime/100:

STUDENT > print(evalf(i/100),evalf(1e-7*subs(x=L,stlate)));

STUDENT > od:

STUDENT > writeto(terminal);

STUDENT >
```

E. Direct Stiffness Solution of the Undamped Wave Equation using Maple

(not included in Internet Edition except for figures below)





```
STUDENT > #Thesis -- Direct Stiffness Solution of Undamped Wave
          Equation Example -- Don C. Warrington

STUDENT > restart;

STUDENT > with(linalg):
Warning: new definition for   norm
Warning: new definition for   trace

STUDENT > #Model has twenty elements of equal length L; E,A,
          constant along length; Lumped Mass Matrices Used; Element
          21 is Soil Spring 1/mass

STUDENT > #Rigid End at Node 22

STUDENT > Nstep:=10;Ncycle:=4;#Number of Steps Per Minimum deltat
          and Number of "L/c" cycles for full pile length
```

$$Nstep := 10$$

$$Ncycle := 4$$

```
STUDENT > Steps:=20*Ncycle*Nstep;#Total Number of Steps
```

$$Steps := 800$$

```
STUDENT > plotable1:=matrix(Steps,2,[seq(0,i=1..2*Steps)]):plotable2
          :=evalm(plotable1):plotable3:=evalm(plotable1):plotable4:=
          evalm(plotable1):plotable5:=evalm(plotable1):plotable6:=ev
          alm(plotable1):

STUDENT > #Input Variables and Basic Equations

STUDENT > rho:=7800;A:=0.12064;L:=5/2;E1:=210000000000;#Pile Element
          Parameters
```

$$\rho := 7800$$

$$A := .12064$$

$$L := \frac{5}{2}$$

$$E1 := 210000000000$$

```
STUDENT > cx:=sqrt(E1/rho);deltatest:=L/cx;deltat:=floor(numer(delta
          test))/ceil(denom(deltatest))/Nstep;beta:=1/6;gamma1:=1/2;
          #Integration Parameters
```

$$cx := \frac{5000}{13}\sqrt{182}$$

$$deltatest := \frac{1}{28000}\sqrt{182}$$

$$deltat := \frac{13}{280000}$$

$$\beta := \frac{1}{6}$$

$$\gamma 1 := \frac{1}{2}$$

```
STUDENT > F0:=37.5e6*exp(-250.8985*t)*sin(316.833*t);piletime:=20*deltatest;#Pile Top Force Parameter
```

$$F0 := .375\ 10^{8}\ e^{(-250.8985\ t)}\ \sin(316.833\ t)$$

$$piletime := \frac{1}{1400}\sqrt{182}$$

```
STUDENT > k3:=evalf(600000000/11);#Pile Toe Stiffness
```

$$k3 := .5454545455\ 10^{8}$$

```
STUDENT > t:=0;
```

$$t := 0$$

```
STUDENT > k:=evalm((E1*A/L)*matrix(2,2,[1,-1,-1,1]));m:=evalm((rho*A*L/2)*matrix(2,2,[1,0,0,1]));
```

$$k := \begin{bmatrix} .1013376000\ 10^{11} & -.1013376000\ 10^{11} \\ -.1013376000\ 10^{11} & .1013376000\ 10^{11} \end{bmatrix}$$

$$m := \begin{bmatrix} 1176.240000 & 0 \\ 0 & 1176.240000 \end{bmatrix}$$

```
STUDENT > #Element 1;

STUDENT > K:=matrix(22,22,[seq(0,i=1..484)]):M:=evalm(K):

STUDENT > copyinto(k,K,1,1):copyinto(m,M,1,1):K1:=evalm(K):M1:=evalm(M):

STUDENT > #Element 2

STUDENT > K:=matrix(22,22,[seq(0,i=1..484)]):M:=evalm(K):

STUDENT > copyinto(k,K,2,2):copyinto(m,M,2,2):K2:=evalm(K):M2:=evalm
```

```
(M):

STUDENT > #Element 3

STUDENT > K:=matrix(22,22,[seq(0,i=1..484)]):M:=evalm(K):

STUDENT > copyinto(k,K,3,3):copyinto(m,M,3,3):K3:=evalm(K):M3:=evalm
          (M):

STUDENT > #Element 4

STUDENT > K:=matrix(22,22,[seq(0,i=1..484)]):M:=evalm(K):

STUDENT > copyinto(k,K,4,4):copyinto(m,M,4,4):K4:=evalm(K):M4:=evalm
          (M):

STUDENT > #Element 5

STUDENT > K:=matrix(22,22,[seq(0,i=1..484)]):M:=evalm(K):

STUDENT > copyinto(k,K,5,5):copyinto(m,M,5,5):K5:=evalm(K):M5:=evalm
          (M):

STUDENT > #Element 6

STUDENT > K:=matrix(22,22,[seq(0,i=1..484)]):M:=evalm(K):

STUDENT > copyinto(k,K,6,6):copyinto(m,M,6,6):K6:=evalm(K):M6:=evalm
          (M):

STUDENT > #Element 7

STUDENT > K:=matrix(22,22,[seq(0,i=1..484)]):M:=evalm(K):

STUDENT > copyinto(k,K,7,7):copyinto(m,M,7,7):K7:=evalm(K):M7:=evalm
          (M):
```

```
STUDENT > #Element 8

STUDENT > K:=matrix(22,22,[seq(0,i=1..484)]):M:=evalm(K):

STUDENT > copyinto(k,K,8,8):copyinto(m,M,8,8):K8:=evalm(K):M8:=evalm
          (M):

STUDENT > #Element 9

STUDENT > K:=matrix(22,22,[seq(0,i=1..484)]):M:=evalm(K):

STUDENT > copyinto(k,K,9,9):copyinto(m,M,9,9):K9:=evalm(K):M9:=evalm
          (M):

STUDENT > #Element 10

STUDENT > K:=matrix(22,22,[seq(0,i=1..484)]):M:=evalm(K):

STUDENT > copyinto(k,K,10,10):copyinto(m,M,10,10):K10:=evalm(K):M10:
          =evalm(M):

STUDENT > #Element 11;

STUDENT > K:=matrix(22,22,[seq(0,i=1..484)]):M:=evalm(K):

STUDENT > copyinto(k,K,11,11):copyinto(m,M,11,11):K11:=evalm(K):M11:
          =evalm(M):

STUDENT > #Element 12

STUDENT > K:=matrix(22,22,[seq(0,i=1..484)]):M:=evalm(K):

STUDENT > copyinto(k,K,12,12):copyinto(m,M,12,12):K12:=evalm(K):M12:
          =evalm(M):
```

```
STUDENT > #Element 13

STUDENT > K:=matrix(22,22,[seq(0,i=1..484)]):M:=evalm(K):

STUDENT > copyinto(k,K,13,13):copyinto(m,M,13,13):K13:=evalm(K):M13:
          =evalm(M):

STUDENT > #Element 14

STUDENT > K:=matrix(22,22,[seq(0,i=1..484)]):M:=evalm(K):

STUDENT > copyinto(k,K,14,14):copyinto(m,M,14,14):K14:=evalm(K):M14:
          =evalm(M):

STUDENT > #Element 15

STUDENT > K:=matrix(22,22,[seq(0,i=1..484)]):M:=evalm(K):

STUDENT > copyinto(k,K,15,15):copyinto(m,M,15,15):K15:=evalm(K):M15:
          =evalm(M):

STUDENT > #Element 16

STUDENT > K:=matrix(22,22,[seq(0,i=1..484)]):M:=evalm(K):

STUDENT > copyinto(k,K,16,16):copyinto(m,M,16,16):K16:=evalm(K):M16:
          =evalm(M):

STUDENT > #Element 17

STUDENT > K:=matrix(22,22,[seq(0,i=1..484)]):M:=evalm(K):

STUDENT > copyinto(k,K,17,17):copyinto(m,M,17,17):K17:=evalm(K):M17:
          =evalm(M):
```

```
STUDENT > #Element 18

STUDENT > K:=matrix(22,22,[seq(0,i=1..484)]):M:=evalm(K):

STUDENT > copyinto(k,K,18,18):copyinto(m,M,18,18):K18:=evalm(K):M18:
          =evalm(M):

STUDENT > #Element 19

STUDENT > K:=matrix(22,22,[seq(0,i=1..484)]):M:=evalm(K):

STUDENT > copyinto(k,K,19,19):copyinto(m,M,19,19):K19:=evalm(K):M19:
          =evalm(M):

STUDENT > #Element 20

STUDENT > K:=matrix(22,22,[seq(0,i=1..484)]):M:=evalm(K):

STUDENT > copyinto(k,K,20,20):copyinto(m,M,20,20):K20:=evalm(K):M20:
          =evalm(M):

STUDENT > #Element 11 -- Pile Toe Matrix

STUDENT > ktm:=matrix(2,2,[k3,-k3,-k3,k3]);kmm:=matrix(2,2,[0,0,0,0]
          );
```

$$ktm := \begin{bmatrix} .5454545455\ 10^{8} & -.5454545455\ 10^{8} \\ -.5454545455\ 10^{8} & .5454545455\ 10^{8} \end{bmatrix}$$

$$kmm := \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

```
STUDENT > K:=matrix(22,22,[seq(0,i=1..484)]):M:=evalm(K):

STUDENT > copyinto(ktm,K,21,21):copyinto(kmm,M,21,21):K21:=evalm(K):
          M21:=evalm(M):

STUDENT > #Evaluate Global Mass and Stiffness Matrices and Solve
```

STUDENT > K:=evalm(K1+K2+K3+K4+K5+K6+K7+K8+K9+K10+K11+K12+K13+K14+K15+K16+K17+K18+K19+K20+K21);

K :=

[.1013376000 10^{11} , -.1013376000 10^{11} , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0]

[-.1013376000 10^{11} , .2026752000 10^{11} , -.1013376000 10^{11} , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0]

[0 , -.1013376000 10^{11} , .2026752000 10^{11} , -.1013376000 10^{11} , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0]

[0 , 0 , -.1013376000 10^{11} , .2026752000 10^{11} , -.1013376000 10^{11} , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0]

[0 , 0 , 0 , -.1013376000 10^{11} , .2026752000 10^{11} , -.1013376000 10^{11} , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0]

[0 , 0 , 0 , 0 , -.1013376000 10^{11} , .2026752000 10^{11} , -.1013376000 10^{11} , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0]

[0 , 0 , 0 , 0 , 0 , -.1013376000 10^{11} , .2026752000 10^{11} , -.1013376000 10^{11} , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0]

[0 , 0 , 0 , 0 , 0 , 0 , -.1013376000 10^{11} , .2026752000 10^{11} , -.1013376000 10^{11} , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0]

[0 , 0 , 0 , 0 , 0 , 0 , 0 , -.1013376000 10^{11} , .2026752000 10^{11} , -.1013376000 10^{11} , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0]

[0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , -.1013376000 10^{11} , .2026752000 10^{11} , -.1013376000 10^{11} , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0]

[0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , -.1013376000 10^{11} , .2026752000 10^{11} , -.1013376000 10^{11} , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0]

[0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , -.1013376000 10^{11} , .2026752000 10^{11} , -.1013376000 10^{11} , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0]

[0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , -.1013376000 10^{11} , .2026752000 10^{11} , -.1013376000 10^{11} , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0]

[0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , -.1013376000 10^{11} , .2026752000 10^{11} , -.1013376000 10^{11} , 0 , 0 , 0 , 0 , 0 , 0 , 0]

[0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , -.1013376000 10^{11} , .2026752000 10^{11} , -.1013376000 10^{11} , 0 , 0 , 0 , 0 , 0 , 0]

[0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , -.1013376000 10^{11} , .2026752000 10^{11}

-.1013376000 10^{11} , 0 , 0 , 0 , 0 , 0]

[0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , -.1013376000 10^{11} , .2026752000 10^{11} ,
-.1013376000 10^{11} , 0 , 0 , 0 , 0]

[0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , -.1013376000 10^{11} , .2026752000 10^{11} ,
-.1013376000 10^{11} , 0 , 0 , 0]

[0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , -.1013376000 10^{11} , .2026752000 10^{11} ,
-.1013376000 10^{11} , 0 , 0]

[0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , -.1013376000 10^{11} , .2026752000 10^{11} ,
-.1013376000 10^{11} , 0]

[0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , -.1013376000 10^{11} ,
.1018830545 10^{11} , -.5454545455 10^{8}]

[0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , -.5454545455 10^{8} ,
.5454545455 10^{8}]

```
STUDENT > M:=evalm(M1+M2+M3+M4+M5+M6+M7+M8+M9+M10+M11+M12+M13+M14+M1
          5+M16+M17+M18+M19+M20+M21);
```

M :=

1176.240000 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0
0 ,2352.480000 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0
0 ,0 ,2352.480000 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0
0 ,0 ,0 ,2352.480000 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0
0 ,0 ,0 ,0 ,2352.480000 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0
0 ,0 ,0 ,0 ,0 ,2352.480000 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0
0 ,0 ,0 ,0 ,0 ,0 ,2352.480000 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0
0 ,0 ,0 ,0 ,0 ,0 ,0 ,2352.480000 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0
0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,2352.480000 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0
0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,2352.480000 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0
0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,2352.480000 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0
0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,2352.480000 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0
0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,2352.480000 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0
0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,2352.480000 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0
0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,2352.480000 ,0 ,0 ,0 ,0 ,0 ,0 ,0
0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,2352.480000 ,0 ,0 ,0 ,0 ,0 ,0
0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,2352.480000 ,0 ,0 ,0 ,0 ,0
0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,2352.480000 ,0 ,0 ,0 ,0
0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,2352.480000 ,0 ,0 ,0
0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,2352.480000 ,0 ,0
0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,0 ,1176.240000 ,0
0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0

```
STUDENT > #Partition Matrix As Displacement and Velocity of Last
          Node is 0 for all time
```

```
STUDENT > K:=delrows(K,22..22):M:=delrows(M,22..22):K:=delcols(K,22..22);M:=delcols(M,22..22);
```

K :=

[.1013376000 10^{11} , -.1013376000 10^{11} , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 ,
0]

[-.1013376000 10^{11} , .2026752000 10^{11} , -.1013376000 10^{11} , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0
, 0 , 0 , 0 , 0 , 0 , 0]

[0 , -.1013376000 10^{11} , .2026752000 10^{11} , -.1013376000 10^{11} , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0
, 0 , 0 , 0 , 0 , 0 , 0]

[0 , 0 , -.1013376000 10^{11} , .2026752000 10^{11} , -.1013376000 10^{11} , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0
, 0 , 0 , 0 , 0 , 0 , 0]

[0 , 0 , 0 , -.1013376000 10^{11} , .2026752000 10^{11} , -.1013376000 10^{11} , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0
, 0 , 0 , 0 , 0 , 0 , 0]

[0 , 0 , 0 , 0 , -.1013376000 10^{11} , .2026752000 10^{11} , -.1013376000 10^{11} , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0
, 0 , 0 , 0 , 0 , 0 , 0]

[0 , 0 , 0 , 0 , 0 , -.1013376000 10^{11} , .2026752000 10^{11} , -.1013376000 10^{11} , 0 , 0 , 0 , 0 , 0 , 0 , 0
, 0 , 0 , 0 , 0 , 0 , 0]

[0 , 0 , 0 , 0 , 0 , 0 , -.1013376000 10^{11} , .2026752000 10^{11} , -.1013376000 10^{11} , 0 , 0 , 0 , 0 , 0 , 0
, 0 , 0 , 0 , 0 , 0 , 0]

[0 , 0 , 0 , 0 , 0 , 0 , 0 , -.1013376000 10^{11} , .2026752000 10^{11} , -.1013376000 10^{11} , 0 , 0 , 0 , 0 , 0
, 0 , 0 , 0 , 0 , 0 , 0]

[0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , -.1013376000 10^{11} , .2026752000 10^{11} , -.1013376000 10^{11} , 0 , 0 , 0 , 0
, 0 , 0 , 0 , 0 , 0 , 0]

[0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , -.1013376000 10^{11} , .2026752000 10^{11} , -.1013376000 10^{11} , 0 , 0 , 0
, 0 , 0 , 0 , 0 , 0 , 0]

[0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , -.1013376000 10^{11} , .2026752000 10^{11} , -.1013376000 10^{11} , 0 , 0
, 0 , 0 , 0 , 0 , 0 , 0]

[0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , -.1013376000 10^{11} , .2026752000 10^{11} , -.1013376000 10^{11} , 0
, 0 , 0 , 0 , 0 , 0 , 0]

[0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , -.1013376000 10^{11} , .2026752000 10^{11} , -.1013376000 10^{11}
, 0 , 0 , 0 , 0 , 0 , 0]

[0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , -.1013376000 10^{11} , .2026752000 10^{11} ,
-.1013376000 10^{11} , 0 , 0 , 0 , 0 , 0]

[0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , -.1013376000 10^{11} , .2026752000 10^{11} ,

$-.1013376000\ 10^{11}, 0, 0, 0, 0]$

$[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -.1013376000\ 10^{11}, .2026752000\ 10^{11},$
$-.1013376000\ 10^{11}, 0, 0, 0]$

$[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -.1013376000\ 10^{11}, .2026752000\ 10^{11},$
$-.1013376000\ 10^{11}, 0, 0]$

$[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -.1013376000\ 10^{11}, .2026752000\ 10^{11},$
$-.1013376000\ 10^{11}, 0]$

$[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -.1013376000\ 10^{11}, .2026752000\ 10^{11},$
$-.1013376000\ 10^{11}]$

$[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -.1013376000\ 10^{11},$
$.1018830545\ 10^{11}]$

$$M := \left[\begin{array}{l}
1176.240000, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 \\
0, 2352.480000, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 \\
0, 0, 2352.480000, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 \\
0, 0, 0, 2352.480000, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 \\
0, 0, 0, 0, 2352.480000, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 \\
0, 0, 0, 0, 0, 2352.480000, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 \\
0, 0, 0, 0, 0, 0, 2352.480000, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 \\
0, 0, 0, 0, 0, 0, 0, 2352.480000, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 \\
0, 0, 0, 0, 0, 0, 0, 0, 2352.480000, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 \\
0, 0, 0, 0, 0, 0, 0, 0, 0, 2352.480000, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 \\
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 2352.480000, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 \\
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 2352.480000, 0, 0, 0, 0, 0, 0, 0, 0, 0 \\
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 2352.480000, 0, 0, 0, 0, 0, 0, 0, 0 \\
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 2352.480000, 0, 0, 0, 0, 0, 0, 0 \\
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 2352.480000, 0, 0, 0, 0, 0, 0 \\
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 2352.480000, 0, 0, 0, 0, 0 \\
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 2352.480000, 0, 0, 0, 0 \\
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 2352.480000, 0, 0, 0 \\
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 2352.480000, 0, 0 \\
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 2352.480000, 0 \\
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1176.240000
\end{array}\right]$$

```
STUDENT > #Set Initial Conditions
```

```
STUDENT > dm:=vector([seq(0,i=1..21)]);
```

$dm := [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]$

```
STUDENT > dvelm:=vector([seq(0,i=1..21)]);
```

$dvelm := [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]$

```
STUDENT > Fm:=vector([F0,seq(0,i=1..20)]);
```

$Fm := [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]$

```
STUDENT > dacclm:=evalm(inverse(M)*(Fm-K*dm));
```

$dacclm := [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]$

```
STUDENT > Kmprime:=evalm(K+M/(beta*deltat^2));
```

Kmprime :=

[.3284117760 10^{13} , -.1013376000 10^{11} , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0]

[-.1013376000 10^{11} , .6568235521 10^{13} , -.1013376000 10^{11} , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0]

[0 , -.1013376000 10^{11} , .6568235521 10^{13} , -.1013376000 10^{11} , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0]

[0 , 0 , -.1013376000 10^{11} , .6568235521 10^{13} , -.1013376000 10^{11} , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0]

[0 , 0 , 0 , -.1013376000 10^{11} , .6568235521 10^{13} , -.1013376000 10^{11} , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0]

[0 , 0 , 0 , 0 , -.1013376000 10^{11} , .6568235521 10^{13} , -.1013376000 10^{11} , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0]

[0 , 0 , 0 , 0 , 0 , -.1013376000 10^{11} , .6568235521 10^{13} , -.1013376000 10^{11} , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0]

[0 , 0 , 0 , 0 , 0 , 0 , -.1013376000 10^{11} , .6568235521 10^{13} , -.1013376000 10^{11} , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0]

[0 , 0 , 0 , 0 , 0 , 0 , 0 , -.1013376000 10^{11} , .6568235521 10^{13} , -.1013376000 10^{11} , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0]

[0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , -.1013376000 10^{11} , .6568235521 10^{13} , -.1013376000 10^{11} , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0]

[0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , -.1013376000 10^{11} , .6568235521 10^{13} , -.1013376000 10^{11} , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0]

[0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , -.1013376000 10^{11} , .6568235521 10^{13} , -.1013376000 10^{11} , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0]

[0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , -.1013376000 10^{11} , .6568235521 10^{13} , -.1013376000 10^{11} , 0 , 0 , 0 , 0 , 0 , 0 , 0]

[0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , -.1013376000 10^{11} , .6568235521 10^{13} , -.1013376000 10^{11} , 0 , 0 , 0 , 0 , 0 , 0]

[0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , -.1013376000 10^{11} , .6568235521 10^{13} , -.1013376000 10^{11} , 0 , 0 , 0 , 0 , 0]

$[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -.1013376000\ 10^{11}, .6568235521\ 10^{13},$
$-.1013376000\ 10^{11}, 0, 0, 0, 0]$
$[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -.1013376000\ 10^{11}, .6568235521\ 10^{13},$
$-.1013376000\ 10^{11}, 0, 0, 0]$
$[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -.1013376000\ 10^{11}, .6568235521\ 10^{13},$
$-.1013376000\ 10^{11}, 0, 0]$
$[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -.1013376000\ 10^{11}, .6568235521\ 10^{13},$
$-.1013376000\ 10^{11}, 0]$
$[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -.1013376000\ 10^{11}, .6568235521\ 10^{13},$
$-.1013376000\ 10^{11}]$
$[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -.1013376000\ 10^{11},$
$.3284172305\ 10^{13}]$

```
STUDENT > #Cycle Loop for displacements, velocities and
          accelerations

STUDENT > for i from 1 by 1 while i<Steps+1 do

STUDENT > Fm:=vector([F0,seq(0,j=1..20)]);

STUDENT > Fmprime:=evalm(Fm+((M/(beta*deltat^2))*(dm+deltat*dvelm+(1
          /2-beta)*deltat^2*dacclm)));

STUDENT > dm1:=evalm(inverse(Kmprime)*Fmprime);

STUDENT > dacclm1:=evalm((dm1-dm-deltat*dvelm-deltat^2*(1/2-beta)*da
          cclm)/(beta*deltat^2));

STUDENT > dvelm1:=evalm(dvelm+deltat*((1-gamma1)*dacclm+gamma1*daccl
          m1));

STUDENT > t:=t+deltat;

STUDENT > dm:=evalm(dm1):

STUDENT > dvelm:=evalm(dvelm1):
```

```
STUDENT > dacclm:=evalm(dacclm1):

STUDENT > plotable1[i,1]:=evalf(t/piletime);

STUDENT > plotable2[i,1]:=evalf(t/piletime);

STUDENT > plotable3[i,1]:=evalf(t/piletime);

STUDENT > plotable4[i,1]:=evalf(t/piletime);

STUDENT > plotable5[i,1]:=evalf(t/piletime);

STUDENT > plotable6[i,1]:=evalf(t/piletime);

STUDENT > plotable1[i,2]:=dm[1];

STUDENT > plotable2[i,2]:=dm[11];

STUDENT > plotable3[i,2]:=dm[21];

STUDENT > ftm:=evalm(k*vector([dm[1],dm[2]]));

STUDENT > fmm:=evalm(k*vector([dm[11],dm[12]]));

STUDENT > fbm:=evalm(k*vector([dm[20],dm[21]]));

STUDENT > plotable4[i,2]:=ftm[2];

STUDENT > plotable5[i,2]:=fmm[2];

STUDENT > plotable6[i,2]:=fbm[2];

STUDENT > od:
```

```
STUDENT > with(plots):

STUDENT > P1:=plot(plotable1,linestyle=1):

STUDENT > P2:=plot(plotable2,linestyle=2):

STUDENT > P3:=plot(plotable3,linestyle=3):

STUDENT > display({P1,P2,P3},title=`Plot of Wave Equation
          Displacements vs. Time, Newmark Solution`);

STUDENT > P4:=plot(plotable4,linestyle=1):

STUDENT > P5:=plot(plotable5,linestyle=2):

STUDENT > P6:=plot(plotable6,linestyle=3):

STUDENT > display({P4,P5,P6},title=`Plot of Wave Equation Forces vs.
          Time, Newmark Solution`);

STUDENT >
```

F. Damped Wave Equation Solution using ANSYS

Appendix F – ANSYS Solution for Damped Case

```
*---------------------------------------------------------------------------*
|                                                                           |
|          W E L C O M E   T O   T H E   A N S Y S   P R O G R A M          |
|                                                                           |
*---------------------------------------------------------------------------*



         *****  ANSYS COMMAND LINE ARGUMENTS  *****
 NONE


         *****  ANSYS DYNAMIC MEMORY ALLOCATION  *****
 WORK SPACE REQUESTED               =    4194304      16.000  MB  COMMAND LINE
 MINIMUM WORK SPACE REQUIRED        =    1564800       5.969  MB
 MINIMUM WORK SPACE RECOMMENDED     =    3548704      13.537  MB
 WORK SPACE OBTAINED                =    4194302      16.000  MB
 BYTES PER WORD                     =          4




   *****  NOTICE  *****  THIS IS THE ANSYS GENERAL PURPOSE
   FINITE ELEMENT COMPUTER PROGRAM.  NEITHER SWANSON ANALYSIS
   SYSTEMS, INC. NOR THE DISTRIBUTOR SUPPLYING THIS PROGRAM
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   APPLICABILITY OF ANY RESULTS OBTAINED FROM THE ANSYS SYSTEM.
   USERS MUST VERIFY THEIR OWN RESULTS.

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   POSSIBLE.  SUGGESTIONS AND COMMENTS ARE WELCOMED.  ANY
   ERRORS ENCOUNTERED IN EITHER THE DOCUMENTATION OR THE
   RESULTS SHOULD BE IMMEDIATELY BROUGHT TO OUR ATTENTION.


ENTER  /SHOW,device TO SET THE GRAPHICS DISPLAY TO device(e.g. VGA,EGA,ETC.)
ENTER  /MENU,ON     TO START THE ANSYS MENU SYSTEM
ENTER  HELP         FOR GENERAL ANSYS HELP INFORMATION

                  VERSION=PC 386/486     REVISION= 5.0ED56
CURRENT JOBNAME=file      21:56:10  OCT 22, 1996 CP=      0.000


BEGIN:
    1  /FILNAM,thesis2
    2  /TITLE,Damped Case
    3  /PREP7
    4  ET,1,link1
    5  et,2,combin14,,1
    6  R,1,.12064
    7  r,2,2.95e7,23137
    8  r,3,2.175e7,261000
    9  r,4,4.350e7,522000
   10  MP,EX,1,210e9
   11  MP,DENS,1,7800
   12  N,1,0,0
   13  N,41,50,0
   14  FILL
   15  N,42,50,0
   16  n,43,0,0
   17  n,83,50,0
   18  fill
   19  E,1,2
   20  EGEN,40,1,-1
```

```
type,2
REAL,2
E,41,42
real,3
e,1,43
e,41,83
real,4
e,2,44
egen,39,1,-1
nlist
ELIST
FINISH
/SOLU
ANTYPE,tran
D,1,UY,0,,42,1
D,42,UX,0,,83,1
f,1,fx,15.566e6
trnopt,full
lumpm,on
autots,on
alphad,.0001
betad,.0001
kbc,1
outres,all,all
deltim,1.205e-5
time,4.6735e-3
f,1,fx,15.566e6
lswrite
time,3.855e-2
f,1,fx,0
lswrite
save
lssolve,1,2
finish
/post26
time
nstore,1
/eshape,1.0
/show,thesis2a,grp
nsol,2,1,u,x,PileTop
nsol,3,21,u,x,PileMid
nsol,4,41,u,x,PileToe
/axlab,y,Displacement
/axlab,x,Time
/xrange,0,3.855e-2
/yrange,0,10e-3
plvar,2,3,4
/show,thesis2b,grp
esol,5,1,1,f,x,PileTop
esol,6,20,21,f,x,PileMid
esol,7,40,41,f,x,PileToe
/axlab,y,Element Force
/yrange,-18.096e6,18.096e6
plvar,5,6,7
finish
/xrange,default
/yrange,default
/POST1
/pnum,node,0
/pbc,u,1
/pbc,f,1
/show,thesis2c,grp
eplot
PRITER
FINISH
/EXIT
```

```
CURRENT JOBNAME REDEFINED AS thesis2

TITLE=
Damped Case
```

```
                  VERSION=PC 386/486    21:56:12  OCT 22, 1996 CP=      1.590

Damped Case

**ANSYS VERSION FOR EDUCATIONAL PURPOSES ONLY**

         ***** ANSYS ANALYSIS DEFINITION (PREP7) *****

ELEMENT TYPE    1 IS LINK1        2-D SPAR  ( OR TRUSS )
 KEYOPT(1-12)=    0  0  0    0  0  0    0  0  0    0  0  0

CURRENT NODAL DOF SET IS  UX    UY
 TWO-DIMENSIONAL MODEL

ELEMENT TYPE    2 IS COMBIN14     SPRING-DAMPER
 KEYOPT(1-12)=    0  1  0    0  0  0    0  0  0    0  0  0

CURRENT NODAL DOF SET IS  UX    UY
 TWO-DIMENSIONAL MODEL

REAL CONSTANT SET       1  ITEMS    1 TO    6
  0.12064      0.00000E+00  0.00000E+00  0.00000E+00  0.00000E+00  0.00000E+00

REAL CONSTANT SET       2  ITEMS    1 TO    6
  0.29500E+08   23137.      0.00000E+00  0.00000E+00  0.00000E+00  0.00000E+00

REAL CONSTANT SET       3  ITEMS    1 TO    6
  0.21750E+08  0.26100E+06  0.00000E+00  0.00000E+00  0.00000E+00  0.00000E+00

REAL CONSTANT SET       4  ITEMS    1 TO    6
  0.43500E+08  0.52200E+06  0.00000E+00  0.00000E+00  0.00000E+00  0.00000E+00

MATERIAL    1          EX   =   0.2100000E+12

MATERIAL    1          DENS =    7800.000

NODE       1  KCS=   0  X,Y,Z= 0.00000E+00 0.00000E+00 0.00000E+00

NODE      41  KCS=   0  X,Y,Z=  50.000     0.00000E+00 0.00000E+00

FILL   39 POINTS BETWEEN NODE     1 AND NODE    41
 START WITH NODE     2 AND INCREMENT BY     1

NODE      42  KCS=   0  X,Y,Z=  50.000     0.00000E+00 0.00000E+00

NODE      43  KCS=   0  X,Y,Z= 0.00000E+00 0.00000E+00 0.00000E+00

NODE      83  KCS=   0  X,Y,Z=  50.000     0.00000E+00 0.00000E+00

FILL   39 POINTS BETWEEN NODE    43 AND NODE    83
 START WITH NODE    44 AND INCREMENT BY     1

ELEMENT      1       1      2

GENERATE   40 TOTAL SETS OF ELEMENTS WITH NODE INCREMENT OF     1
  SET IS SELECTED ELEMENTS IN RANGE     1 TO     1 IN STEPS OF     1
MAXIMUM ELEMENT NUMBER=    40

ELEMENT TYPE SET TO      2

REAL CONSTANT NUMBER=      2

ELEMENT     41      41     42

REAL CONSTANT NUMBER=      3

ELEMENT     42       1     43

ELEMENT     43      41     83

REAL CONSTANT NUMBER=      4

ELEMENT     44       2     44
```

```
GENERATE   39 TOTAL SETS OF ELEMENTS WITH NODE INCREMENT OF     1
  SET IS SELECTED ELEMENTS IN RANGE    44 TO    44 IN STEPS OF     1
MAXIMUM ELEMENT NUMBER=    82

LIST ALL SELECTED NODES.   DSYS=  0
```

VERSION=PC 386/486   21:56:13 OCT 22, 1996 CP=   3.350

Damped Case

ANSYS VERSION FOR EDUCATIONAL PURPOSES ONLY

NODE	X	Y	Z	THXY	THYZ	THZX
1	0.00000E+00	0.00000E+00	0.00000E+00	0.00	0.00	0.00
2	1.2500	0.00000E+00	0.00000E+00	0.00	0.00	0.00
3	2.5000	0.00000E+00	0.00000E+00	0.00	0.00	0.00
4	3.7500	0.00000E+00	0.00000E+00	0.00	0.00	0.00
5	5.0000	0.00000E+00	0.00000E+00	0.00	0.00	0.00
6	6.2500	0.00000E+00	0.00000E+00	0.00	0.00	0.00
7	7.5000	0.00000E+00	0.00000E+00	0.00	0.00	0.00
8	8.7500	0.00000E+00	0.00000E+00	0.00	0.00	0.00
9	10.000	0.00000E+00	0.00000E+00	0.00	0.00	0.00
10	11.250	0.00000E+00	0.00000E+00	0.00	0.00	0.00
11	12.500	0.00000E+00	0.00000E+00	0.00	0.00	0.00
12	13.750	0.00000E+00	0.00000E+00	0.00	0.00	0.00
13	15.000	0.00000E+00	0.00000E+00	0.00	0.00	0.00
14	16.250	0.00000E+00	0.00000E+00	0.00	0.00	0.00
15	17.500	0.00000E+00	0.00000E+00	0.00	0.00	0.00
16	18.750	0.00000E+00	0.00000E+00	0.00	0.00	0.00
17	20.000	0.00000E+00	0.00000E+00	0.00	0.00	0.00
18	21.250	0.00000E+00	0.00000E+00	0.00	0.00	0.00
19	22.500	0.00000E+00	0.00000E+00	0.00	0.00	0.00
20	23.750	0.00000E+00	0.00000E+00	0.00	0.00	0.00
21	25.000	0.00000E+00	0.00000E+00	0.00	0.00	0.00
22	26.250	0.00000E+00	0.00000E+00	0.00	0.00	0.00
23	27.500	0.00000E+00	0.00000E+00	0.00	0.00	0.00
24	28.750	0.00000E+00	0.00000E+00	0.00	0.00	0.00
25	30.000	0.00000E+00	0.00000E+00	0.00	0.00	0.00
26	31.250	0.00000E+00	0.00000E+00	0.00	0.00	0.00
27	32.500	0.00000E+00	0.00000E+00	0.00	0.00	0.00
28	33.750	0.00000E+00	0.00000E+00	0.00	0.00	0.00
29	35.000	0.00000E+00	0.00000E+00	0.00	0.00	0.00
30	36.250	0.00000E+00	0.00000E+00	0.00	0.00	0.00
31	37.500	0.00000E+00	0.00000E+00	0.00	0.00	0.00
32	38.750	0.00000E+00	0.00000E+00	0.00	0.00	0.00
33	40.000	0.00000E+00	0.00000E+00	0.00	0.00	0.00
34	41.250	0.00000E+00	0.00000E+00	0.00	0.00	0.00
35	42.500	0.00000E+00	0.00000E+00	0.00	0.00	0.00
36	43.750	0.00000E+00	0.00000E+00	0.00	0.00	0.00
37	45.000	0.00000E+00	0.00000E+00	0.00	0.00	0.00
38	46.250	0.00000E+00	0.00000E+00	0.00	0.00	0.00
39	47.500	0.00000E+00	0.00000E+00	0.00	0.00	0.00
40	48.750	0.00000E+00	0.00000E+00	0.00	0.00	0.00
41	50.000	0.00000E+00	0.00000E+00	0.00	0.00	0.00
42	50.000	0.00000E+00	0.00000E+00	0.00	0.00	0.00
43	0.00000E+00	0.00000E+00	0.00000E+00	0.00	0.00	0.00
44	1.2500	0.00000E+00	0.00000E+00	0.00	0.00	0.00
45	2.5000	0.00000E+00	0.00000E+00	0.00	0.00	0.00
46	3.7500	0.00000E+00	0.00000E+00	0.00	0.00	0.00
47	5.0000	0.00000E+00	0.00000E+00	0.00	0.00	0.00
48	6.2500	0.00000E+00	0.00000E+00	0.00	0.00	0.00
49	7.5000	0.00000E+00	0.00000E+00	0.00	0.00	0.00
50	8.7500	0.00000E+00	0.00000E+00	0.00	0.00	0.00

VERSION=PC 386/486   21:56:14 OCT 22, 1996 CP=   3.460

Damped Case

ANSYS VERSION FOR EDUCATIONAL PURPOSES ONLY

NODE	X	Y	Z	THXY	THYZ	THZX
51	10.000	0.00000E+00	0.00000E+00	0.00	0.00	0.00
52	11.250	0.00000E+00	0.00000E+00	0.00	0.00	0.00
53	12.500	0.00000E+00	0.00000E+00	0.00	0.00	0.00
54	13.750	0.00000E+00	0.00000E+00	0.00	0.00	0.00
55	15.000	0.00000E+00	0.00000E+00	0.00	0.00	0.00
56	16.250	0.00000E+00	0.00000E+00	0.00	0.00	0.00
57	17.500	0.00000E+00	0.00000E+00	0.00	0.00	0.00
58	18.750	0.00000E+00	0.00000E+00	0.00	0.00	0.00
59	20.000	0.00000E+00	0.00000E+00	0.00	0.00	0.00
60	21.250	0.00000E+00	0.00000E+00	0.00	0.00	0.00
61	22.500	0.00000E+00	0.00000E+00	0.00	0.00	0.00
62	23.750	0.00000E+00	0.00000E+00	0.00	0.00	0.00
63	25.000	0.00000E+00	0.00000E+00	0.00	0.00	0.00
64	26.250	0.00000E+00	0.00000E+00	0.00	0.00	0.00
65	27.500	0.00000E+00	0.00000E+00	0.00	0.00	0.00
66	28.750	0.00000E+00	0.00000E+00	0.00	0.00	0.00
67	30.000	0.00000E+00	0.00000E+00	0.00	0.00	0.00
68	31.250	0.00000E+00	0.00000E+00	0.00	0.00	0.00
69	32.500	0.00000E+00	0.00000E+00	0.00	0.00	0.00
70	33.750	0.00000E+00	0.00000E+00	0.00	0.00	0.00
71	35.000	0.00000E+00	0.00000E+00	0.00	0.00	0.00
72	36.250	0.00000E+00	0.00000E+00	0.00	0.00	0.00
73	37.500	0.00000E+00	0.00000E+00	0.00	0.00	0.00
74	38.750	0.00000E+00	0.00000E+00	0.00	0.00	0.00
75	40.000	0.00000E+00	0.00000E+00	0.00	0.00	0.00
76	41.250	0.00000E+00	0.00000E+00	0.00	0.00	0.00
77	42.500	0.00000E+00	0.00000E+00	0.00	0.00	0.00
78	43.750	0.00000E+00	0.00000E+00	0.00	0.00	0.00
79	45.000	0.00000E+00	0.00000E+00	0.00	0.00	0.00
80	46.250	0.00000E+00	0.00000E+00	0.00	0.00	0.00
81	47.500	0.00000E+00	0.00000E+00	0.00	0.00	0.00
82	48.750	0.00000E+00	0.00000E+00	0.00	0.00	0.00
83	50.000	0.00000E+00	0.00000E+00	0.00	0.00	0.00

LIST ALL SELECTED ELEMENTS. (LIST NODES)

VERSION=PC 386/486 21:56:14 OCT 22, 1996 CP= 3.460

Damped Case

ANSYS VERSION FOR EDUCATIONAL PURPOSES ONLY

ELEM	MAT	TYP	REL	ESY	NODES	
1	1	1	1	0	1	2
2	1	1	1	0	2	3
3	1	1	1	0	3	4
4	1	1	1	0	4	5
5	1	1	1	0	5	6
6	1	1	1	0	6	7
7	1	1	1	0	7	8
8	1	1	1	0	8	9
9	1	1	1	0	9	10
10	1	1	1	0	10	11
11	1	1	1	0	11	12
12	1	1	1	0	12	13
13	1	1	1	0	13	14
14	1	1	1	0	14	15
15	1	1	1	0	15	16
16	1	1	1	0	16	17
17	1	1	1	0	17	18
18	1	1	1	0	18	19
19	1	1	1	0	19	20
20	1	1	1	0	20	21
21	1	1	1	0	21	22
22	1	1	1	0	22	23
23	1	1	1	0	23	24
24	1	1	1	0	24	25
25	1	1	1	0	25	26
26	1	1	1	0	26	27
27	1	1	1	0	27	28
28	1	1	1	0	28	29
29	1	1	1	0	29	30
30	1	1	1	0	30	31
31	1	1	1	0	31	32
32	1	1	1	0	32	33
33	1	1	1	0	33	34
34	1	1	1	0	34	35
35	1	1	1	0	35	36
36	1	1	1	0	36	37
37	1	1	1	0	37	38
38	1	1	1	0	38	39
39	1	1	1	0	39	40
40	1	1	1	0	40	41
41	1	2	2	0	41	42
42	1	2	3	0	1	43
43	1	2	3	0	41	83
44	1	2	4	0	2	44
45	1	2	4	0	3	45
46	1	2	4	0	4	46
47	1	2	4	0	5	47
48	1	2	4	0	6	48
49	1	2	4	0	7	49
50	1	2	4	0	8	50

```
***** ANSYS - ENGINEERING ANALYSIS SYSTEM  REVISION 5.0ED56 *****
                  VERSION=PC 386/486    21:56:14  OCT 22, 1996 CP=      3.460

Damped Case

**ANSYS VERSION FOR EDUCATIONAL PURPOSES ONLY**

   ELEM MAT TYP REL ESY        NODES

     51   1   2   4   0      9     51
     52   1   2   4   0     10     52
     53   1   2   4   0     11     53
     54   1   2   4   0     12     54
     55   1   2   4   0     13     55
     56   1   2   4   0     14     56
     57   1   2   4   0     15     57
     58   1   2   4   0     16     58
     59   1   2   4   0     17     59
     60   1   2   4   0     18     60
     61   1   2   4   0     19     61
     62   1   2   4   0     20     62
     63   1   2   4   0     21     63
     64   1   2   4   0     22     64
     65   1   2   4   0     23     65
     66   1   2   4   0     24     66
     67   1   2   4   0     25     67
     68   1   2   4   0     26     68
     69   1   2   4   0     27     69
     70   1   2   4   0     28     70
     71   1   2   4   0     29     71
     72   1   2   4   0     30     72
     73   1   2   4   0     31     73
     74   1   2   4   0     32     74
     75   1   2   4   0     33     75
     76   1   2   4   0     34     76
     77   1   2   4   0     35     77
     78   1   2   4   0     36     78
     79   1   2   4   0     37     79
     80   1   2   4   0     38     80
     81   1   2   4   0     39     81
     82   1   2   4   0     40     82

***** ROUTINE COMPLETED *****  CP =         3.510


*****  ANSYS SOLUTION ROUTINE  *****


PERFORM A TRANSIENT ANALYSIS
 THIS WILL BE A NEW ANALYSIS

SPECIFIED CONSTRAINT UY   FOR SELECTED NODES        1 TO      42 BY       1
REAL= 0.000000000E+00   IMAG= 0.000000000E+00

SPECIFIED CONSTRAINT UX   FOR SELECTED NODES       42 TO      83 BY       1
REAL= 0.000000000E+00   IMAG= 0.000000000E+00

SPECIFIED NODAL LOAD FX   FOR SELECTED NODES        1 TO       1 BY       1
 REAL=  15566000.0       IMAG= 0.000000000E+00

*** NOTE ***                           CP=      3.900   TIME= 21:56:14
Damping (if present) may be ignored only in reduced transient analyses.


PERFORM A FULL TRANSIENT ANALYSIS

USE LUMPED MASS MATRIX APPROXIMATION

USE AUTOMATIC TIME STEPPING THIS LOAD STEP

MASS MATRIX DAMPING MULTIPLIER= 0.10000E-03
```

```
STIFFNESS MATRIX DAMPING MULTIPLIER= 0.10000E-03

STEP BOUNDARY CONDITION KEY= 1

WRITE ALL  ITEMS TO THE DATABASE WITH A FREQUENCY OF ALL
  FOR ALL APPLICABLE ENTITIES

USE INITIAL TIME STEP SIZE OF 0.1205000E-04 FOR ALL  DOFS

TIME= 0.46735E-02

SPECIFIED NODAL LOAD FX   FOR SELECTED NODES        1 TO       1 BY       1
 REAL=  15566000.0       IMAG= 0.000000000E+00

WRITE ANSYS LOADS DATA AS FILE=thesis2.s01

TIME= 0.38550E-01

SPECIFIED NODAL LOAD FX   FOR SELECTED NODES        1 TO       1 BY       1
 REAL= 0.000000000E+00   IMAG= 0.000000000E+00

WRITE ANSYS LOADS DATA AS FILE=thesis2.s02

ALL CURRENT ANSYS DATA WRITTEN TO FILE NAME= thesis2.db
 FOR POSSIBLE RESUME FROM THIS POINT
ANSYS REVISION  5.0             ED56    21:56:15    10/22/1996

PRINTOUT RESUMED BY /GOP

Load step file number 1. Begin solution ...

*****  ANSYS SOLVE    COMMAND  *****
```

```
                  VERSION=PC 386/486    21:56:18  OCT 22, 1996 CP=      8.340

Damped Case

**ANSYS VERSION FOR EDUCATIONAL PURPOSES ONLY**

                      S O L U T I O N   O P T I O N S

  PROBLEM DIMENSIONALITY. . . . . . . . . . . . .2-D
  DEGREES OF FREEDOM. . . . . . UX   UY
  ANALYSIS TYPE . . . . . . . . . . . . . . . . .TRANSIENT
     SOLUTION METHOD. . . . . . . . . . . . . . .FULL
  LUMPED MASS MATRICES. . . . . . . . . . . . . .ON

                     L O A D   S T E P   O P T I O N S

  LOAD STEP NUMBER. . . . . . . . . . . . . . . .     1
  TIME AT END OF THE LOAD STEP. . . . . . . . . . 0.46735E-02
  AUTOMATIC TIME STEPPING . . . . . . . . . . . .    ON
     STARTING TIME STEP SIZE. . . . . . . . . . . 0.12045E-04
     MINIMUM TIME STEP SIZE . . . . . . . . . . . 0.12045E-04
     MAXIMUM TIME STEP SIZE . . . . . . . . . . . 0.46735E-02
  STEP CHANGE BOUNDARY CONDITIONS . . . . . . . .   YES
  TRANSIENT (INERTIA) EFFECTS
     STRUCTURAL DOFS. . . . . . . . . . . . . . .    ON
  TRANSIENT INTEGRATION PARAMETERS
     ALPHA. . . . . . . . . . . . . . . . . . . . 0.25251
     DELTA. . . . . . . . . . . . . . . . . . . . 0.50500
  RAYLEIGH DAMPING MULTIPLIERS
     ALPHA (MASS) . . . . . . . . . . . . . . . . 0.10000E-03
     BETA (STIFFNESS) . . . . . . . . . . . . . . 0.10000E-03
  PRINT OUTPUT CONTROLS . . . . . . . . . . . . .NO PRINTOUT
  DATABASE OUTPUT CONTROLS
     ITEM     FREQUENCY   COMPONENT
      ALL        ALL



           ***** CENTROID, MASS, AND MASS MOMENTS OF INERTIA *****

 CALCULATIONS ASSUME ELEMENT MASS AT ELEMENT CENTROID

 TOTAL MASS =   47050.

                           MOM. OF INERTIA         MOM. OF INERTIA
    CENTROID                ABOUT ORIGIN            ABOUT CENTROID

 XC =   25.000          IXX =   0.0000E+00      IXX =   0.0000E+00
 YC =  0.00000E+00      IYY =   0.3920E+08      IYY =   0.9796E+07
 ZC =  0.00000E+00      IZZ =   0.3920E+08      IZZ =   0.9796E+07
                        IXY =   0.0000E+00      IXY =   0.0000E+00
                        IYZ =   0.0000E+00      IYZ =   0.0000E+00
                        IZX =   0.0000E+00      IZX =   0.0000E+00


 *** MASS SUMMARY BY ELEMENT TYPE ***

 TYPE      MASS
    1   47049.6

Range of element maximum matrix coefficients in global coordinates
Maximum= 2.026752E+10 at element 40.
Minimum= 21750000.0 at element 43.

  *** ELEMENT MATRIX FORMULATION TIMES
 TYPE  NUMBER   ENAME      TOTAL CP  AVE CP

    1      40  LINK1         0.550    0.014
    2      42  COMBIN14      0.440    0.010
Time at end of element matrix formulation CP= 11.4799997.
```

```
Estimated number of active DOF= 41.
Maximum wavefront= 6.

Time at end of matrix triangularization CP= 12.2999997.
Equation solver maximum pivot= 3.290436666E+13 at node 40 UX.
Equation solver minimum pivot= 1.645411767E+13 at node 1 UX.

  *** ELEMENT RESULT CALCULATION TIMES
 TYPE  NUMBER   ENAME      TOTAL CP  AVE CP

    1      40  LINK1         0.210    0.005
    2      42  COMBIN14      0.300    0.007

  *** NODAL LOAD CALCULATION TIMES
 TYPE  NUMBER   ENAME      TOTAL CP  AVE CP

    1      40  LINK1         0.000    0.000
    2      42  COMBIN14      0.050    0.001
*** LOAD STEP     1   SUBSTEP     1  COMPLETED.    CUM ITER =      1
*** TIME =  0.120451E-04     TIME INC =  0.120451E-04  NEW TRIANG MATRIX
*** RESPONSE FREQ =  924.6       PERIOD=  0.1082E-02  PTS/CYC =  90.
*** AUTO STEP TIME:  NEXT TIME INC = 0.12045E-04  UNCHANGED

*** LOAD STEP     1   SUBSTEP     2  COMPLETED.    CUM ITER =      2
*** TIME =  0.240902E-04     TIME INC =  0.120451E-04  OLD TRIANG MATRIX
*** RESPONSE FREQ =  919.3       PERIOD=  0.1088E-02  PTS/CYC =  90.
*** AUTO TIME STEP:  NEXT TIME INC = 0.36135E-04  INCREASED (FACTOR = 3.0000)

*** LOAD STEP     1   SUBSTEP     3  COMPLETED.    CUM ITER =      3
*** TIME =  0.602255E-04     TIME INC =  0.361353E-04  NEW TRIANG MATRIX
*** RESPONSE FREQ =  892.2       PERIOD=  0.1121E-02  PTS/CYC =  31.
*** AUTO STEP TIME:  NEXT TIME INC = 0.36135E-04  UNCHANGED

*** LOAD STEP     1   SUBSTEP     4  COMPLETED.    CUM ITER =      4
*** TIME =  0.963608E-04     TIME INC =  0.361353E-04  OLD TRIANG MATRIX
*** RESPONSE FREQ =  857.6       PERIOD=  0.1166E-02  PTS/CYC =  32.
*** AUTO STEP TIME:  NEXT TIME INC = 0.36135E-04  UNCHANGED

*** LOAD STEP     1   SUBSTEP     5  COMPLETED.    CUM ITER =      5
*** TIME =  0.132496E-03     TIME INC =  0.361353E-04  OLD TRIANG MATRIX
*** RESPONSE FREQ =  813.7       PERIOD=  0.1229E-02  PTS/CYC =  34.
*** AUTO STEP TIME:  NEXT TIME INC = 0.36135E-04  UNCHANGED

*** LOAD STEP     1   SUBSTEP     6  COMPLETED.    CUM ITER =      6
*** TIME =  0.168631E-03     TIME INC =  0.361353E-04  OLD TRIANG MATRIX
*** RESPONSE FREQ =  762.6       PERIOD=  0.1311E-02  PTS/CYC =  36.
*** AUTO STEP TIME:  NEXT TIME INC = 0.36135E-04  UNCHANGED

*** LOAD STEP     1   SUBSTEP     7  COMPLETED.    CUM ITER =      7
*** TIME =  0.204767E-03     TIME INC =  0.361353E-04  OLD TRIANG MATRIX
*** RESPONSE FREQ =  705.9       PERIOD=  0.1417E-02  PTS/CYC =  39.
*** AUTO STEP TIME:  NEXT TIME INC = 0.36135E-04  UNCHANGED

*** LOAD STEP     1   SUBSTEP     8  COMPLETED.    CUM ITER =      8
*** TIME =  0.240902E-03     TIME INC =  0.361353E-04  OLD TRIANG MATRIX
*** RESPONSE FREQ =  645.4       PERIOD=  0.1549E-02  PTS/CYC =  43.
*** AUTO TIME STEP:  NEXT TIME INC = 0.77469E-04  INCREASED (FACTOR = 2.1439)

*** LOAD STEP     1   SUBSTEP     9  COMPLETED.    CUM ITER =      9
*** TIME =  0.318371E-03     TIME INC =  0.774694E-04  NEW TRIANG MATRIX
*** RESPONSE FREQ =  547.0       PERIOD=  0.1828E-02  PTS/CYC =  24.
*** AUTO STEP TIME:  NEXT TIME INC = 0.77469E-04  UNCHANGED

*** LOAD STEP     1   SUBSTEP    10  COMPLETED.    CUM ITER =     10
*** TIME =  0.395841E-03     TIME INC =  0.774694E-04  OLD TRIANG MATRIX
*** RESPONSE FREQ =  434.3       PERIOD=  0.2302E-02  PTS/CYC =  30.
*** AUTO STEP TIME:  NEXT TIME INC = 0.77469E-04  UNCHANGED

*** LOAD STEP     1   SUBSTEP    11  COMPLETED.    CUM ITER =     11
*** TIME =  0.473310E-03     TIME INC =  0.774694E-04  OLD TRIANG MATRIX
*** RESPONSE FREQ =  353.6       PERIOD=  0.2828E-02  PTS/CYC =  37.
*** AUTO STEP TIME:  NEXT TIME INC = 0.77469E-04  UNCHANGED

*** LOAD STEP     1   SUBSTEP    12  COMPLETED.    CUM ITER =     12
*** TIME =  0.550780E-03     TIME INC =  0.774694E-04  OLD TRIANG MATRIX
```

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*** RESPONSE FREQ =  304.1       PERIOD=  0.3288E-02  PTS/CYC =  42.
*** AUTO TIME STEP:  NEXT TIME INC = 0.16442E-03  INCREASED (FACTOR = 2.1224)

*** LOAD STEP     1   SUBSTEP    13  COMPLETED.    CUM ITER =     13
*** TIME =  0.715202E-03     TIME INC =  0.164422E-03  NEW TRIANG MATRIX
*** RESPONSE FREQ =  254.4       PERIOD=  0.3931E-02  PTS/CYC =  24.
*** AUTO STEP TIME:  NEXT TIME INC = 0.16442E-03  UNCHANGED

*** LOAD STEP     1   SUBSTEP    14  COMPLETED.    CUM ITER =     14
*** TIME =  0.879624E-03     TIME INC =  0.164422E-03  OLD TRIANG MATRIX
*** RESPONSE FREQ =  211.2       PERIOD=  0.4736E-02  PTS/CYC =  29.
*** AUTO STEP TIME:  NEXT TIME INC = 0.16442E-03  UNCHANGED

*** LOAD STEP     1   SUBSTEP    15  COMPLETED.    CUM ITER =     15
*** TIME =  0.104405E-02     TIME INC =  0.164422E-03  OLD TRIANG MATRIX
*** RESPONSE FREQ =  180.6       PERIOD=  0.5537E-02  PTS/CYC =  34.
*** AUTO STEP TIME:  NEXT TIME INC = 0.16442E-03  UNCHANGED

*** LOAD STEP     1   SUBSTEP    16  COMPLETED.    CUM ITER =     16
*** TIME =  0.120847E-02     TIME INC =  0.164422E-03  OLD TRIANG MATRIX
*** RESPONSE FREQ =  159.5       PERIOD=  0.6270E-02  PTS/CYC =  38.
*** AUTO STEP TIME:  NEXT TIME INC = 0.16442E-03  UNCHANGED

*** LOAD STEP     1   SUBSTEP    17  COMPLETED.    CUM ITER =     17
*** TIME =  0.137289E-02     TIME INC =  0.164422E-03  OLD TRIANG MATRIX
*** RESPONSE FREQ =  143.2       PERIOD=  0.6982E-02  PTS/CYC =  42.
*** AUTO TIME STEP:  NEXT TIME INC = 0.34909E-03  INCREASED (FACTOR = 2.1231)

*** LOAD STEP     1   SUBSTEP    18  COMPLETED.    CUM ITER =     18
*** TIME =  0.172198E-02     TIME INC =  0.349092E-03  NEW TRIANG MATRIX
*** RESPONSE FREQ =  122.2       PERIOD=  0.8183E-02  PTS/CYC =  23.
*** AUTO STEP TIME:  NEXT TIME INC = 0.34909E-03  UNCHANGED

*** LOAD STEP     1   SUBSTEP    19  COMPLETED.    CUM ITER =     19
*** TIME =  0.207107E-02     TIME INC =  0.349092E-03  OLD TRIANG MATRIX
*** RESPONSE FREQ =  106.0       PERIOD=  0.9430E-02  PTS/CYC =  27.
*** AUTO STEP TIME:  NEXT TIME INC = 0.34909E-03  UNCHANGED

*** LOAD STEP     1   SUBSTEP    20  COMPLETED.    CUM ITER =     20
*** TIME =  0.242017E-02     TIME INC =  0.349092E-03  OLD TRIANG MATRIX
*** RESPONSE FREQ =  94.42       PERIOD=  0.1059E-01  PTS/CYC =  30.
*** AUTO STEP TIME:  NEXT TIME INC = 0.34909E-03  UNCHANGED

*** LOAD STEP     1   SUBSTEP    21  COMPLETED.    CUM ITER =     21
*** TIME =  0.276926E-02     TIME INC =  0.349092E-03  OLD TRIANG MATRIX
*** RESPONSE FREQ =  85.60       PERIOD=  0.1168E-01  PTS/CYC =  33.
*** AUTO STEP TIME:  NEXT TIME INC = 0.34909E-03  UNCHANGED

*** LOAD STEP     1   SUBSTEP    22  COMPLETED.    CUM ITER =     22
*** TIME =  0.311835E-02     TIME INC =  0.349092E-03  OLD TRIANG MATRIX
*** RESPONSE FREQ =  78.69       PERIOD=  0.1271E-01  PTS/CYC =  36.
*** AUTO STEP TIME:  NEXT TIME INC = 0.34909E-03  UNCHANGED

*** LOAD STEP     1   SUBSTEP    23  COMPLETED.    CUM ITER =     23
*** TIME =  0.346744E-02     TIME INC =  0.349092E-03  OLD TRIANG MATRIX
*** RESPONSE FREQ =  73.13       PERIOD=  0.1368E-01  PTS/CYC =  39.
*** AUTO STEP TIME:  NEXT TIME INC = 0.34909E-03  UNCHANGED

*** LOAD STEP     1   SUBSTEP    24  COMPLETED.    CUM ITER =     24
*** TIME =  0.381653E-02     TIME INC =  0.349092E-03  OLD TRIANG MATRIX
*** RESPONSE FREQ =  68.55       PERIOD=  0.1459E-01  PTS/CYC =  42.
*** AUTO TIME STEP:  NEXT TIME INC = 0.72941E-03  INCREASED (FACTOR = 2.0895)

*** LOAD STEP     1   SUBSTEP    25  COMPLETED.    CUM ITER =     25
*** TIME =  0.454595E-02     TIME INC =  0.729414E-03  NEW TRIANG MATRIX
*** RESPONSE FREQ =  61.37       PERIOD=  0.1630E-01  PTS/CYC =  22.
*** AUTO TIME STEP:  NEXT TIME INC = 0.12755E-03  DECREASED (FACTOR = 0.1749)

*** LOAD STEP     1   SUBSTEP    26  COMPLETED.    CUM ITER =     26
*** TIME =  0.467350E-02     TIME INC =  0.127552E-03  NEW TRIANG MATRIX
*** RESPONSE FREQ =  60.85       PERIOD=  0.1644E-01  PTS/CYC = 0.13E+03


  *** PROBLEM STATISTICS
 ACTUAL NO. OF ACTIVE DEGREES OF FREEDOM =     41
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 R.M.S. WAVEFRONT SIZE =     3.6


*** ANSYS BINARY FILE STATISTICS
 BUFFER SIZE USED=  4096
       0.031 MB WRITTEN ON ELEMENT MATRIX FILE: thesis2.ema
       0.031 MB WRITTEN ON ELEMENT SAVED DATA FILE: thesis2.esa
       0.016 MB WRITTEN ON TRIANGULARIZED MATRIX FILE: thesis2.tri
       0.844 MB WRITTEN ON RESULTS FILE: thesis2.rst
ANSYS REVISION  5.0             ED56    21:56:15    10/22/1996

PRINTOUT RESUMED BY /GOP

Load step file number 2. Begin solution ...

*****  ANSYS SOLVE    COMMAND  *****
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                  VERSION=PC 386/486    21:57:22  OCT 22, 1996 CP=     72.170

Damped Case

**ANSYS VERSION FOR EDUCATIONAL PURPOSES ONLY**

                     L O A D   S T E P   O P T I O N S

  LOAD STEP NUMBER. . . . . . . . . . . . . . . .      2
  TIME AT END OF THE LOAD STEP. . . . . . . . . . 0.38550E-01
  AUTOMATIC TIME STEPPING . . . . . . . . . . . .     ON
     STARTING TIME STEP SIZE. . . . . . . . . . . 0.12051E-04
     MINIMUM TIME STEP SIZE . . . . . . . . . . . 0.12051E-04
     MAXIMUM TIME STEP SIZE . . . . . . . . . . . 0.33877E-01
  STEP CHANGE BOUNDARY CONDITIONS . . . . . . . .    YES
  TRANSIENT (INERTIA) EFFECTS
     STRUCTURAL DOFS. . . . . . . . . . . . . . .     ON
  TRANSIENT INTEGRATION PARAMETERS
     ALPHA. . . . . . . . . . . . . . . . . . . . 0.25251
     DELTA. . . . . . . . . . . . . . . . . . . . 0.50500
  RAYLEIGH DAMPING MULTIPLIERS
     ALPHA (MASS) . . . . . . . . . . . . . . . . 0.10000E-03
     BETA (STIFFNESS) . . . . . . . . . . . . . . 0.10000E-03
  PRINT OUTPUT CONTROLS . . . . . . . . . . . . .NO PRINTOUT
  DATABASE OUTPUT CONTROLS
     ITEM     FREQUENCY   COMPONENT
      ALL        ALL


*** LOAD STEP     2   SUBSTEP     1  COMPLETED.    CUM ITER =     27
*** TIME =  0.468555E-02     TIME INC =  0.120514E-04  NEW TRIANG MATRIX
*** RESPONSE FREQ =  61.28       PERIOD=  0.1632E-01  PTS/CYC = 0.14E+04
*** AUTO STEP TIME:  NEXT TIME INC = 0.12051E-04  UNCHANGED

*** LOAD STEP     2   SUBSTEP     2  COMPLETED.    CUM ITER =     28
*** TIME =  0.469760E-02     TIME INC =  0.120514E-04  OLD TRIANG MATRIX
*** RESPONSE FREQ =  72.69       PERIOD=  0.1376E-01  PTS/CYC = 0.11E+04
*** AUTO TIME STEP:  NEXT TIME INC = 0.36154E-04  INCREASED (FACTOR = 3.0000)

*** LOAD STEP     2   SUBSTEP     3  COMPLETED.    CUM ITER =     29
*** TIME =  0.473376E-02     TIME INC =  0.361542E-04  NEW TRIANG MATRIX
*** RESPONSE FREQ =  126.7       PERIOD=  0.7890E-02  PTS/CYC = 0.22E+03
*** AUTO TIME STEP:  NEXT TIME INC = 0.10846E-03  INCREASED (FACTOR = 3.0000)

*** LOAD STEP     2   SUBSTEP     4  COMPLETED.    CUM ITER =     30
*** TIME =  0.484222E-02     TIME INC =  0.108463E-03  NEW TRIANG MATRIX
*** RESPONSE FREQ =  250.9       PERIOD=  0.3985E-02  PTS/CYC =  37.
*** AUTO STEP TIME:  NEXT TIME INC = 0.10846E-03  UNCHANGED

*** LOAD STEP     2   SUBSTEP     5  COMPLETED.    CUM ITER =     31
*** TIME =  0.495068E-02     TIME INC =  0.108463E-03  OLD TRIANG MATRIX
*** RESPONSE FREQ =  326.7       PERIOD=  0.3061E-02  PTS/CYC =  28.
*** AUTO STEP TIME:  NEXT TIME INC = 0.10846E-03  UNCHANGED

*** LOAD STEP     2   SUBSTEP     6  COMPLETED.    CUM ITER =     32
*** TIME =  0.505914E-02     TIME INC =  0.108463E-03  OLD TRIANG MATRIX
*** RESPONSE FREQ =  299.1       PERIOD=  0.3344E-02  PTS/CYC =  31.
*** AUTO STEP TIME:  NEXT TIME INC = 0.10846E-03  UNCHANGED

*** LOAD STEP     2   SUBSTEP     7  COMPLETED.    CUM ITER =     33
*** TIME =  0.516761E-02     TIME INC =  0.108463E-03  OLD TRIANG MATRIX
*** RESPONSE FREQ =  255.1       PERIOD=  0.3921E-02  PTS/CYC =  36.
*** AUTO STEP TIME:  NEXT TIME INC = 0.10846E-03  UNCHANGED

*** LOAD STEP     2   SUBSTEP     8  COMPLETED.    CUM ITER =     34
*** TIME =  0.527607E-02     TIME INC =  0.108463E-03  OLD TRIANG MATRIX
*** RESPONSE FREQ =  232.3       PERIOD=  0.4304E-02  PTS/CYC =  40.
*** AUTO STEP TIME:  NEXT TIME INC = 0.10846E-03  UNCHANGED

*** LOAD STEP     2   SUBSTEP     9  COMPLETED.    CUM ITER =     35
*** TIME =  0.538453E-02     TIME INC =  0.108463E-03  OLD TRIANG MATRIX
*** RESPONSE FREQ =  219.5       PERIOD=  0.4555E-02  PTS/CYC =  42.
*** AUTO TIME STEP:  NEXT TIME INC = 0.22775E-03  INCREASED (FACTOR = 2.0998)
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*** LOAD STEP     2   SUBSTEP    10  COMPLETED.    CUM ITER =     36
*** TIME =  0.561229E-02     TIME INC =  0.227754E-03  NEW TRIANG MATRIX
*** RESPONSE FREQ =  196.1       PERIOD=  0.5100E-02  PTS/CYC =  22.
*** AUTO STEP TIME:  NEXT TIME INC = 0.22775E-03  UNCHANGED

*** LOAD STEP     2   SUBSTEP    11  COMPLETED.    CUM ITER =     37
*** TIME =  0.584004E-02     TIME INC =  0.227754E-03  OLD TRIANG MATRIX
*** RESPONSE FREQ =  177.4       PERIOD=  0.5636E-02  PTS/CYC =  25.
*** AUTO STEP TIME:  NEXT TIME INC = 0.22775E-03  UNCHANGED

*** LOAD STEP     2   SUBSTEP    12  COMPLETED.    CUM ITER =     38
*** TIME =  0.606780E-02     TIME INC =  0.227754E-03  OLD TRIANG MATRIX
*** RESPONSE FREQ =  163.4       PERIOD=  0.6121E-02  PTS/CYC =  27.
*** AUTO STEP TIME:  NEXT TIME INC = 0.22775E-03  UNCHANGED

*** LOAD STEP     2   SUBSTEP    13  COMPLETED.    CUM ITER =     39
*** TIME =  0.629555E-02     TIME INC =  0.227754E-03  OLD TRIANG MATRIX
*** RESPONSE FREQ =  151.6       PERIOD=  0.6595E-02  PTS/CYC =  29.
*** AUTO STEP TIME:  NEXT TIME INC = 0.22775E-03  UNCHANGED

*** LOAD STEP     2   SUBSTEP    14  COMPLETED.    CUM ITER =     40
*** TIME =  0.652330E-02     TIME INC =  0.227754E-03  OLD TRIANG MATRIX
*** RESPONSE FREQ =  142.0       PERIOD=  0.7044E-02  PTS/CYC =  31.
*** AUTO STEP TIME:  NEXT TIME INC = 0.22775E-03  UNCHANGED

*** LOAD STEP     2   SUBSTEP    15  COMPLETED.    CUM ITER =     41
*** TIME =  0.675106E-02     TIME INC =  0.227754E-03  OLD TRIANG MATRIX
*** RESPONSE FREQ =  133.7       PERIOD=  0.7479E-02  PTS/CYC =  33.
*** AUTO STEP TIME:  NEXT TIME INC = 0.22775E-03  UNCHANGED

*** LOAD STEP     2   SUBSTEP    16  COMPLETED.    CUM ITER =     42
*** TIME =  0.697881E-02     TIME INC =  0.227754E-03  OLD TRIANG MATRIX
*** RESPONSE FREQ =  126.6       PERIOD=  0.7898E-02  PTS/CYC =  35.
*** AUTO STEP TIME:  NEXT TIME INC = 0.22775E-03  UNCHANGED

*** LOAD STEP     2   SUBSTEP    17  COMPLETED.    CUM ITER =     43
*** TIME =  0.720657E-02     TIME INC =  0.227754E-03  OLD TRIANG MATRIX
*** RESPONSE FREQ =  120.4       PERIOD=  0.8304E-02  PTS/CYC =  36.
*** AUTO STEP TIME:  NEXT TIME INC = 0.22775E-03  UNCHANGED

*** LOAD STEP     2   SUBSTEP    18  COMPLETED.    CUM ITER =     44
*** TIME =  0.743432E-02     TIME INC =  0.227754E-03  OLD TRIANG MATRIX
*** RESPONSE FREQ =  115.0       PERIOD=  0.8698E-02  PTS/CYC =  38.
*** AUTO STEP TIME:  NEXT TIME INC = 0.22775E-03  UNCHANGED

*** LOAD STEP     2   SUBSTEP    19  COMPLETED.    CUM ITER =     45
*** TIME =  0.766208E-02     TIME INC =  0.227754E-03  OLD TRIANG MATRIX
*** RESPONSE FREQ =  110.1       PERIOD=  0.9082E-02  PTS/CYC =  40.
*** AUTO STEP TIME:  NEXT TIME INC = 0.22775E-03  UNCHANGED

*** LOAD STEP     2   SUBSTEP    20  COMPLETED.    CUM ITER =     46
*** TIME =  0.788983E-02     TIME INC =  0.227754E-03  OLD TRIANG MATRIX
*** RESPONSE FREQ =  105.7       PERIOD=  0.9457E-02  PTS/CYC =  42.
*** AUTO TIME STEP:  NEXT TIME INC = 0.47284E-03  INCREASED (FACTOR = 2.0761)

*** LOAD STEP     2   SUBSTEP    21  COMPLETED.    CUM ITER =     47
*** TIME =  0.836267E-02     TIME INC =  0.472844E-03  NEW TRIANG MATRIX
*** RESPONSE FREQ =  98.35       PERIOD=  0.1017E-01  PTS/CYC =  22.
*** AUTO STEP TIME:  NEXT TIME INC = 0.47284E-03  UNCHANGED

*** LOAD STEP     2   SUBSTEP    22  COMPLETED.    CUM ITER =     48
*** TIME =  0.883552E-02     TIME INC =  0.472844E-03  OLD TRIANG MATRIX
*** RESPONSE FREQ =  91.85       PERIOD=  0.1089E-01  PTS/CYC =  23.
*** AUTO STEP TIME:  NEXT TIME INC = 0.47284E-03  UNCHANGED

*** LOAD STEP     2   SUBSTEP    23  COMPLETED.    CUM ITER =     49
*** TIME =  0.930836E-02     TIME INC =  0.472844E-03  OLD TRIANG MATRIX
*** RESPONSE FREQ =  85.59       PERIOD=  0.1168E-01  PTS/CYC =  25.
*** AUTO STEP TIME:  NEXT TIME INC = 0.47284E-03  UNCHANGED

*** LOAD STEP     2   SUBSTEP    24  COMPLETED.    CUM ITER =     50
*** TIME =  0.978121E-02     TIME INC =  0.472844E-03  OLD TRIANG MATRIX
*** RESPONSE FREQ =  79.57       PERIOD=  0.1257E-01  PTS/CYC =  27.
*** AUTO STEP TIME:  NEXT TIME INC = 0.47284E-03  UNCHANGED
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*** LOAD STEP     2   SUBSTEP    25  COMPLETED.    CUM ITER =     51
*** TIME =  0.102541E-01     TIME INC =  0.472844E-03  OLD TRIANG MATRIX
*** RESPONSE FREQ =  74.56       PERIOD=  0.1341E-01  PTS/CYC =  28.
*** AUTO STEP TIME:  NEXT TIME INC = 0.47284E-03  UNCHANGED

*** LOAD STEP     2   SUBSTEP    26  COMPLETED.    CUM ITER =     52
*** TIME =  0.107269E-01     TIME INC =  0.472844E-03  OLD TRIANG MATRIX
*** RESPONSE FREQ =  71.55       PERIOD=  0.1398E-01  PTS/CYC =  30.
*** AUTO STEP TIME:  NEXT TIME INC = 0.47284E-03  UNCHANGED

*** LOAD STEP     2   SUBSTEP    27  COMPLETED.    CUM ITER =     53
*** TIME =  0.111997E-01     TIME INC =  0.472844E-03  OLD TRIANG MATRIX
*** RESPONSE FREQ =  70.85       PERIOD=  0.1411E-01  PTS/CYC =  30.
*** AUTO STEP TIME:  NEXT TIME INC = 0.47284E-03  UNCHANGED

*** LOAD STEP     2   SUBSTEP    28  COMPLETED.    CUM ITER =     54
*** TIME =  0.116726E-01     TIME INC =  0.472844E-03  OLD TRIANG MATRIX
*** RESPONSE FREQ =  72.19       PERIOD=  0.1385E-01  PTS/CYC =  29.
*** AUTO STEP TIME:  NEXT TIME INC = 0.47284E-03  UNCHANGED

*** LOAD STEP     2   SUBSTEP    29  COMPLETED.    CUM ITER =     55
*** TIME =  0.121454E-01     TIME INC =  0.472844E-03  OLD TRIANG MATRIX
*** RESPONSE FREQ =  74.56       PERIOD=  0.1341E-01  PTS/CYC =  28.
*** AUTO STEP TIME:  NEXT TIME INC = 0.47284E-03  UNCHANGED

*** LOAD STEP     2   SUBSTEP    30  COMPLETED.    CUM ITER =     56
*** TIME =  0.126183E-01     TIME INC =  0.472844E-03  OLD TRIANG MATRIX
*** RESPONSE FREQ =  75.86       PERIOD=  0.1318E-01  PTS/CYC =  28.
*** AUTO STEP TIME:  NEXT TIME INC = 0.47284E-03  UNCHANGED

*** LOAD STEP     2   SUBSTEP    31  COMPLETED.    CUM ITER =     57
*** TIME =  0.130911E-01     TIME INC =  0.472844E-03  OLD TRIANG MATRIX
*** RESPONSE FREQ =  73.92       PERIOD=  0.1353E-01  PTS/CYC =  29.
*** AUTO STEP TIME:  NEXT TIME INC = 0.47284E-03  UNCHANGED

*** LOAD STEP     2   SUBSTEP    32  COMPLETED.    CUM ITER =     58
*** TIME =  0.135640E-01     TIME INC =  0.472844E-03  OLD TRIANG MATRIX
*** RESPONSE FREQ =  68.13       PERIOD=  0.1468E-01  PTS/CYC =  31.
*** AUTO STEP TIME:  NEXT TIME INC = 0.47284E-03  UNCHANGED

*** LOAD STEP     2   SUBSTEP    33  COMPLETED.    CUM ITER =     59
*** TIME =  0.140368E-01     TIME INC =  0.472844E-03  OLD TRIANG MATRIX
*** RESPONSE FREQ =  59.22       PERIOD=  0.1689E-01  PTS/CYC =  36.
*** AUTO STEP TIME:  NEXT TIME INC = 0.47284E-03  UNCHANGED

*** LOAD STEP     2   SUBSTEP    34  COMPLETED.    CUM ITER =     60
*** TIME =  0.145096E-01     TIME INC =  0.472844E-03  OLD TRIANG MATRIX
*** RESPONSE FREQ =  48.09       PERIOD=  0.2080E-01  PTS/CYC =  44.
*** AUTO TIME STEP:  NEXT TIME INC = 0.10398E-02  INCREASED (FACTOR = 2.1990)

*** LOAD STEP     2   SUBSTEP    35  COMPLETED.    CUM ITER =     61
*** TIME =  0.155494E-01     TIME INC =  0.103979E-02  NEW TRIANG MATRIX
*** RESPONSE FREQ =  40.22       PERIOD=  0.2486E-01  PTS/CYC =  24.
*** AUTO STEP TIME:  NEXT TIME INC = 0.10398E-02  UNCHANGED

*** LOAD STEP     2   SUBSTEP    36  COMPLETED.    CUM ITER =     62
*** TIME =  0.165892E-01     TIME INC =  0.103979E-02  OLD TRIANG MATRIX
*** RESPONSE FREQ =  53.65       PERIOD=  0.1864E-01  PTS/CYC =  18.
*** AUTO STEP TIME:  NEXT TIME INC = 0.10398E-02  UNCHANGED

*** LOAD STEP     2   SUBSTEP    37  COMPLETED.    CUM ITER =     63
*** TIME =  0.176290E-01     TIME INC =  0.103979E-02  OLD TRIANG MATRIX
*** RESPONSE FREQ =  49.69       PERIOD=  0.2012E-01  PTS/CYC =  19.
*** AUTO STEP TIME:  NEXT TIME INC = 0.10398E-02  UNCHANGED

*** LOAD STEP     2   SUBSTEP    38  COMPLETED.    CUM ITER =     64
*** TIME =  0.186688E-01     TIME INC =  0.103979E-02  OLD TRIANG MATRIX
*** RESPONSE FREQ =  44.87       PERIOD=  0.2229E-01  PTS/CYC =  21.
*** AUTO STEP TIME:  NEXT TIME INC = 0.10398E-02  UNCHANGED

*** LOAD STEP     2   SUBSTEP    39  COMPLETED.    CUM ITER =     65
*** TIME =  0.197086E-01     TIME INC =  0.103979E-02  OLD TRIANG MATRIX
*** RESPONSE FREQ =  41.74       PERIOD=  0.2396E-01  PTS/CYC =  23.
*** AUTO STEP TIME:  NEXT TIME INC = 0.10398E-02  UNCHANGED
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*** LOAD STEP     2   SUBSTEP    40  COMPLETED.    CUM ITER =     66
*** TIME =  0.207484E-01     TIME INC =  0.103979E-02  OLD TRIANG MATRIX
*** RESPONSE FREQ =  39.95       PERIOD=  0.2503E-01  PTS/CYC =  24.
*** AUTO STEP TIME:  NEXT TIME INC = 0.10398E-02  UNCHANGED

*** LOAD STEP     2   SUBSTEP    41  COMPLETED.    CUM ITER =     67
*** TIME =  0.217882E-01     TIME INC =  0.103979E-02  OLD TRIANG MATRIX
*** RESPONSE FREQ =  40.22       PERIOD=  0.2487E-01  PTS/CYC =  24.
*** AUTO STEP TIME:  NEXT TIME INC = 0.10398E-02  UNCHANGED

*** LOAD STEP     2   SUBSTEP    42  COMPLETED.    CUM ITER =     68
*** TIME =  0.228280E-01     TIME INC =  0.103979E-02  OLD TRIANG MATRIX
*** RESPONSE FREQ =  40.60       PERIOD=  0.2463E-01  PTS/CYC =  24.
*** AUTO STEP TIME:  NEXT TIME INC = 0.10398E-02  UNCHANGED

*** LOAD STEP     2   SUBSTEP    43  COMPLETED.    CUM ITER =     69
*** TIME =  0.238678E-01     TIME INC =  0.103979E-02  OLD TRIANG MATRIX
*** RESPONSE FREQ =  37.86       PERIOD=  0.2641E-01  PTS/CYC =  25.
*** AUTO STEP TIME:  NEXT TIME INC = 0.10398E-02  UNCHANGED

*** LOAD STEP     2   SUBSTEP    44  COMPLETED.    CUM ITER =     70
*** TIME =  0.249076E-01     TIME INC =  0.103979E-02  OLD TRIANG MATRIX
*** RESPONSE FREQ =  33.18       PERIOD=  0.3014E-01  PTS/CYC =  29.
*** AUTO STEP TIME:  NEXT TIME INC = 0.10398E-02  UNCHANGED

*** LOAD STEP     2   SUBSTEP    45  COMPLETED.    CUM ITER =     71
*** TIME =  0.259473E-01     TIME INC =  0.103979E-02  OLD TRIANG MATRIX
*** RESPONSE FREQ =  32.00       PERIOD=  0.3125E-01  PTS/CYC =  30.
*** AUTO STEP TIME:  NEXT TIME INC = 0.10398E-02  UNCHANGED

*** LOAD STEP     2   SUBSTEP    46  COMPLETED.    CUM ITER =     72
*** TIME =  0.269871E-01     TIME INC =  0.103979E-02  OLD TRIANG MATRIX
*** RESPONSE FREQ =  33.60       PERIOD=  0.2977E-01  PTS/CYC =  29.
*** AUTO STEP TIME:  NEXT TIME INC = 0.10398E-02  UNCHANGED

*** LOAD STEP     2   SUBSTEP    47  COMPLETED.    CUM ITER =     73
*** TIME =  0.280269E-01     TIME INC =  0.103979E-02  OLD TRIANG MATRIX
*** RESPONSE FREQ =  33.37       PERIOD=  0.2996E-01  PTS/CYC =  29.
*** AUTO STEP TIME:  NEXT TIME INC = 0.10398E-02  UNCHANGED

*** LOAD STEP     2   SUBSTEP    48  COMPLETED.    CUM ITER =     74
*** TIME =  0.290667E-01     TIME INC =  0.103979E-02  OLD TRIANG MATRIX
*** RESPONSE FREQ =  32.20       PERIOD=  0.3106E-01  PTS/CYC =  30.
*** AUTO STEP TIME:  NEXT TIME INC = 0.10398E-02  UNCHANGED

*** LOAD STEP     2   SUBSTEP    49  COMPLETED.    CUM ITER =     75
*** TIME =  0.301065E-01     TIME INC =  0.103979E-02  OLD TRIANG MATRIX
*** RESPONSE FREQ =  32.06       PERIOD=  0.3119E-01  PTS/CYC =  30.
*** AUTO STEP TIME:  NEXT TIME INC = 0.10398E-02  UNCHANGED

*** LOAD STEP     2   SUBSTEP    50  COMPLETED.    CUM ITER =     76
*** TIME =  0.311463E-01     TIME INC =  0.103979E-02  OLD TRIANG MATRIX
*** RESPONSE FREQ =  32.11       PERIOD=  0.3115E-01  PTS/CYC =  30.
*** AUTO STEP TIME:  NEXT TIME INC = 0.10398E-02  UNCHANGED

*** LOAD STEP     2   SUBSTEP    51  COMPLETED.    CUM ITER =     77
*** TIME =  0.321861E-01     TIME INC =  0.103979E-02  OLD TRIANG MATRIX
*** RESPONSE FREQ =  32.01       PERIOD=  0.3124E-01  PTS/CYC =  30.
*** AUTO STEP TIME:  NEXT TIME INC = 0.10398E-02  UNCHANGED

*** LOAD STEP     2   SUBSTEP    52  COMPLETED.    CUM ITER =     78
*** TIME =  0.332259E-01     TIME INC =  0.103979E-02  OLD TRIANG MATRIX
*** RESPONSE FREQ =  32.07       PERIOD=  0.3119E-01  PTS/CYC =  30.
*** AUTO STEP TIME:  NEXT TIME INC = 0.10398E-02  UNCHANGED

*** LOAD STEP     2   SUBSTEP    53  COMPLETED.    CUM ITER =     79
*** TIME =  0.342657E-01     TIME INC =  0.103979E-02  OLD TRIANG MATRIX
*** RESPONSE FREQ =  31.54       PERIOD=  0.3171E-01  PTS/CYC =  30.
*** AUTO STEP TIME:  NEXT TIME INC = 0.10398E-02  UNCHANGED

*** LOAD STEP     2   SUBSTEP    54  COMPLETED.    CUM ITER =     80
*** TIME =  0.353055E-01     TIME INC =  0.103979E-02  OLD TRIANG MATRIX
*** RESPONSE FREQ =  31.10       PERIOD=  0.3216E-01  PTS/CYC =  31.
*** AUTO STEP TIME:  NEXT TIME INC = 0.10398E-02  UNCHANGED
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```
*** LOAD STEP     2   SUBSTEP    55  COMPLETED.    CUM ITER =     81
*** TIME =  0.363452E-01     TIME INC =  0.103979E-02  OLD TRIANG MATRIX
*** RESPONSE FREQ =  31.00       PERIOD=  0.3226E-01  PTS/CYC =  31.
*** AUTO STEP TIME:  NEXT TIME INC = 0.10398E-02  UNCHANGED

*** LOAD STEP     2   SUBSTEP    56  COMPLETED.    CUM ITER =     82
*** TIME =  0.373850E-01     TIME INC =  0.103979E-02  OLD TRIANG MATRIX
*** RESPONSE FREQ =  31.18       PERIOD=  0.3207E-01  PTS/CYC =  31.
*** AUTO TIME STEP:  NEXT TIME INC = 0.58248E-03  DECREASED (FACTOR = 0.5602)

*** LOAD STEP     2   SUBSTEP    57  COMPLETED.    CUM ITER =     83
*** TIME =  0.379675E-01     TIME INC =  0.582482E-03  NEW TRIANG MATRIX
*** RESPONSE FREQ =  31.19       PERIOD=  0.3207E-01  PTS/CYC =  55.
*** AUTO STEP TIME:  NEXT TIME INC = 0.58248E-03  UNCHANGED

*** LOAD STEP     2   SUBSTEP    58  COMPLETED.    CUM ITER =     84
*** TIME =  0.385500E-01     TIME INC =  0.582482E-03  OLD TRIANG MATRIX
*** RESPONSE FREQ =  31.07       PERIOD=  0.3218E-01  PTS/CYC =  55.

FINISH SOLUTION PROCESSING


***** ROUTINE COMPLETED *****  CP =       213.820
```

```
                 VERSION=PC 386/486    21:59:44  OCT 22, 1996 CP=    214.150

Damped Case

**ANSYS VERSION FOR EDUCATIONAL PURPOSES ONLY**


         ***** TIME-HISTORY POSTPROCESSOR (POST26) *****

ALL POST26 SPECIFICATIONS ARE RESET TO INITIAL DEFAULTS

INCLUDE ALL TIME POINTS IN RANGE FOR STORAGE

DATA STORAGE INCREMENT=     1

ELEMENT DISPLAYS USING REAL CONSTANT DATA WITH FACTOR     1.00
DISPLAY TYPE DEFAULT SET TO FACE HIDDEN

/SHOW SWITCH PLOTS TO FILE THESIS2A.grp      - RASTER MODE.

VARIABLE  2  IS AT NODE     1  ITEM= U    COMP= X     NAME= PileTop

VARIABLE  3  IS AT NODE    21  ITEM= U    COMP= X     NAME= PileMid

VARIABLE  4  IS AT NODE    41  ITEM= U    COMP= X     NAME= PileToe

Y AXIS LABEL = Displacement

X AXIS LABEL = Time
CURVE X RANGE SET FROM   0.0000E+00 TO   0.3855E-01
CURVE  1 Y RANGE SET FROM   0.0000E+00 TO   0.1000E-01

STORAGE COMPLETE FOR     84 DATA POINTS

        SUMMARY OF VARIABLES STORED THIS STEP AND EXTREME VALUES
VARI TYPE    IDENTIFIERS    NAME   MINIMUM    AT TIME     MAXIMUM   AT TIME

  2 NSOL     1 UX         PileTop   0.9462E-06 0.1205E-04 0.9373E-02 0.4734E-02
  3 NSOL    21 UX         PileMid   0.4776E-45 0.1205E-04 0.3088E-02 0.9308E-02
  4 NSOL    41 UX         PileToe   0.4821E-84 0.1205E-04 0.2136E-02 0.1451E-01

PLOT DEFINITION
CURVE  VARIABLE   NAME
   1        2   PileTop
   2        3   PileMid
   3        4   PileToe

CUMULATIVE DISPLAY NUMBER   1 WRITTEN TO FILE THESIS2A.grp      - RASTER MODE.
DISPLAY TITLE=
Damped Case

/SHOW SWITCH PLOTS TO FILE THESIS2B.grp      - RASTER MODE.

VARIABLE  5  IS AT ELEMENT       1  NODE=        1
   ITEM= F    COMP= X     NAME= PileTop

*** WARNING ***                         CP=    230.960   TIME= 22:00:01
ESOL command erases all previously stored or calculated data
 unless STORE,MERGE is used.

VARIABLE  6  IS AT ELEMENT      20  NODE=       21
   ITEM= F    COMP= X     NAME= PileMid

VARIABLE  7  IS AT ELEMENT      40  NODE=       41
   ITEM= F    COMP= X     NAME= PileToe

Y AXIS LABEL = Element Force
CURVE  1 Y RANGE SET FROM  -0.1810E+08 TO   0.1810E+08

STORAGE COMPLETE FOR     84 DATA POINTS

        SUMMARY OF VARIABLES STORED THIS STEP AND EXTREME VALUES
VARI TYPE    IDENTIFIERS    NAME   MINIMUM    AT TIME     MAXIMUM   AT TIME
```

```
  2 NSOL     1 UX         PileTop   0.9462E-06 0.1205E-04 0.9373E-02 0.4734E-02
  3 NSOL    21 UX         PileMid   0.4776E-45 0.1205E-04 0.3088E-02 0.9308E-02
  4 NSOL    41 UX         PileToe   0.4821E-84 0.1205E-04 0.2136E-02 0.1451E-01
  5 ESOL     1 F    X     PileTop  -0.2912E+08 0.4698E-02 0.1380E+08 0.2409E-04
  6 ESOL    20 F    X     PileMid  -0.5375E+06 0.1763E-01 0.5732E+07 0.6296E-02
  7 ESOL    40 F    X     PileToe  -0.2204E+06 0.1451E-01 0.3947E+06 0.1025E-01

PLOT DEFINITION
CURVE  VARIABLE   NAME
   1        5   PileTop
   2        6   PileMid
   3        7   PileToe

CUMULATIVE DISPLAY NUMBER   1 WRITTEN TO FILE THESIS2B.grp      - RASTER MODE.
DISPLAY TITLE=
Damped Case


***** ROUTINE COMPLETED *****  CP =       247.220


CURVE X RANGE SET FROM DEFAULT VALUES
ALL CURVE Y RANGES SET FROM DEFAULT VALUES
```

```
                  VERSION=PC 386/486    22:00:17  OCT 22, 1996 CP=    247.220

Damped Case

**ANSYS VERSION FOR EDUCATIONAL PURPOSES ONLY**

         ***** ANSYS RESULTS INTERPRETATION (POST1) *****

NODE NUMBERING KEY =  0

U    BOUNDARY CONDITION DISPLAY KEY =  1

F    BOUNDARY CONDITION DISPLAY KEY =  1

/SHOW SWITCH PLOTS TO FILE THESIS2C.grp      - RASTER MODE.

PRODUCE ELEMENT PLOT IN DSYS =   0

CUMULATIVE DISPLAY NUMBER   1 WRITTEN TO FILE THESIS2C.grp      - RASTER MODE.
DISPLAY TITLE=
Damped Case

PRINT ITERATION SUMMARY

          **** POST1 ITERATION SUMMARY ****

LOAD STEP    2  SUBSTEP   58        CUMULATIVE ITERATION   84
TIME =   0.385500E-01           TIME INCREMENT =   0.582482E-03
 NUMBER OF EQUILIBRIUM ITERATIONS =          1
 CONVERGENCE INDICATOR = 0
 MAXIMUM DISPLACEMENT VALUE =   0.123620E-03
 RESPONSE FREQUENCY FOR 2ND ORDER SYSTEMS =    31.0710
 DESCENT PARAMETER =   0.000000E+00
 FORCE CONVERGENCE VALUE =   0.000000E+00
 MOMENT CONVERGENCE VALUE =   0.000000E+00
 DISPLACEMENT CONVERGENCE VALUE =   0.000000E+00
 ROTATION CONVERGENCE VALUE =   0.000000E+00


NUMBER OF WARNING MESSAGES ENCOUNTERED=          1
NUMBER OF ERROR   MESSAGES ENCOUNTERED=          0

EXIT THE ANSYS POST1 DATABASE PROCESSOR


***** ROUTINE COMPLETED *****  CP =       249.190



PURGE ALL SOLUTION AND POST DATA
SAVE ALL MODEL DATA

ALL CURRENT ANSYS DATA WRITTEN TO FILE NAME= thesis2.db
 FOR POSSIBLE RESUME FROM THIS POINT


NUMBER OF WARNING MESSAGES ENCOUNTERED=          1
NUMBER OF ERROR   MESSAGES ENCOUNTERED=          0


*---------------------------------------------------------------------------*
|                                                                           |
|                            ANSYS RUN COMPLETED                            |
|                                                                           |
|---------------------------------------------------------------------------|
|                                                                           |
|               REV. 5.0 ED56                     PC 386/486                 |
|                                                                           |
|        CP TIME      (sec) =        257.000       TIME  =  22:00:21        |
|        ELAPSED TIME (sec) =        257.000       DATE  =  10/22/96        |
|                                                                           |
*---------------------------------------------------------------------------*
```

G. Damped Wave Equation Solution using WEAP87

```
1                        ECHO PRINT OF INPUT DATA

     Thesis -- Damped Pile Case
    6   1 328   0   0   0  32   0   0   0  98   1   6   0   0   0  -1 500
     9.920      .000        .0      .000     1.000      .010   14000.0
      .000        .0      .000      .500      .010        .0
   162.040   186.990 30000.000   492.000     1.000      .010
 Thesis  Sample     3   1   0
   33.0690   83.0000   34.9000    4.9213     .0000     .8000
      .000      .000      .800      .010    2
    12.580    12.580      .000   39753.4   39753.4        .0
     .0000     .0000     .0000     .0000     .0000
      .500      .500      .288      .019
         5052.2        .0        .0        .0        .0        .0        .0
            .0        .0        .0
    1   16   32    0    0    0    0    0    0    0    0    0    0
1      WEAP87: WAVE EQUATION ANALYSIS OF PILE FOUNDATIONS
                    1987, VERSION 3.00

           Thesis -- Damped Pile Case

            HAMMER MODEL OF: Sample   MADE BY: Thesis

       ELEMENT     WEIGHT  STIFFNESS  COEFF. OF    D-NL.   CAP DAMPG
                   (KIPS)   (K/IN)   RESTITUTION    FT      (K/FT/S)
             1     33.069
       CAP/RAM      9.920   13437.2     1.000     .0100        .0

       ASSEMBLY     WEIGHT STIFFNESS   COEFF. OF   D-NL.
                   (KIPS)   (K/IN)   RESTITUTION   FT
             1     12.580   39753.4
             2     12.580   39753.4      .800     .0100

       HAMMER OPTIONS:
        HAMMER NO. FUEL SETTG. STROKE OPT. HAMMER TYPE DAMPNG-HAMR
         328          1           0           3           0

       HAMMER PERFORMANCE DATA
        RAM WEIGHT  RAM LENGTH   MAX STROKE     STROKE   EFFICIENCY
          (KIPS)         (IN)        (FT)         (FT)
             33.07       83.00        4.92        4.92        .800

        RTD PRESS.   ACT PRESS.   EFF. AREA  IMPACT VEL.
          (PSI)         (PSI)       (IN2)       (FT/S)
               .00         .00         .00       15.92


        HAMMER CUSHION    AREA   E-MODULUS    THICKNESS   STIFFNESS
                         (IN2)       (KSI)         (IN)   (KIPS/IN)
                           .00          .0         .000     14000.0
```

PILE PROFILE:

LBT (FT)	AREA (IN2)	E-MOD (KSI)	SP.W. (LB/FT3)	WAVE SP (FT/S)	EA/C (K/FT/S)
.00	187.0	30000.	492.000	16806.8	333.8
162.04	187.0	30000.	492.000	16806.8	333.8

WAVE TRAVEL TIME - 2L/C - = 19.283 MS

PILE AND SOIL MODEL FOR RULT = 5052.2 KIPS

NO	WEIGHT (KIPS)	STIFFN (K/IN)	D-NL (FT)	SPLICE (FT)	COR	SOIL-S (KIPS)	SOIL-D (S/FT)	QUAKE (IN)	L BT (FT)	AREA (IN**2)
1	3.235	92318.	.010	.000	1.000	154.2	.288	.500	5.06	187.0
2	3.235	92318.	.000	.000	1.000	154.7	.288	.500	10.13	187.0
3	3.235	92318.	.000	.000	1.000	154.7	.288	.500	15.19	187.0
4	3.235	92318.	.000	.000	1.000	154.7	.288	.500	20.25	187.0
5	3.235	92318.	.000	.000	1.000	154.7	.288	.500	25.32	187.0
6	3.235	92318.	.000	.000	1.000	154.7	.288	.500	30.38	187.0
7	3.235	92318.	.000	.000	1.000	154.7	.288	.500	35.45	187.0
8	3.235	92318.	.000	.000	1.000	154.7	.288	.500	40.51	187.0
9	3.235	92318.	.000	.000	1.000	154.7	.288	.500	45.57	187.0
10	3.235	92318.	.000	.000	1.000	154.7	.288	.500	50.64	187.0
11	3.235	92318.	.000	.000	1.000	154.7	.288	.500	55.70	187.0
12	3.235	92318.	.000	.000	1.000	154.7	.288	.500	60.77	187.0
13	3.235	92318.	.000	.000	1.000	154.7	.288	.500	65.83	187.0
14	3.235	92318.	.000	.000	1.000	154.7	.288	.500	70.89	187.0
15	3.235	92318.	.000	.000	1.000	154.7	.288	.500	75.96	187.0
16	3.235	92318.	.000	.000	1.000	154.7	.288	.500	81.02	187.0
17	3.235	92318.	.000	.000	1.000	154.7	.288	.500	86.08	187.0
18	3.235	92318.	.000	.000	1.000	154.7	.288	.500	91.15	187.0
19	3.235	92318.	.000	.000	1.000	154.7	.288	.500	96.21	187.0
20	3.235	92318.	.000	.000	1.000	154.7	.288	.500	101.28	187.0
21	3.235	92318.	.000	.000	1.000	154.7	.288	.500	106.34	187.0
22	3.235	92318.	.000	.000	1.000	154.7	.288	.500	111.40	187.0
23	3.235	92318.	.000	.000	1.000	154.7	.288	.500	116.47	187.0
24	3.235	92318.	.000	.000	1.000	154.7	.288	.500	121.53	187.0
25	3.235	92318.	.000	.000	1.000	154.7	.288	.500	126.59	187.0
26	3.235	92318.	.000	.000	1.000	154.7	.288	.500	131.66	187.0
27	3.235	92318.	.000	.000	1.000	154.7	.288	.500	136.72	187.0
28	3.235	92318.	.000	.000	1.000	154.7	.288	.500	141.79	187.0
29	3.235	92318.	.000	.000	1.000	154.7	.288	.500	146.85	187.0
30	3.235	92318.	.000	.000	1.000	154.7	.288	.500	151.91	187.0
31	3.235	92318.	.000	.000	1.000	154.7	.288	.500	156.98	187.0
32	3.235	92318.	.000	.000	1.000	154.7	.288	.500	162.04	187.0
TOE						101.0	.019	.500		

PILE OPTIONS:

N/UNIFORM	AUTO S.G.	SPLICES	DAMPNG-P	D-P VALUE (K/FT/S)
0	0	0	1	6.676

SOIL OPTIONS:

% SKIN FR	% END BG	DIS. NO.	S DAMPING
98	2	6	SMITH-1

ANALYSIS/OUTPUT OPTIONS:

ITERATNS	DTCR/DT(%)	RES STRESS	IOUT	AUTO SGMNT	OUTPT INCR	MAX T(MS)
0	160	0	6	1	1	500

PC-WEAP87 REVISED JUNE, 1988 FHWA Thesis -- Damped Pile Case

RULT = 5052.2, RTOE = 101.0 KIPS

J	TIME (MS)	F AS (KIPS)	FTOP (KIPS)	VTOP (FT/S)	DTOP (IN)	FTOE (KIPS)	VTOE (FT/S)	DTOE (IN)	RSUM
1	.2	25.2	34.1	.0	.006	.1	.0	.000	.0
2	.4	25.2	34.0	.0	.006	.8	.0	.000	.1
3	.6	25.2	34.0	.0	.006	1.5	.0	.000	.1
4	.7	25.2	34.0	.0	.006	1.7	.0	.000	.3
5	.9	25.2	34.0	.0	.006	1.5	.0	.000	.5
6	1.1	25.2	33.9	.0	.006	1.1	.0	.000	.7
7	1.3	25.0	33.9	.0	.006	.9	.0	.000	1.0
8	1.5	23.8	34.4	.0	.006	1.0	.0	.000	1.3
9	1.7	19.8	36.5	.0	.006	1.1	.0	.000	1.7
10	1.9	11.4	43.1	.1	.006	1.2	.0	.000	2.1
11	2.0	2.0	58.7	.2	.006	1.2	.0	.000	2.7
12	2.2	.0	91.7	.3	.007	1.1	.0	.000	3.4
13	2.4	.0	156.0	.7	.008	1.1	.0	.000	4.4
14	2.6	.0	272.4	1.2	.010	1.1	.0	.000	5.9
15	2.8	.0	467.0	2.0	.014	1.2	.0	.000	8.2
16	3.0	.0	762.5	3.2	.019	1.2	.0	.001	11.9
17	3.2	.0	1164.7	4.7	.028	1.2	.0	.001	17.5
18	3.3	.0	1648.1	6.2	.040	1.2	.0	.001	25.8
19	3.5	.0	2155.0	7.6	.056	1.2	.0	.001	37.2
20	3.7	.0	2613.6	8.7	.074	1.2	.0	.001	52.2
21	3.9	.0	2968.3	9.4	.094	1.2	.0	.001	71.2
22	4.1	.0	3201.1	9.7	.116	1.2	.0	.001	94.3
23	4.3	.0	3331.6	9.9	.138	1.3	.0	.001	121.4
24	4.5	.0	3397.2	9.9	.160	1.3	.0	.001	152.7
25	4.7	.0	3430.4	9.8	.182	1.3	.0	.001	188.0
26	4.8	.0	3448.1	9.7	.203	1.3	.0	.001	227.1
27	5.0	.0	3454.4	9.5	.225	1.3	.0	.001	270.0
28	5.2	.0	3448.1	9.2	.246	1.4	.0	.002	316.4
29	5.4	.0	3429.3	8.9	.266	1.4	.0	.002	366.1
30	5.6	.0	3399.3	8.6	.285	1.4	.0	.002	419.0
31	5.8	.0	3359.7	8.2	.304	1.4	.1	.002	474.7
32	6.0	.0	3311.1	7.8	.322	1.4	.1	.002	532.9
33	6.1	.0	3253.5	7.3	.339	1.5	.1	.002	593.5
34	6.3	.0	3187.5	6.9	.355	1.5	.1	.002	656.0
35	6.5	.0	3113.8	6.4	.369	1.5	.1	.002	720.2
36	6.7	.0	3033.5	5.9	.383	1.5	.1	.002	785.9
37	6.9	.0	2947.5	5.4	.396	1.5	.1	.003	852.6
38	7.1	.0	2856.9	4.8	.407	1.6	.1	.003	920.1
39	7.3	.0	2762.6	4.3	.417	1.6	.1	.003	988.2
40	7.4	.0	2665.7	3.8	.426	1.6	.1	.003	1056.5
41	7.6	.0	2567.3	3.3	.434	1.6	.1	.003	1124.9
42	7.8	.0	2468.4	2.9	.441	1.7	.1	.003	1193.0
43	8.0	.0	2369.7	2.4	.447	1.7	.1	.003	1260.7
44	8.2	.0	2272.1	2.0	.452	1.7	.1	.003	1327.7
45	8.4	.0	2176.3	1.6	.456	1.8	.1	.004	1393.8
46	8.6	.0	2083.0	1.2	.459	1.8	.1	.004	1459.0
47	8.7	.0	1992.5	.8	.461	1.8	.1	.004	1522.9
48	8.9	.0	1905.3	.5	.463	1.9	.1	.004	1585.6
49	9.1	.0	1821.6	.2	.463	2.1	.1	.004	1646.8
50	9.3	.0	1741.6	-.1	.463	2.4	.1	.004	1706.5
51	9.5	.0	1665.3	-.4	.463	3.0	.1	.004	1764.5

RULT = 5052.2, RTOE = 101.0 KIPS

J	TIME (MS)	F AS (KIPS)	FTOP (KIPS)	VTOP (FT/S)	DTOP (IN)	FTOE (KIPS)	VTOE (FT/S)	DTOE (IN)	RSUM
52	9.7	.0	1592.8	-.6	.462	4.0	.1	.004	1820.9
53	9.9	.0	1523.8	-.9	.460	5.9	.1	.005	1875.4
54	10.0	.0	1458.1	-1.1	.458	9.2	.1	.005	1928.2
55	10.2	.0	1395.6	-1.2	.455	14.7	.1	.005	1979.0
56	10.4	.0	1335.9	-1.4	.452	23.7	.1	.005	2028.0
57	10.6	.0	1278.7	-1.6	.449	37.8	.2	.006	2075.0
58	10.8	.0	1223.8	-1.7	.445	59.1	.3	.006	2120.0
59	11.0	.0	1170.7	-1.9	.441	90.1	.4	.007	2163.1
60	11.2	.0	1119.1	-2.0	.437	133.1	.6	.008	2204.2
61	11.3	.0	1068.8	-2.1	.432	190.4	.9	.009	2243.4
62	11.5	.0	1019.4	-2.2	.427	262.9	1.3	.012	2280.7
63	11.7	.0	970.7	-2.4	.422	349.3	1.8	.015	2316.1
64	11.9	.0	922.5	-2.5	.417	445.8	2.6	.020	2349.7
65	12.1	.0	874.6	-2.6	.411	544.7	3.5	.027	2381.5
66	12.3	.0	827.1	-2.7	.406	634.5	4.5	.036	2411.7
67	12.5	.0	779.8	-2.8	.399	701.1	5.7	.047	2440.2
68	12.6	.0	733.0	-2.9	.393	729.5	6.9	.061	2467.1
69	12.8	.0	686.8	-3.0	.387	707.0	8.2	.078	2492.5
70	13.0	.0	641.3	-3.0	.380	626.6	9.2	.098	2516.1
71	13.2	.0	596.6	-3.1	.373	490.7	10.1	.119	2537.9
72	13.4	.0	552.9	-3.2	.366	311.8	10.5	.142	2557.7
73	13.6	.0	510.3	-3.3	.359	111.9	10.6	.166	2575.3
74	13.8	.0	469.0	-3.3	.351	-81.7	10.2	.189	2590.3
75	14.0	.0	429.1	-3.4	.344	-242.6	9.4	.211	2602.5
76	14.1	.0	390.7	-3.5	.336	-350.8	8.3	.231	2611.7
77	14.3	.0	353.8	-3.5	.328	-398.1	7.1	.248	2617.7
78	14.5	.0	318.5	-3.6	.321	-388.4	5.8	.262	2620.5
79	14.7	.0	284.9	-3.6	.313	-336.2	4.6	.274	2619.9
80	14.9	.0	253.0	-3.6	.304	-261.4	3.6	.283	2616.0
81	15.1	.0	222.9	-3.7	.296	-183.8	2.7	.290	2609.0
82	15.3	.0	194.4	-3.7	.288	-118.4	2.0	.295	2598.9
83	15.4	.0	167.7	-3.7	.280	-72.7	1.4	.299	2585.9
84	15.6	.0	142.7	-3.7	.271	-47.0	.9	.301	2570.0
85	15.8	.0	119.5	-3.8	.263	-36.3	.5	.303	2551.4
86	16.0	.0	97.9	-3.8	.255	-33.4	.1	.304	2530.1
87	16.2	.0	78.1	-3.8	.246	-32.0	-.2	.303	2506.3
88	16.4	.0	60.0	-3.8	.238	-28.1	-.6	.303	2480.1
89	16.6	.0	43.5	-3.8	.229	-20.9	-.9	.301	2451.5
90	16.7	.0	28.8	-3.8	.221	-11.5	-1.2	.299	2420.6
91	16.9	.0	15.8	-3.7	.213	-1.9	-1.4	.296	2387.6
92	17.1	.0	4.6	-3.7	.204	6.3	-1.6	.292	2352.6
93	17.3	.0	-4.9	-3.7	.196	12.4	-1.8	.289	2315.6
94	17.5	.0	-12.8	-3.7	.188	16.6	-2.0	.284	2276.8
95	17.7	.0	-19.2	-3.7	.180	19.5	-2.1	.280	2236.2
96	17.9	.0	-23.2	-3.6	.171	22.0	-2.2	.275	2194.1
97	18.0	.0	-24.7	-3.6	.163	24.4	-2.4	.270	2150.3
98	18.2	.0	-24.9	-3.5	.155	26.9	-2.5	.264	2105.2
99	18.4	.0	-24.7	-3.5	.148	29.2	-2.5	.259	2058.6
100	18.6	.0	-25.0	-3.5	.140	31.1	-2.6	.253	2010.8
101	18.8	.0	-25.8	-3.4	.132	32.5	-2.7	.247	1961.8
102	19.0	1.9	-26.9	-3.4	.124	33.5	-2.8	.241	1911.7
103	19.2	38.3	-28.2	-3.4	.117	34.0	-2.8	.235	1860.6

PC-WEAP87 REVISED JUNE, 1988 FHWA Thesis -- Damped Pile Case

RULT = 5052.2, RTOE = 101.0 KIPS

J	TIME (MS)	F AS (KIPS)	FTOP (KIPS)	VTOP (FT/S)	DTOP (IN)	FTOE (KIPS)	VTOE (FT/S)	DTOE (IN)	RSUM
104	19.3	120.6	-30.1	-3.3	.109	34.2	-2.9	.228	1808.4
105	19.5	245.0	-32.9	-3.3	.102	34.0	-2.9	.222	1755.4
106	19.7	403.4	-36.8	-3.2	.095	33.6	-2.9	.215	1701.6
107	19.9	582.3	-42.4	-3.2	.088	32.8	-3.0	.209	1647.0
108	20.1	763.5	-49.9	-3.1	.081	31.7	-3.0	.202	1591.7
109	20.3	926.2	-60.0	-3.0	.074	30.3	-3.0	.195	1535.8
110	20.5	1050.4	-73.3	-2.9	.067	28.5	-3.1	.189	1479.2
111	20.6	1121.1	-90.5	-2.7	.061	26.5	-3.1	.182	1422.2
112	20.8	1132.3	-111.8	-2.5	.055	24.2	-3.1	.175	1364.6
113	21.0	1072.5	-137.0	-2.2	.050	21.8	-3.2	.168	1306.7
114	21.2	954.9	-165.2	-1.9	.045	19.1	-3.2	.161	1248.3
115	21.4	801.8	-194.7	-1.6	.041	16.4	-3.2	.153	1189.7
116	21.6	636.5	-222.6	-1.2	.038	13.6	-3.3	.146	1130.7
117	21.8	478.3	-244.4	-.7	.036	10.8	-3.3	.139	1071.4
118	21.9	340.6	-251.4	-.2	.035	8.1	-3.3	.131	1011.9
119	22.1	229.3	-232.7	.4	.035	5.4	-3.4	.124	952.3
120	22.3	144.9	-182.9	1.0	.037	2.9	-3.4	.116	892.5
121	22.5	84.3	-106.3	1.5	.040	.5	-3.5	.109	832.7
122	22.7	42.9	-14.6	1.9	.044	-1.7	-3.5	.101	773.0
123	22.9	16.6	79.2	2.1	.048	-3.7	-3.5	.093	713.3
124	23.1	2.7	165.5	2.1	.053	-5.5	-3.6	.085	653.6
125	23.3	.0	239.6	2.0	.058	-7.1	-3.6	.077	593.9
126	23.4	.0	300.0	1.7	.062	-8.6	-3.7	.069	534.3
127	23.6	.0	346.4	1.4	.065	-9.9	-3.7	.061	474.7
128	23.8	.0	379.4	1.0	.068	-11.0	-3.7	.052	415.2
129	24.0	.0	401.1	.6	.070	-12.0	-3.8	.044	355.7
130	24.2	.0	415.4	.2	.070	-12.8	-3.8	.036	296.3
131	24.4	.0	427.9	-.1	.070	-13.5	-3.9	.027	237.1
132	24.6	.0	444.4	-.3	.070	-14.0	-3.9	.018	178.1
133	24.7	.0	468.7	-.4	.069	-14.4	-3.9	.010	119.5
134	24.9	.0	501.5	-.5	.068	-14.6	-4.0	.001	61.3
135	25.1	.0	539.3	-.5	.067	-14.6	-4.0	-.008	5.3
136	25.3	.0	576.0	-.5	.066	-14.1	-4.0	-.017	-49.9
137	25.5	.0	604.7	-.6	.065	-13.0	-4.0	-.026	-104.2
138	25.7	.0	620.0	-.7	.063	-11.1	-4.0	-.035	-157.4
139	25.9	.0	619.2	-.8	.062	-8.6	-4.0	-.044	-209.5
140	26.0	.0	602.7	-1.0	.060	-5.6	-4.0	-.053	-260.4
141	26.2	.0	572.9	-1.2	.057	-2.5	-3.9	-.061	-310.0
142	26.4	.0	533.5	-1.4	.054	.6	-3.8	-.070	-358.3
143	26.6	.0	488.2	-1.6	.051	3.8	-3.7	-.078	-405.2
144	26.8	.0	440.4	-1.8	.048	7.1	-3.7	-.087	-450.6
145	27.0	.0	392.4	-1.9	.043	10.3	-3.5	-.095	-494.6
146	27.2	.0	346.1	-2.1	.039	13.4	-3.4	-.102	-537.1
147	27.3	.0	302.5	-2.2	.034	16.3	-3.3	-.110	-578.1
148	27.5	.0	262.1	-2.4	.029	18.9	-3.1	-.117	-617.6
149	27.7	.0	224.8	-2.5	.023	21.0	-3.0	-.124	-655.5
150	27.9	.0	190.6	-2.6	.018	22.8	-2.8	-.130	-692.0
151	28.1	.0	159.2	-2.7	.012	24.1	-2.6	-.136	-726.9
152	28.3	.0	130.5	-2.8	.006	25.2	-2.4	-.142	-760.3
153	28.5	.0	104.4	-2.8	-.001	26.2	-2.2	-.147	-792.3
154	28.6	.0	80.8	-2.9	-.007	27.4	-2.1	-.152	-822.7
155	28.8	.0	59.6	-2.9	-.013	29.0	-1.9	-.156	-851.7

PC-WEAP87 REVISED JUNE, 1988 FHWA Thesis -- Damped Pile Case

RULT = 5052.2, RTOE = 101.0 KIPS

J	TIME (MS)	F AS (KIPS)	FTOP (KIPS)	VTOP (FT/S)	DTOP (IN)	FTOE (KIPS)	VTOE (FT/S)	DTOE (IN)	RSUM
156	29.0	.0	40.8	-2.9	-.020	31.5	-1.7	-.160	-879.1
157	29.2	.0	24.1	-2.9	-.026	35.1	-1.5	-.164	-905.1
158	29.4	1.7	9.5	-2.9	-.033	40.0	-1.3	-.167	-929.6
159	29.6	7.7	-2.8	-2.8	-.039	46.4	-1.1	-.169	-952.7
160	29.8	21.0	-12.6	-2.8	-.045	54.5	-.8	-.171	-974.3
161	29.9	45.4	-19.4	-2.7	-.051	64.3	-.6	-.173	-994.5
162	30.1	84.9	-23.1	-2.6	-.057	75.7	-.4	-.174	-1013.2
163	30.3	142.8	-24.0	-2.5	-.063	88.6	-.1	-.175	-1030.4
164	30.5	220.0	-23.2	-2.4	-.068	102.8	.2	-.175	-1046.2
165	30.7	313.9	-21.6	-2.3	-.074	117.7	.5	-.174	-1060.4
166	30.9	416.9	-19.9	-2.2	-.079	132.7	.8	-.172	-1073.1
167	31.1	517.5	-18.0	-2.1	-.084	146.8	1.2	-.170	-1084.1
168	31.2	601.2	-15.9	-2.0	-.088	158.6	1.5	-.167	-1093.5
169	31.4	654.2	-13.3	-1.9	-.093	166.7	1.9	-.163	-1101.4
170	31.6	667.4	-10.5	-1.8	-.097	169.2	2.2	-.159	-1107.6
171	31.8	628.7	-7.5	-1.7	-.101	164.4	2.6	-.153	-1112.3
172	32.0	542.8	-4.6	-1.6	-.104	150.9	2.9	-.147	-1115.4
173	32.2	426.5	-2.0	-1.5	-.108	128.3	3.1	-.141	-1116.9
174	32.4	300.9	.1	-1.4	-.111	97.0	3.3	-.133	-1116.9
175	32.6	185.5	1.7	-1.2	-.114	58.9	3.5	-.126	-1115.3
176	32.7	94.4	2.6	-1.1	-.116	17.1	3.5	-.118	-1112.4
177	32.9	34.2	3.0	-1.0	-.119	-24.3	3.5	-.110	-1108.0
178	33.1	4.8	3.6	-.9	-.121	-60.6	3.4	-.102	-1102.2
179	33.3	.0	5.7	-.8	-.123	-87.7	3.2	-.095	-1095.1
180	33.5	.0	10.5	-.7	-.125	-102.3	3.0	-.088	-1086.7
181	33.7	.0	17.7	-.6	-.126	-103.5	2.8	-.081	-1077.1
182	33.9	.0	26.0	-.4	-.127	-92.1	2.6	-.075	-1066.1
183	34.0	.0	34.0	-.3	-.128	-71.4	2.4	-.070	-1054.0
184	34.2	.0	40.6	-.2	-.129	-46.3	2.3	-.064	-1040.5
185	34.4	.0	45.0	-.1	-.129	-22.0	2.2	-.059	-1025.9
186	34.6	.0	46.8	.0	-.129	-3.6	2.1	-.055	-1010.0
187	34.8	.0	46.1	.1	-.129	5.5	2.1	-.050	-992.9
188	35.0	.0	42.9	.2	-.129	3.8	2.1	-.045	-974.6
189	35.2	.0	37.4	.2	-.128	-8.0	2.0	-.040	-955.2
190	35.3	.0	30.0	.3	-.128	-27.5	2.0	-.036	-934.8
191	35.5	.0	21.4	.4	-.127	-50.9	1.9	-.032	-913.4
192	35.7	.0	12.1	.4	-.126	-74.1	1.7	-.028	-891.1
193	35.9	.0	2.7	.5	-.125	-93.5	1.5	-.024	-868.0
194	36.1	.0	-6.3	.6	-.124	-106.3	1.3	-.021	-844.1
195	36.3	.0	-14.2	.7	-.122	-111.0	1.0	-.019	-819.5
196	36.5	.0	-20.3	.7	-.121	-107.8	.8	-.017	-794.3
197	36.6	.0	-24.3	.8	-.119	-97.7	.5	-.015	-768.5
198	36.8	.0	-26.5	.9	-.117	-82.7	.3	-.014	-742.2
199	37.0	.0	-27.6	1.0	-.115	-65.2	.1	-.014	-715.3
200	37.2	.0	-28.3	1.1	-.112	-47.3	-.1	-.014	-688.0

PC-WEAP87 REVISED JUNE, 1988 FHWA Thesis -- Damped Pile Case

RULT = 5052.2, RTOE = 101.0 KIPS, DEL T = .186 MS

NO.	FMIN,JMN (K)	FMAX,JMX (K)	STRMIN,JSN (KSI)	STRMAX,JSX (KSI)	VMAX,JVX (F/S)	DMAX,JDX (IN)
1	.0, 0	3569.6, 26	.00, 0	19.09, 26	9.9, 24	.463, 50
2	-251.4,118	3454.4, 27	-1.34,118	18.47, 27	9.7, 25	.445, 51
3	-495.8,118	3355.5, 27	-2.65,118	17.94, 27	9.6, 26	.427, 52
4	-672.4,117	3284.5, 28	-3.60,117	17.57, 28	9.4, 28	.411, 54
5	-789.7,116	3227.3, 29	-4.22,116	17.26, 29	9.3, 29	.395, 55
6	-864.9,115	3166.8, 30	-4.63,115	16.94, 30	9.1, 31	.380, 56
7	-913.0,114	3117.2, 32	-4.88,114	16.67, 32	9.0, 32	.366, 57
8	-946.7,112	3055.0, 33	-5.06,112	16.34, 33	8.8, 34	.353, 59
9	-984.2,111	3009.6, 35	-5.26,111	16.09, 35	8.6, 35	.340, 60
10	-1017.9,109	2949.6, 37	-5.44,109	15.77, 37	8.5, 37	.328, 61
11	-1042.0,108	2901.2, 38	-5.57,108	15.52, 38	8.3, 39	.317, 62
12	-1054.1,106	2847.2, 40	-5.64,106	15.23, 40	8.2, 40	.306, 63
13	-1058.7,105	2791.0, 41	-5.66,105	14.93, 41	8.1, 42	.296, 65
14	-1055.6,103	2744.3, 43	-5.65,103	14.68, 43	7.9, 43	.286, 66
15	-1047.5,102	2683.3, 45	-5.60,102	14.35, 45	7.8, 45	.277, 67
16	-1038.3,100	2641.0, 46	-5.55,100	14.12, 46	7.6, 47	.268, 68
17	-1023.9, 99	2587.6, 48	-5.48, 99	13.84, 48	7.5, 48	.260, 69
18	-1019.2, 97	2537.4, 49	-5.45, 97	13.57, 49	7.3, 50	.252, 71
19	-1014.4, 95	2492.0, 51	-5.42, 95	13.33, 51	7.2, 51	.244, 72
20	-1018.9, 94	2434.1, 52	-5.45, 94	13.02, 52	7.1, 53	.242, 96
21	-1035.0, 92	2396.7, 54	-5.53, 92	12.82, 54	6.9, 55	.251, 95
22	-1050.4, 90	2345.4, 56	-5.62, 90	12.54, 56	6.8, 56	.259, 94
23	-1075.9, 89	2301.9, 57	-5.75, 89	12.31, 57	6.6, 58	.266, 93
24	-1092.0, 87	2258.5, 59	-5.84, 87	12.08, 59	6.5, 59	.273, 92
25	-1105.1, 86	2208.2, 60	-5.91, 86	11.81, 60	6.4, 61	.280, 91
26	-1101.5, 84	2171.8, 62	-5.89, 84	11.61, 62	6.2, 63	.286, 90
27	-1087.6, 83	2119.0, 64	-5.82, 83	11.33, 64	6.2, 64	.291, 89
28	-1042.6, 81	2067.4, 65	-5.58, 81	11.06, 65	6.2, 66	.296, 88
29	-975.7, 80	1955.0, 66	-5.22, 80	10.46, 66	6.5, 69	.299, 87
30	-879.2, 79	1731.2, 67	-4.70, 79	9.26, 67	8.0, 72	.302, 87
31	-720.6, 78	1324.6, 68	-3.85, 78	7.08, 68	9.6, 72	.303, 86
32	-398.1, 77	729.5, 68	-2.13, 77	3.90, 68	10.6, 73	.304, 86

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R ULT	BL CT	STROKE(EQ.)	MINSTR	I,J	MAXSTR	I,J	ENTHRU
KIPS	BPF	FT	KSI		KSI		FT-KIP
5052.2	9999.0	4.92	-5.91	(25, 86)	19.09	(1, 26)	119.0
5052.3							

XI. Vita

Don C. Warrington was born 22 May 1955. After graduating from the St. Andrew's School in Boca Raton, Florida, he attended and graduated magna cum laude from Texas A&M University in 1976 with a Bachelor of Science degree in Mechanical Engineering. While at Texas A&M he was elected to the Pi Tau Sigma, Phi Eta Sigma, Tau Beta Pi and Phi Kappa Phi honor fraternities; he was also a junior member of the Student's Engineers Council and student chapters of the American Society of Mechanical Engineers and the National Society of Professional Engineers. He first worked as an engineer for Texas Instruments, then held several technical and management positions at Vulcan Iron Works Inc. from 1978 to 1996, when the company was merged. He was also chairman of the Vibratory Hammer Committee of the Deep Foundations Institute and was program chairman for two of that organization's annual meetings.

While pursuing his master's degree at the University of Tennessee at Chattanooga, he was selected to *Who's Who Among Students in American Universities and Colleges*. He is presently a consultant to Vulcan Iron Works Inc. and is also the Coordinator of Field Services for the Church of God Department of Lay Ministries. He is a registered Professional Engineer in the state of Tennessee. He is a member of the American Society of Civil Engineers, the Deep Foundations Institute and the Pachyderm Club. He is also a member of the Churchmen's Commission of the National Association of Evangelicals, the Council of 100 of the Christian Broadcasting Network and the Lay Coordinator for the North Cleveland Church of God.