Stresses and Strains in Soils: Elastic and Plastic*

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Many undergraduate civil engineering students find their required geotechnical courses strange. They enter into a new world of soil classifications, granular mechanics and porous materials, and a raft of empirical formulae. There seems to be little connection between the topics and a unifying theory is hard to find.

Other enter geotechnical engineering in the course of their work as equipment suppliers, owners and the like, who may not have specific training in the field and find many of the concepts baffling.

This article attempts to approach one of the important topics in geotechnical engineering–stresses in soils–in a different way. In the past, presentation of theory was just that–presentation–and it was difficult to apply the theory in a practical way except for the simplest of cases. Now, with finite element analysis, this theory can become practical reality. Many practising civil engineers, however, look on FEA as a "black box" where one puts in (hopefully meaningful) data and gets out answers which are at best no more meaningful than the data. Hopefully this article will bridge the gap between the two and make learning the essentials of stresses in soils easier.

1 Stresses in Two Dimensions

When undergraduate civil engineering students enter their first geotechnical class, the materials they have studied are generally linear. But there are two ways which we use this term.

The first is the generally understood meaning that a linear material is one which obeys Hooke's Law, which is

$$F = k\Delta x \tag{1}$$

where

- F =force applied on a body, kN
- k =spring constant, $^{kN}/_{m}$
- Δx =change in deflection from one point to another, m

A spring that obeys Hooke's law will, when extended, resist the extension in a linear relationship with the length the spring is extended. You double the length you pull the spring apart, you double the force the spring will resist. But

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if you go beyond a certain point, that relationship is broken, if not the spring. It's usually possible to shove the spring back to its original length/extension, but we instinctively know that things are not the same. We have exceeded the elastic limit of the spring and have entered the plastic region of the spring's mechanical response. Things are more complicated now than they were before; in some applications, the spring or structural member is no longer functional or dangerous to use.

With some materials (especially metals) we can design a structure so that it will not exceed the elastic limit in normal or intended use (and unfortunately those are not the same, something any designer needs to keep in mind.) With soils, we don't have that luxury; at some point we are generally forced to use them in the plastic, non-linear region.

The other way geotechnical engineering is different is that, for the first time, students are forced to consider a real three-dimensional medium and not just forces for bending moments on lines. We say three-dimensional; realities notwithstanding, even geotechnical engineering people wince at the concept of doing things in three dimensions. They will try to reduce the problem at hand to a two- or one-dimensional problem to make things easier. Two examples reduction to a one-dimensional representation are effective stresses (which we discuss briefly in this article) and consolidation (which is beyond our scope).

Two-dimensional representations, however, are commonplace in geotechnical engineering. It makes like a good deal simpler if we can visualise (and make calculations based on a two-dimensional situation. Let's consider the foundation shown in Figure 1.

Figure 1: Continuous Foundations (from [Kimmerling (2002)]

There are two rectangular slabs that are sitting flat on the ground. We could analyse either or both of these as rectangles. However, the longer a foundation or structural member is (and with pavements, they can turn into many kilometres!) the less the effects of length have on the performance. With shallow foundations like those shown, if the aspect ratio (i.e., the ratio of the width to the length) is greater than 10, we generally classify them as continuous foundations and analyse then as a two-dimensional structure (and in many ways a one-dimensional one.) One consequence of this is that the loading, instead of being in units of force like kN, is now in units of load per unit length, say $^{kN}/_{m}$. The stresses under the foundation can certainly be analysed two-dimensionally.

To illustrate what this might look like, now consider the diagram in Figure 2.

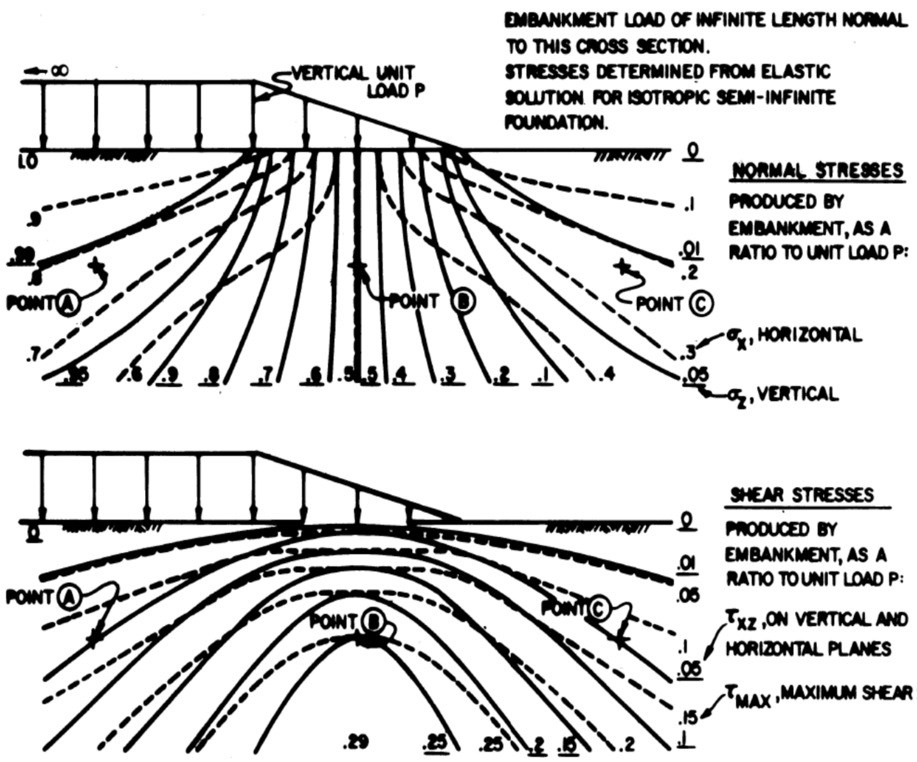


Figure 2: Soil Stresses Under an Embankment (from [NAVFAC DM 7.01 (1982)])

We have the case of a soil, minding its own business, suddenly facing having a large embankment dumped and compacted on top of it. This is obviously a stressful situation, and we can see this with the stress contours in the various parts of the soil. The embankment is semi-infinite in that it extends indefinitely to the left. But it is also infinite in that it is assumed to extend indefinitely in and out of the screen or paper. Thus we can model the already complex response of the soil (made so in part because the slope of the embankment makes the surface stress of the soil non-constant for some length) without having to worry about three-dimensional effects or representations.

Now also note the stresses themselves. The top part of the figure shows normal stresses in both the x- and y-directions. (Actually, geotechnical engineers prefer x- and z-directions, with z downward positive.) In their early formation, most civil engineers primarily consider axial stresses (trusses) and bending stresses (beams.) In both cases the stresses are in one direction, i.e., along the axis of the truss or beam. But now we have normal stresses that are not only in two directions, but their relative magnitudes vary in different parts of the soil mass.

Complicating matters further, we see shear stresses in the lower part of the figure. Although soils experience failure in direct shear in this case the shear stresses are the result of the normal ones. How normal and shear stresses exist at a single point is best illustrated in Figure 3, which looks at the points shown in Figure 2 in detail.

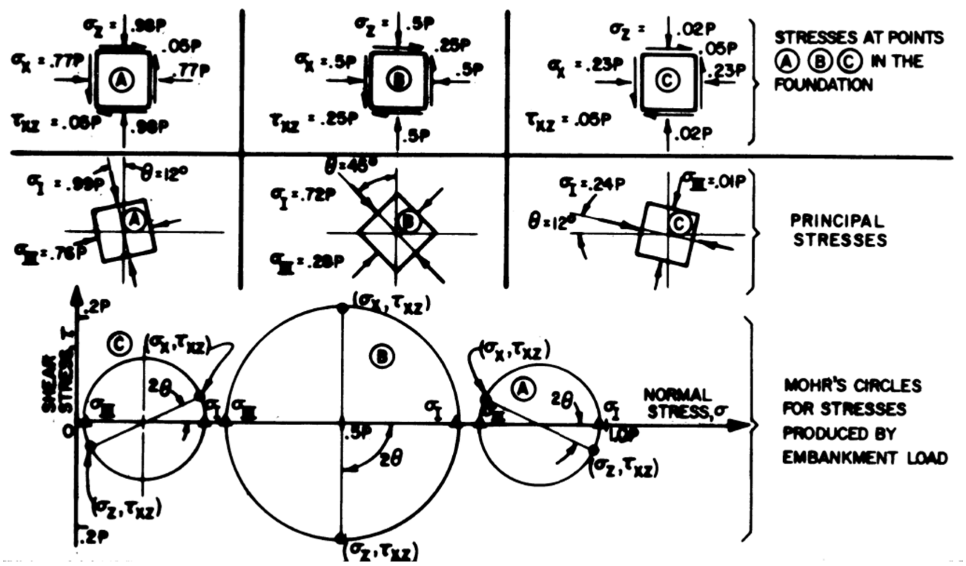


Figure 3: Mohr's Circle and Normal and Shear Stresses at Selected Locations (from [NAVFAC DM 7.01 (1982)])

Mohr's Circle is one of those things we learn about in mechanics of materials and work hard to forget about afterwards. But here it rears its ugly head. The horizontal axis is the normal stress and the vertical axis is the shear stress. If we know the normal stresses in the x- and z-directions and the shear stresses at any point, we can plot Mohr's Circle for any stress point and determine the stress state at any direction θ. There are two directions of special interest.

- The first is along the horizontal axis. The two points where the circle intersects that axis (labelled σ_I and σ_{III}) are the principal stresses.
- The second isn't quite along the vertical axis but at the top and bottom of the circles, and this is the maximum shear stress τ_{max}.

Both of these stress states are important and we will come back to them later. When computational power was limited, Mohr's Circle was actually used to graphically analyse the stress states. Today we do this analytically, using formulae such as

$$\sigma_I, \sigma_{III} = \frac{\sigma_x + \sigma_z}{2} \pm \sqrt{\left(\frac{\sigma_x + \sigma_z}{2}\right)^2 + \tau_{xz}^2} \quad (2)$$

$$\tau_{xz} = \frac{\sigma_I - \sigma_{III}}{2} \quad (3)$$

If we define the deviator stress as

$$\Delta\sigma = \sigma_I - \sigma_{III} \quad (4)$$

Equation 3 can be written as

$$\tau_{xz} = \frac{\Delta\sigma}{2} \quad (5)$$

So how often would a geotechnical engineer have to generate a plot like Figure 2 with all of these stresses? The answer is, not very, not at least by hand. In the past, elastic theory has been used to estimate how these stresses vary with both depth and distance from the load, and various formulae, charts and tables can be used. This is obviously a good problem to which FEA can be applied. We need to have some basic understanding of how stresses induced at the soil surface (or below it) are distributed and the stress levels that result.

But first, we need to back up a little and consider elastic and plastic stresses and the event that separates the two: failure.

2 Failure in Soils

It is evident that there are two things in life that most of us are unprepared for: success and failure. This is especially evident in a cyclical business like construction. If business is slow or non-existent, we go out of business for lack of revenue. If it is very good, we sometimes go out of business because we overextend ourselves.

With soils, we generally do not prepare them for failure except when failure is what's necessary. The best example of this is the installation of driven piles, when what we're trying to do is get the soil mass around the pile to fail with each blow of the hammer (or with the oscillation of a vibratory one.) Preparing soils for success is another matter. The oldest way of doing this is compaction, although now there is an array of soil improvement methods available.

2.1 What is Failure in Soils?

But now we must answer the question: what is failure? How do we define it? And once we've defined it, how do we prevent it? Let us look at Figures 4 and 5.

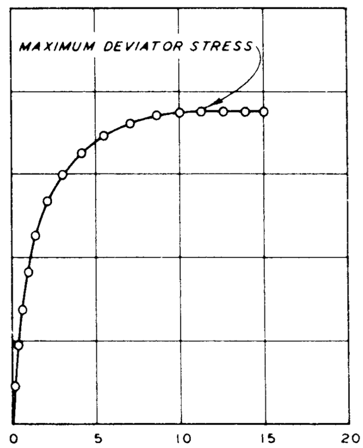


Figure 4: Ductile Failure from Direct Shear Test (from [EM 1110-2-1906 (1986)])

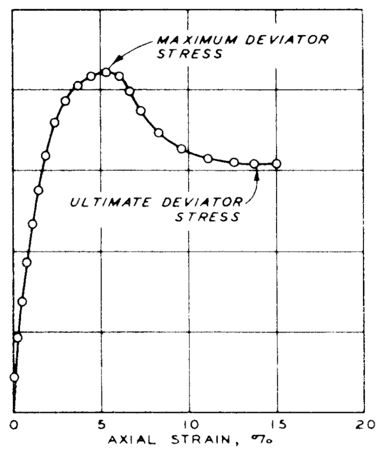


Figure 5: Brittle Failure from Direct Shear Test (from [EM 1110-2-1906 (1986)])

From both of these we get an idea that soil response to loads and stresses begins with "not so bad" and deteriorates to "really bad." There is no real elastic behaviour to be seen in either of these curves, especially with Figure 4. In the case of Figure 5, things start out somewhat better, but then reach a peak stress, from which the material actually experiences a stress drop before failure. In both cases we see a considerable step-up in complexity from simple elastic behaviour. Often our goal is to design something by avoiding stressing the material beyond its yield point. Especially with Figure 4, we would be hard pressed to even find a yield point!

Attempts have been made over the years to directly model this non-linear behaviour. The best known of these is the "hyperbolic" model generally associated with [Duncan and Chang (1970)]. The soil response with such a model is shown in Figure 6.

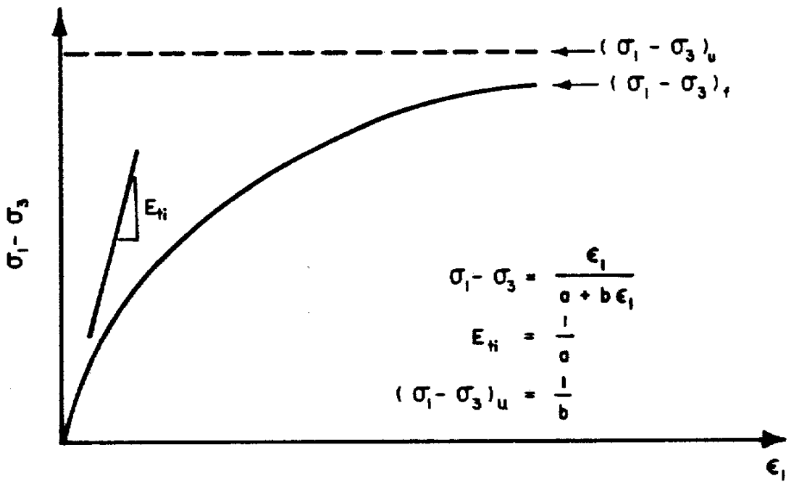


Figure 6: Hyperbolic Soil Model (from [EM 1110-1-1904 (1990)])

We can see the similarity of the response with Figure 4. Although this is so, it is complex to implement.

But can we, say, take a simple model and make it work in spite of its limitations? The answer to this is a qualified "yes," and we can see such a model in Figure 7.

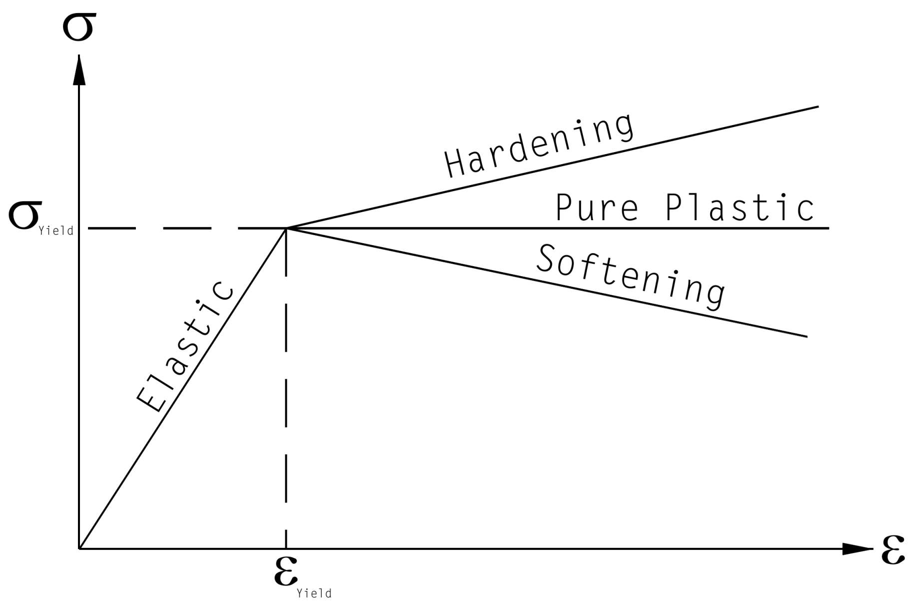


Figure 7: Elastic-Plastic Soil Model

Now we see two clear regions of soil response. To the left of the yield strain we have elastic behaviour, in accordance with Equation 6. As long as we stay in this region, the response is what we call "path independent," i.e., independent of the stress history of the soil in question. If we go to the right, we have plastic deformations, or permanent sets, and Equation 6 no longer applies. The yield strain becomes our failure point. As we saw with the actual results, the stress can continue to rise, remain constant, or fall with increasing strain.

Although this looks like an oversimplification–and in many ways it is–we can use this not only to effectively model soils in computational frameworks like finite element analysis but also use it to conceptually describe soil behaviour on a more simplistic basis.

There are two more things we should say about the plastic region, the one to the right of the yield strain. The first is that the three options we are given are obviously not the only ones we have available. A curved response is certainly possible. Linear responses, however, are obviously simpler to model. To make things even less complex, for the rest of this monograph we will only consider the case where stress does not increase past the yield strain, i.e., the "pure plastic" case. For many soils this is a conservative assumption.

The second is that, once the yield strain is exceeded, we generally assume that, if we decide to have mercy on the soil and stop increasing the strain, the stress will return along an elastic slope. An excellent example of this is the way we model pile shaft and toe static response during driving, as shown in Figure 8.

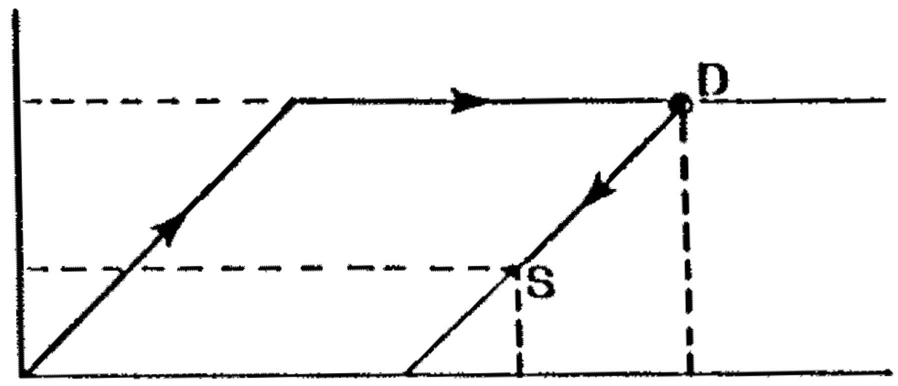


Figure 8: Elasto-Plastic Response After Yield (from [Goble and Rausche (1986)])

We see that the unloading DS is parallel to the elastic line before failure.

2.2 When Does Failure Take Place?

Now that we have at least a vague idea of what failure in soils is, we need to ask the next logical question: when does it take place? Do we need to run tests on every soil to determine that point of failure? Or can we make some estimate?

The question "when" implies time, which isn't what we're looking for at this point. What we are looking for is a stress-strain state at which point the soil goes beyond the elastic limit. Let's start with the simpler failure model.

We saw in Mohr's Circle three points of special interest: the two principal stresses σ_I and σ_{III} and the maximum shear stress τ_{max}. Both of these suggest a failure model.

One obvious candidate would be the maximum principal stress σ_I. For stresses in one direction it is very common to use this; if it exceeds yield, we have experienced yield failure (and sooner or later will experience ultimate failure if we keep increasing the stresses.) With stresses in more than one axis, this is problematic due to the greater complexity of the stress state, so we are more inclined to use other failure criteria.

Another possibility is the maximum shear stress. In fact, this model–generally called the Tresca criterion–is probably the most commonly used failure criterion in materials engineering. It's also easy to compute because, if we look at Mohr's Circle, the radius τ_{max} is half the diameter of $\sigma_I - \sigma_{III}$, or

$$\tau_{max} = \frac{\sigma_I - \sigma_{III}}{2} \tag{6}$$

With some soils, we can use this criterion to estimate the point of failure. However, the Tresca/maximum shear stress criterion does not include the effects of internal friction, i.e., the grains of soil rubbing against each other.

The most straightforward way of doing that brings us back to our friend Mohr, or more specifically the Mohr-Coulomb criterion of failure, which is

$$\tau_f = c + \sigma_n tan\phi \tag{7}$$

This is plotted in Figure 9.

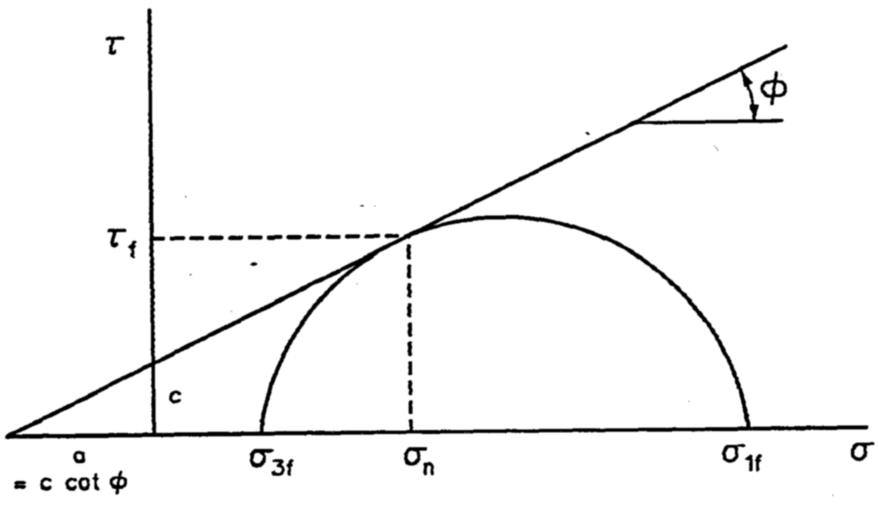


Figure 9: Mohr-Coulomb Failure Line (from [EM 1110-2-2504 (1994)])

The failure line (more commonly called an envelope) is, in theory at least, a solid line. Soils don't always obey this criterion (or any other for that matter) perfectly; variance from the ideal is very much the rule in soil mechanics. But it's a reasonable approximation.

With all of its parameters non-zero, it is a sloped line. There are two special cases that are very important in soil mechanics:

1. $\phi = 0$, the line becomes horizontal. Soils which meet this criterion are referred to as purely cohesive; the Tresca (maximum shear stress) criterion can be used to define their failure. Clays and, to a lesser extent, silts can be categorised in this way.

2. $c = 0$, the line begins at the origin and slopes upward. Soils which meet this criterion are referred to as purely cohesionless. Many sands and gravels (especially if they have no fines) can be categorised in this way.

It is not an overstatement to say that the theory of soil mechanics, as currently practised, is largely based on Mohr-Coulomb failure theory. Much of our soil characterisation activity, and the testing designed to perform that characterisation, is designed to obtain Mohr-Coulomb results. As [Abbo et. al. (2011)] point out:

> The Mohr–Coulomb yield criterion provides a relatively simple model for simulating the plastic behavior of soil. Other more sophisticated constitutive models for predicting the behavior of soil have been developed over the past three decades, however the complexity of these models, as well as the additional testing required to determine the various soil parameters involved, minimizes their utility for practicing geotechnical engineers. The Mohr–Coulomb yield function is also of importance to finite element researchers and practitioners as it forms the basis of many analytical solutions. These analytical solutions serve as crucial benchmarks for validating numerical algorithms and software.

This observation is supported by [McCarron (2013)].

Although it looks simple, Mohr-Coulomb failure theory has its complexities in implementation, whether that takes place on an elementary level or in finite element code. Some of these are addressed by its close cousin Drucker-Prager theory, although the jury is still very much out on whether it should replace Mohr-Coulomb or not. So we proceed with Mohr-Coulomb, limitations and all.

2.3 Putting the "What" and "When" Together

Now that we have defined how we plan to model the soil, what failure is and at what point does it take place, it's time to put this all together, and to do this we can use our Mohr's circle once again. But before we do a little overview of testing is in order.

If we look at Figure 2, we see that the stress state of a given point in the soil can be a very complex business. In fact this figure does not include effects of effective/overburden stress, stress history of the soil (principally preconsolidation) or many other effects. Attempting to simulate the stress state of soils in the ground in a laboratory setting is a difficult business. But try we must.

Looking at Figure 9, we see that failure is defined as a stress state which appears above the Mohr-Coulomb failure line (or beyond the envelope if we mirror image the line around the horizontal axis.) Ultimately this is about elevating the shear stress (vertical axis) to a point where the soil experiences failure. The most straightforward way of doing this is the direct shear tests, whose results can be seen in Figures 4 and 5. A diagram of this test is shown in Figure 10.

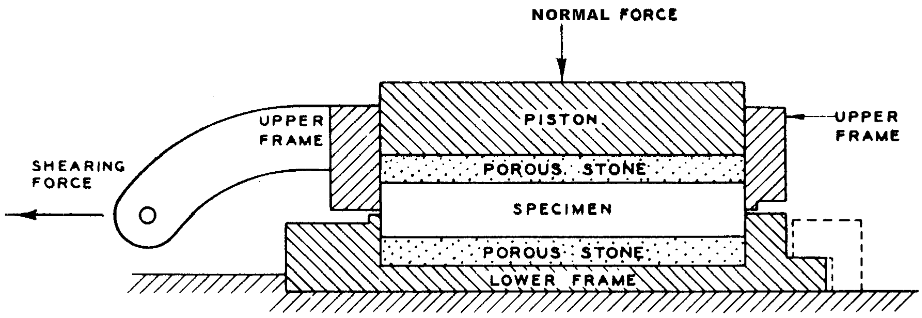


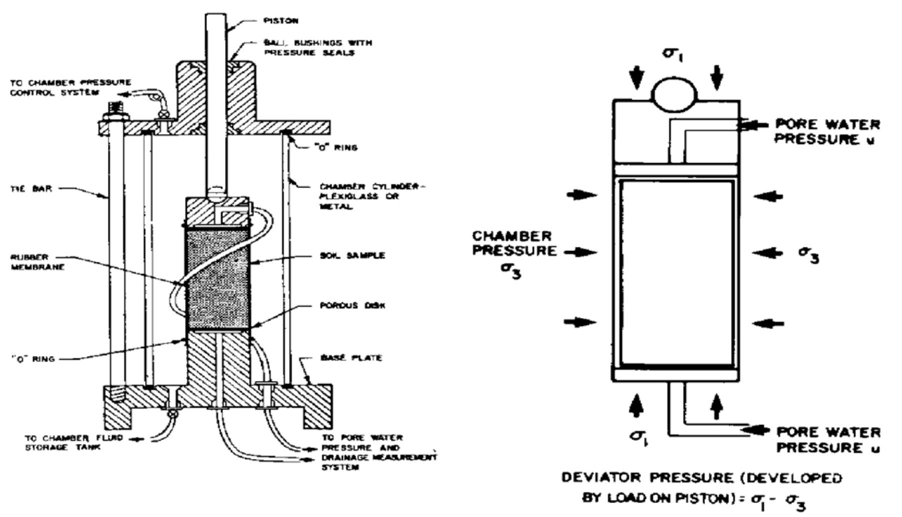
Figure 10: Direct Shear Test Schematic (from [EM 1110-2-1906 (1986)])

Although the direct shear test is simple to run and, in some cases, appropriately simulates the application of shear stresses, in reality it not used extensively. The direct shear test does not do two things which are necessary for most geotechnical applications:

1. Properly model the way in which shear stresses are generated. If we look at the situation in Figure 2, we see that the shear stresses take place because of the increase in normal stresses, not a direct occurrence of shear.

2. Does not simulate confinement as well as one would like. Again with the situation in Figure 2, we have an increase in the vertical stress. But we also have an increase in horizontal stress due to the fact that each point in the soil is confined by the surrounding soil. If you compress a specimen of material, it will bulge. But if it is confined, it cannot bulge, or not as much as it could otherwise.

The answer to these weaknesses is the triaxial test, which is still an important test for determining values of c, ϕ, and other quantities such as the modulus of elasticity. A diagram of the test is shown in Figure 11.



	UNCONSOLIDATED UNDRAINED TEST	CONSOLIDATED TESTS		
		CONSOLIDATION PHASE	CU	CD
σ_3	HELD CONSTANT	HELD CONSTANT	HELD CONSTANT	HELD CONSTANT
σ_1	GRADUALLY INCREASED FROM σ_3	EQUAL TO σ_3*	GRADUALLY INCREASED FROM σ_3	VERY GRADUALLY INCREASED FROM σ_3
u	DRAINAGE LINES CLOSED	DRAINAGE LINES OPEN **	NO WATER PERMITTED TO ESCAPE. PORE PRESSURE MEASURED FOR EFFECTIVE STRESS TESTS.	DRAINAGE LINES OPEN

* UNLESS ANISOTROPIC CONSOLIDATION IS TO BE EFFECTED
** IN BACK PRESSURED TESTS, PRESSURE IS SUPPLIED TO PORE LINES, BUT DRAINAGE IS PERMITTED

Figure 11: Triaxial Test (from [NAVFAC DM 7.01 (1982)])

A cylindrical soil sample is wrapped in a membrane and placed between a base and a piston. The specimen is immersed in water, which is pressurised to the desired confining stress (meant to simulate the lateral stress σ_{III}. The specimen is compressed by the piston to simulate a vertical stress σ_I. This latter stress is increased until failure is achieved. As the lower part of the figure indicates, a wide variety of drainage conditions can be simulated, as can a wide variety of pore pressure conditions. The triaxial test has been so successful that it is simulated frequently by finite element code developers as a test case! The result in either method is a stress-strain curve where we can determine the location of the "break" from what we model as elastic behaviour to plastic, and use the conditions under which that break took place along with Mohr-Coulomb theory to characterise the soil and predict failure under actual conditions.

So what do the results look like? Some samples of this can be shown in Figure 12.

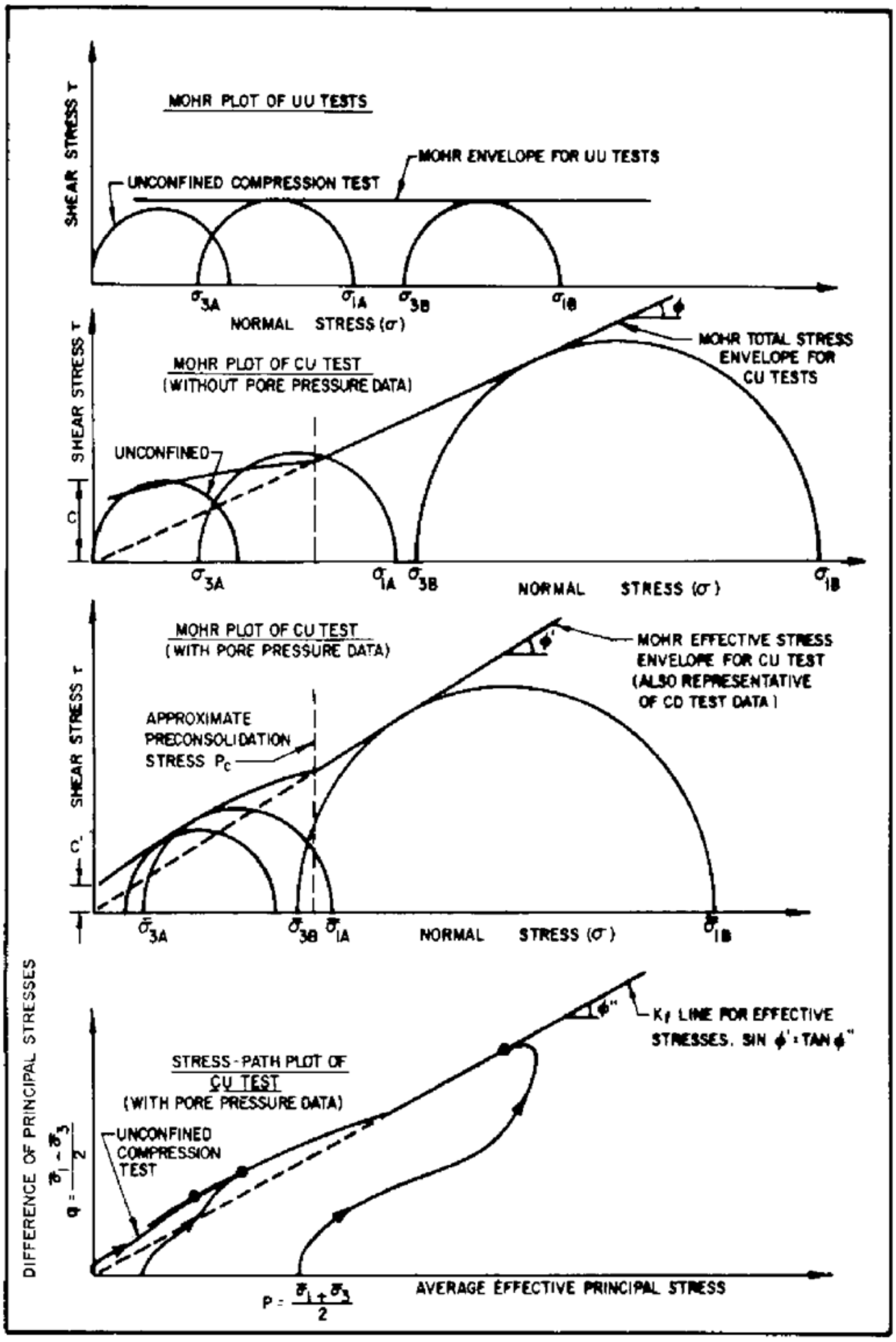


Figure 12: Triaxial Test Results (from [NAVFAC DM 7.01 (1982)])

At the top we see the result for a purely cohesive soil. Remember that we can simulate various values of confining stress σ_{III}. However, for such a soil, the deviator stress (and thus the maximum shear stress, per Equation 6) is constant, thus the failure line is likewise horizontal as Mohr-Coulomb failure theory would predict. If we have reason to believe that the soil is purely cohesive, we can dispense with the confining pressure altogether and determine the cohesion/maximum shear stress with one test. This is referred to as an unconfined compression test, and the result is either the cohesion or the unconfined compression strength (which is double the cohesion.)

The next two are more typical of cohesionless soils. The ideal for cohesionless and mixed soils is for the successive results of the triaxial test (varied by changing σ_{III} and thus σ_I) is to draw a line tangent to all of them and determine both the slope of the line $tan\phi$ and its y-intercept c. Unfortunately factors such as preconsolidation and non-linearities in the failure envelope make this difficult to achieve.

The last one gets us to stress-path theory. This is beyond the scope of this paper, but to model any non-linear (and thus non-path independent) material some understand of this is necessary.

3 Elastic and Plastic Behaviour in Soils

At this point we can make some justification for modelling the soils according to the scheme shown in Figure 7. But how do we implement it in practice? The answer is, "it depends." But first a quick note about effective stresses.

3.1 Effective Stresses

Probably the first type of soil stress civil engineering students are exposed to are effective stresses. These are the stresses that are due to the weight of the soil and its interaction with the pore water. The computation of effective stress is beyond the scope of this article, and more information can be obtained in [FHWA-NHI-06-088 (2006)]. A couple of interesting points can be made here, however,

The first is that it is a very common (if not often explicit) assumption that the confining stress at a point in a soil mass σ_{III} is in fact the effective stress exerted horizontally. If we consider the sequence of the triaxial test, once we load the specimen into the machine, we bring the confining stress σ_{III} up to the level of the effective stress we would like to simulate. In doing that we also have to bring the vertical stress σ_I to the same level to keep equilibrium. Once everything is equal and stable, then we increase σ_I until failure. In real situations, placing a load at or near the surface has the same effect: it raises the vertical stress σ_I to a higher level, possibly inducing failure.

The second is that modelling effective stresses in finite element analysis is one think that makes geotechnical FEA unique. Failure to do so will give disastrous results, so all viable commercial code has a method of simulating this. A description of this simulation is given in [Naylor et.al. (1981)].

3.2 Elasticity and Plasticity: the Forks in the Road

We must directly address the issue of elasticity and plasticity. How do we implement a model that includes both, as we should? The core of the problem is that, although elasto-plastic mechanics have been appreciated for a long time, except for the simplest cases it is very difficult to include both in "closed form" analytic calculations.

Yogi Berra used to say, "When you come to a fork in the road, take it." We actually have two forks in the road to consider.

The first is whether we plan to used "traditional" formula types of calculations or use a numerical method such as finite elements. Finite elements can handle both elasticity and plasticity at the same time by calculating which one is appropriate and how it should be implemented. Although some might consider simply jettisoning all traditional methods, such a solution is not really practical and potentially misleading, since we are now completely at the mercy of a computer program. We should also keep in mind that most finite element code in common use has the same underlying Mohr-Coulomb (and other) principles that the more conventional methods do. We can make the same (or worse) mistakes with finite element analysis as we can with conventional methods, especially if our soil data is spotty.

If we stick with traditional methods, we come to the second fork in the road: do we analyse elastically or plastically? The answer is "one or the other," depending upon the problem at hand. One of the things that baffles newcomers to geotechnical engineering is the mixture of various types of methods with no apparent relationship to each other. But there is method in this madness. Looking at Figure 7 once again, if we have a problem dominated by small deflections, elastic analysis makes sense to use. On the other hand, if large deflections (in many cases catastrophic) are anticipated, then plastic analysis is called for. A surprisingly large variety of problems can be solved by ignoring one regime and concentrating on the other.

3.3 Elastic Solutions

Going back once again to our triaxial test, we first load the ends and the sides of the specimen with the same confining stress σ_{III}, and then apply additional stress $\sigma_I - \sigma_{III}$ to the ends. With soils under a newly applied surface load, it is the same: we apply a load Q (total load) or q (load per unit) area to the surface, and then attempt to estimate the increase in stress Δp at a point of interest in the earth. We are probably not interested in inducing plastic deformation in the soil; we will have enough trouble with other forms of deformation, such as poroelasticity/consolidation, collapse and swelling not to want to risk elastic failure. Thus the change in soil stress at any point in the soil can be estimated using elastic theory.

3.3.1 Boussinesq Theory for Stresses

The most common theory for elastic estimations of changes in soil stress is Boussinesq theory. A simple way to illustrate this is to consider point loading as shown in Figure 13. The diagram on the left shows some basic geometric parameters, the equations on the right show the various stress types (normal or shear) and directions.)

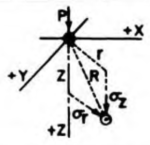
STRESS DIAGRAM	STRESS COMPONENT	EQUATION
	VERTICAL	$\sigma_z = -\frac{P}{2\pi R^2}\left[\frac{-3r^2 Z}{R^3} + \frac{(1-2\mu)R}{R+Z}\right]$
	HORIZONTAL	$\sigma_r = \frac{P}{2\pi}\left[3\frac{r^2 Z}{R^5} - (1-2\mu)\left(\frac{R-Z}{Rr^2}\right)\right]$
	SHEAR	$\tau_{rz} = \frac{3P}{2\pi} \cdot \frac{rZ^2}{R^5}$

Figure 13: Point Loading (from [NAVFAC DM 7.01 (1982)])

We notice several interesting things about this:

1. The vertical stress is at its maximum directly under the load. In this case, it is infinite, as the area is zero.

2. For a given depth Z the vertical stresses are at their maximum value directly under the load.

3. The horizontal and shear stresses directly under the load are zero.

4. The further away from the load (i.e., as R increases) the smaller the stresses.

Most engineers are interested in the increase in the vertical stress component, i.e., the Δp or deviator stress increase with the application of the load. As long as the ground is reasonably homogeneous and the elastic limit is not breached, elastic solutions such as Boussinesq and its relative Westergaard are very useful in computing stress changes in soils, which in turn are useful for analyses such as consolidation and other settlement estimates.

3.3.2 Methods of Implementing Boussinesq Theory

Several ways have been developed to estimate these stress increases.

The first (and traditionally most popular) are the chart solutions. An example of this for circles is found in Figure 14, and more of them (with instructions and examples of use) are found in [NAVFAC DM 7.01 (1982)]. Because theory of elasticity is being used, superposition applies, and solutions (i.e., the cumulative effect of neighbouring structures) can be added or, for voids, subtracted.

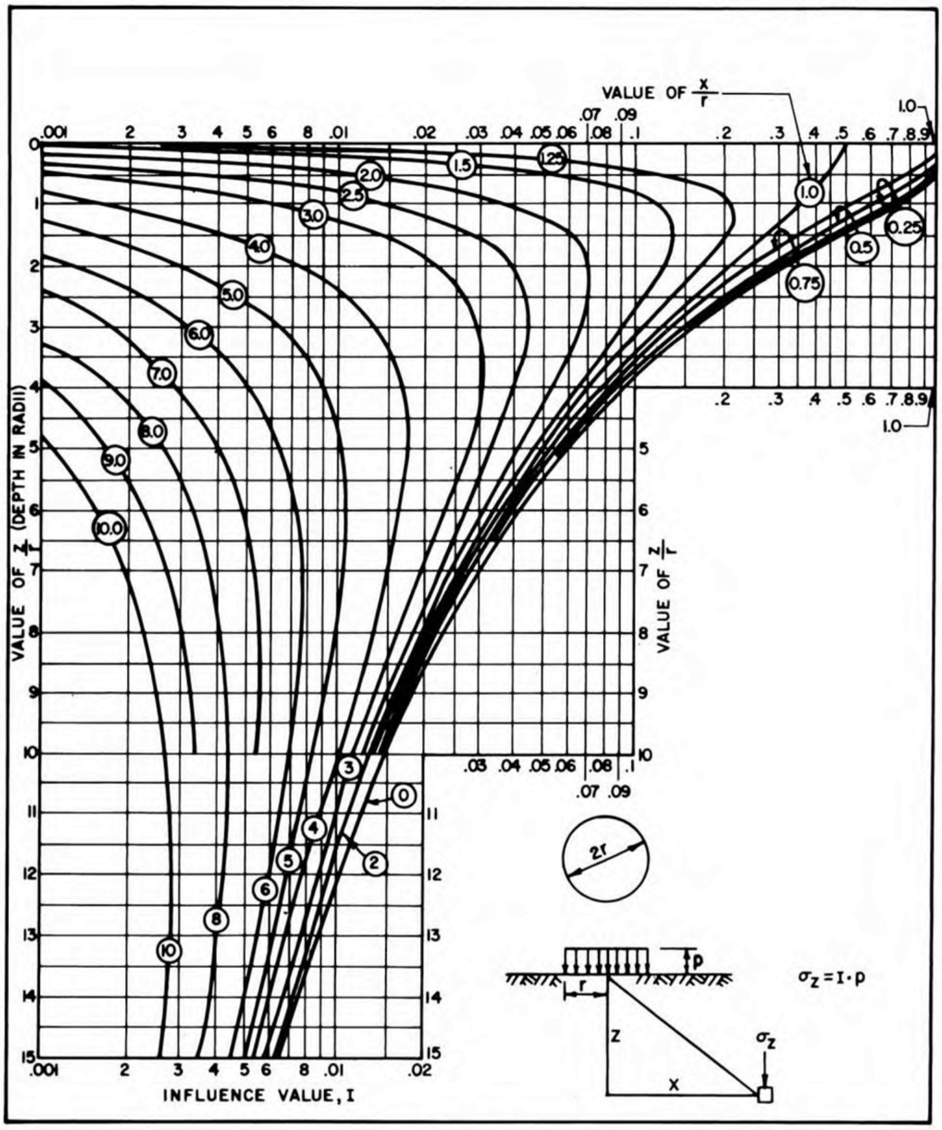


Figure 14: Boussinesq Chart for Circular Foundations (from [NAVFAC DM 7.01 (1982)])

The second is to use equations, as illustrated in Figure 13. While especially useful for, say, a spreadsheet, they are frequently wrong in the literature and thus should be used with care.

The third is Newmark's Method, shown in Figure 15. Although developed in the U.S., it is more popular in Europe, and is the most flexible non-computer method for computing stress increases in soils.

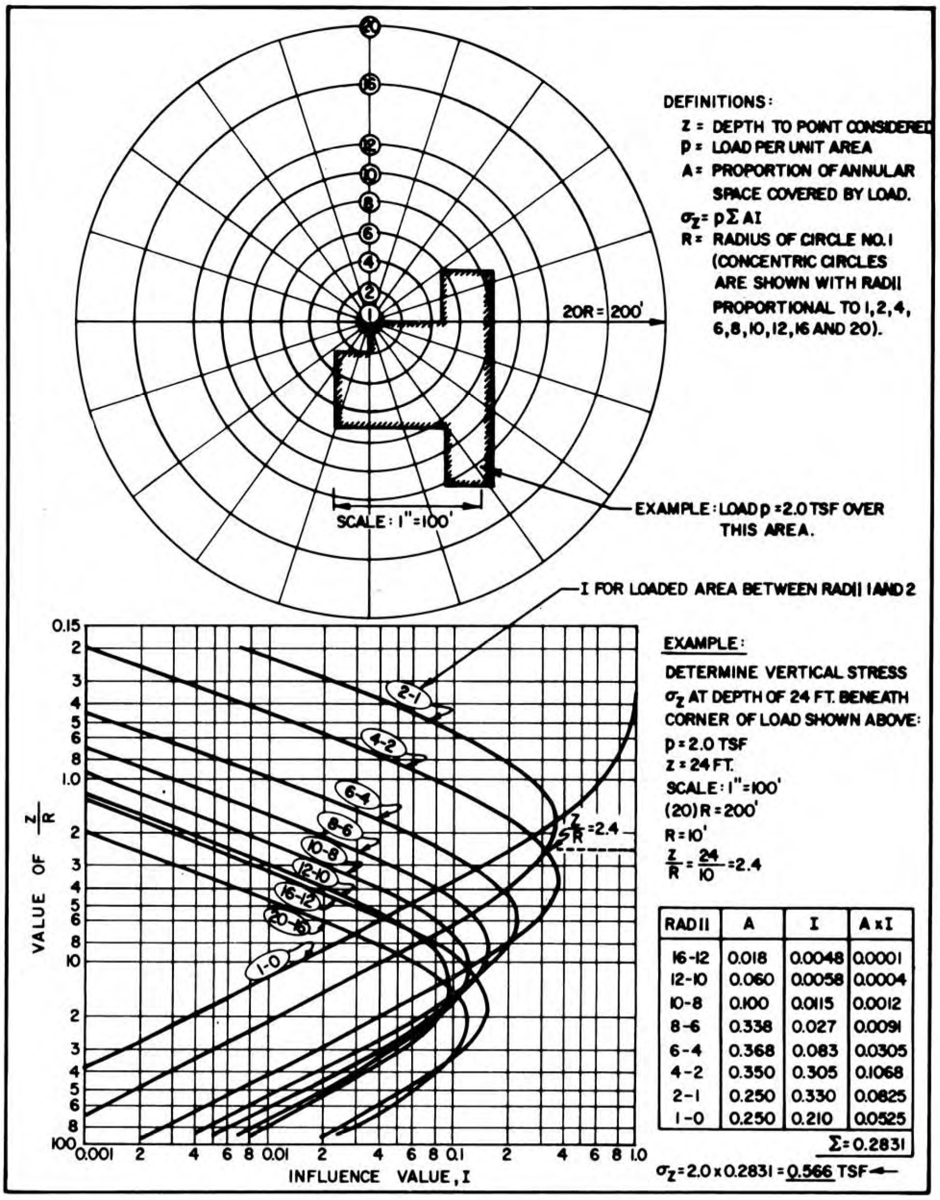


Figure 15: Newmark's Method (from [NAVFAC DM 7.01 (1982)])

Yet another approach is to use computer software. The results of one package (FoSSA) are shown in Figure 16 for an embankment loading. The software can be based on a number of theories: Boussinesq, Westergaard, or even the finite element method itself. The advantage of using software is that varying ground conditions can be modelled, and of course the effective stress can be likewise calculated for the soil profile.

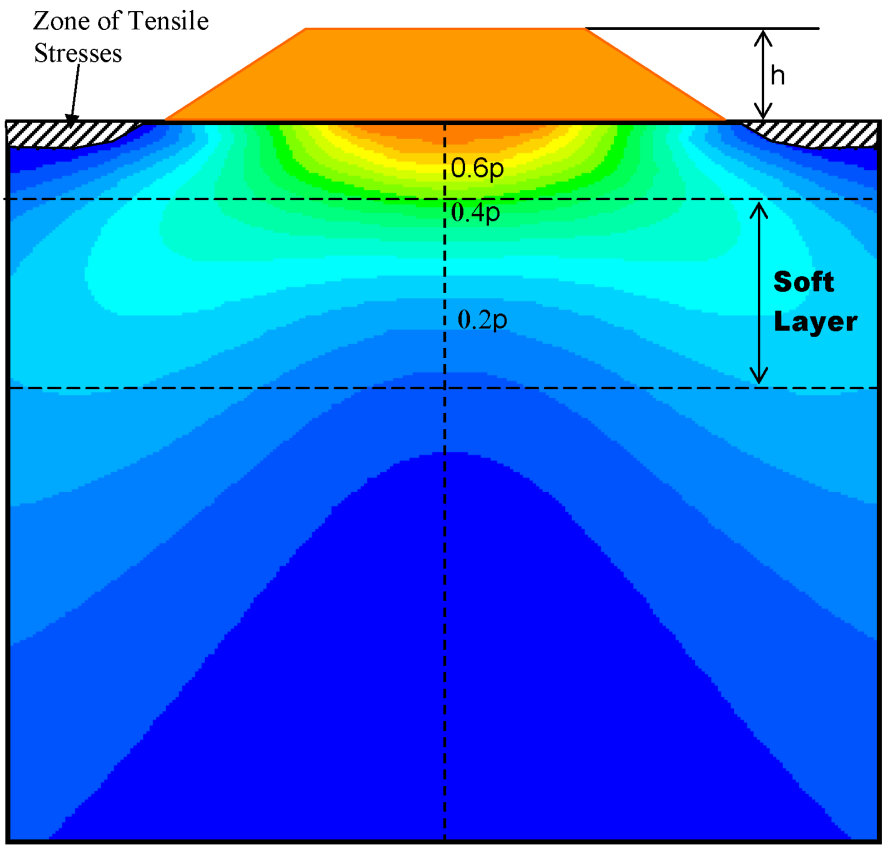


Figure 16: FoSSA Results for Embankment Loading (from [FHWA-NHI-06-088 (2006)])

3.3.3 Elastic Settlements

Up to this point we've talked about elastically induced (and estimated) stresses. But we can also estimate deflections using elastic theory. The elastic settlement induced by a surface load can be estimated using the equation

$$S = \frac{qBI}{E_s}\left(1-\mu^2\right) \quad (8)$$

where q is the load per unit area, B is the basic lateral dimension of the foundation (usually the width,) E_s is the modulus of elasticity, and μ is Poisson's Ratio. I, the influence factor, depends upon the geometry of the foundation and the depth from the surface where an "incompressible" (usually rock) layer is found. For cases where the soil can be assumed to be "infinite" in depth (the same assumption applies to the Boussinesq discussion earlier) influence factors can be found in Table 1.

Table 1: Influence Factors for Elastic Settlement, Infinitely Deep Soil Profile (from [NAVFAC DM 7.01 (1982)])

Shape and Rigidity Factor I for Loaded Areas on an Elastic Half-Space of Infinite Depth				
Shape and Rigidity	Center	Corner	Edge/Middle of Long Side	Average
Circle (flexible)	1.00		0.64	0.85
Circle (rigid)	0.79		0.79	0.79
Square (flexible)	1.12	0.56	0.76	0.95
Square (rigid)	0.82	0.82	0.82	0.82
Rectangle: (flexible) length/width				
2	1.53	0.76	1.12	1.30
5	2.10	1.05	1.68	1.82
10	2.56	1.28	2.10	2.24
Rectangle: (rigid) length/width				
2	1.12	1.12	1.12	1.12
5	1.6	1.6	1.6	1.6
10	2.0	2.0	2.0	2.0

Influence factors are also available for situations where the relative rigid layer is close to the surface.

Elastic methods are generally used to estimate the immediate settlement of a foundation and in conjunction with other types of settlement, such as consolidation. Some methods of settlement estimation, such as Schmertmann's Method for cohesionless soils, attempt to incorporate elastic settlement with pore water drainage/particle rearrangement settlement into one method, with varying degrees of success.

3.3.4 Horizontal Stresses

We mentioned that vertical stress increases caused by applied loads could be analysed by using elastic theory. With large soil masses, horizontal stresses are not of great interest. An entirely different situation exists with retaining walls, where loads that are developed on the surface near a retaining wall can appear in the horizontal pressure profile against the wall itself. Examples of this can be seen in Figure 17.

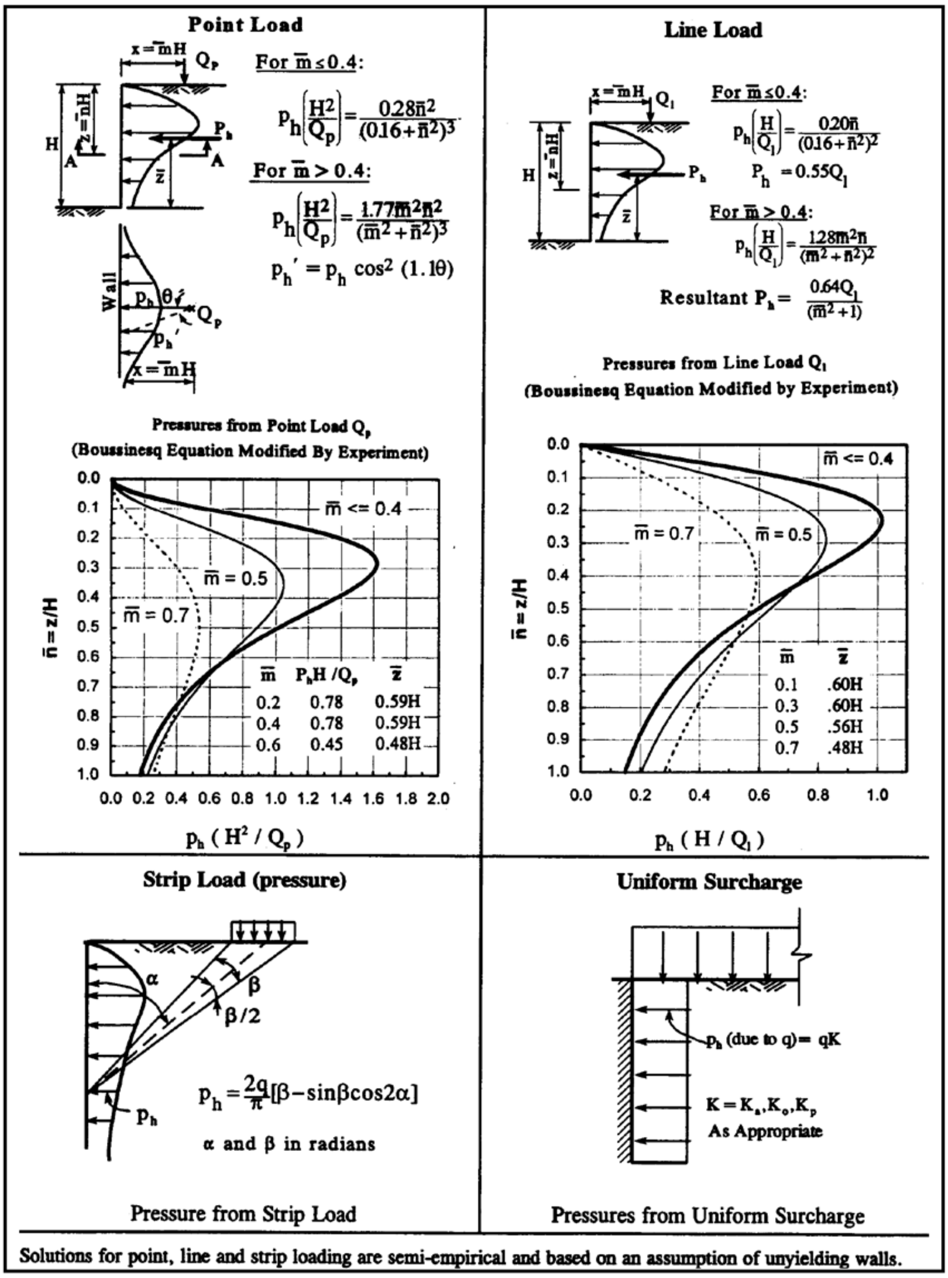


Figure 17: Lateral Stresses Estimated with Elastic Theory (from [FHWA-NHI-06-088 (2006)])

One "short cut" commonly used–and illustrated in Figure 17–is that pressure distributions on retaining walls can be treated as concentrated loads, which makes hand calculations considerably simpler.

While on the subject of lateral earth pressures, the existence of Poisson's Ratio implies that horizontal stresses and deflections in soils can be computed from vertical stresses using theory of elasticity. Although this is theoretically possible, most major failures due to lateral earth pressures are plastic in nature. These will be discussed in the next section.

3.4 Plastic Solutions

The one thing that separates elastic solutions from plastic solutions is the degree of deformation. The elastic solutions discussed earlier imply relatively small deflections. With plastic deformations large deformations follow, and the failure that goes with them is frequently catastrophic.

We have used extensively the triaxial test as an illustrative aid for the concepts presented. With most compressive specimens of soils and other materials, a progressively larger compressive load is applied until the column of material collapses. In practice, soils generally fail plastically along what we call "slip surfaces," i.e., surfaces where two soil masses move relative to each other to the point where the elastic limit is reached, at which point significant relative movement begins.

3.4.1 Slope Stability

The problem which really brought this to the attention of the profession–and the general public–is slope stability. An illustration of that kind of failure is shown in Figure 18.

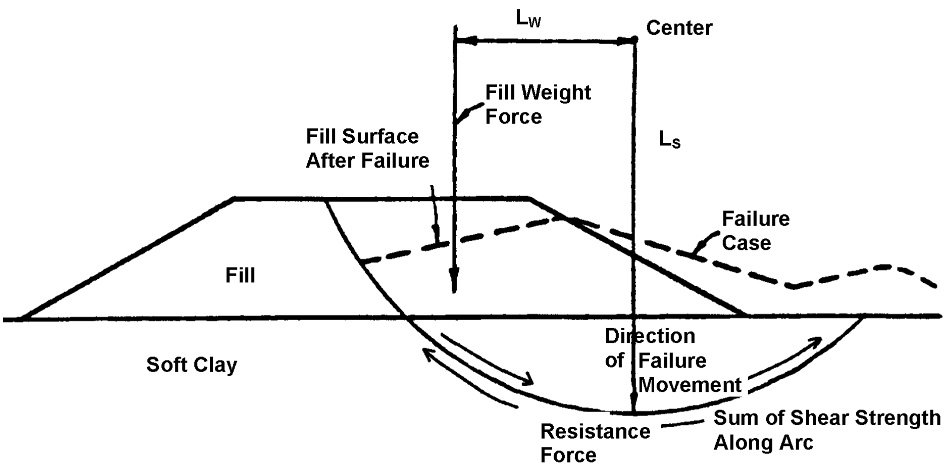


Figure 18: Slope Stability Failure (from [FHWA-NHI-06-088 (2006)])

The main driving force to induce failure in a slope is the weight of the slope soil itself, with help from whatever surcharge loading may be at the top of the slope. For most slope stability problems we assume a circular failure surface.

Equation 7 implies that there are two sources for shear strength at the critical failure surface: the soil cohesion and the product of the horizontal stress and the friction angle. Both of these (especially the latter) can vary along the surface, which complicate calculations considerably. There are several solution techniques to this problem. We can assume that the soil mass acts as a unit, which is fine for cohesive soils in simple profiles but not so good when things get complicated. We also have the method of slices, where we divide the slope into vertical slices which can then be analysed like giant elements. If we choose to use software, this makes finding the failure circle simpler, as it will iterate through the possible cases and find a solution. Detailed descriptions of these methods are beyond this article; a more thorough treatment can be found in [NAVFAC DM 7.01 (1982), FHWA-NHI-06-088 (2006)].

We can also apply finite element software to the problem. Finite element solutions take a different approach to finding the critical surface. Instead of making assumptions about the shape of the surface, the software progressively reduces soil strength parameters until failure is induced. It is not necessary to assume the surface or location of the failure surface, which is especially advantageous in profiles with large non-uniformities.

3.4.2 Bearing Capacity Failure

Another type of slip surface failure that is commonly analysed in geotechnical practice is bearing capacity failure, especially for shallow foundations. An example of this (for general shear failure) is shown in Figure 19.

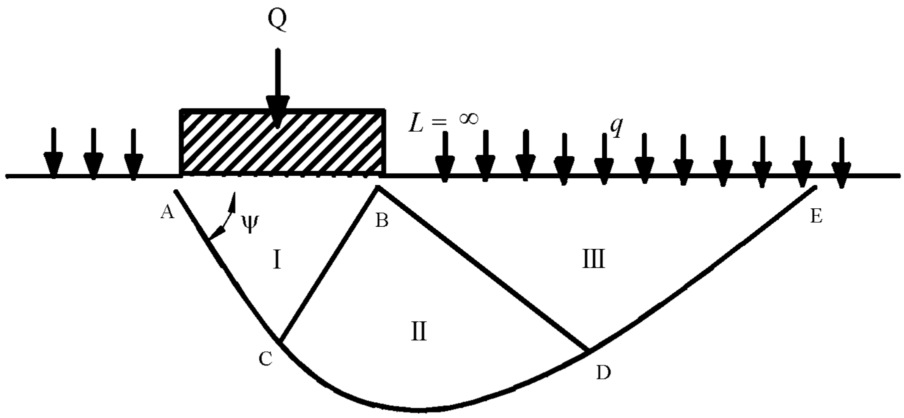


Figure 19: Bearing Capacity Failure Zones (from [Kimmerling (2002)])

As was the case with the elastic analysis, a shallow foundation applies a load to the soil. In addition to elastic and other types of settlement, a failure surface appears in the soil. It is in a log-spiral shape, a shape that appears often in nature. If the applied load results in a stress along the failure surface in the plastic zone, the foundation fails, generally by overturning. So we see that two very different types of responses come from the same applied load, and of course there can be plastic deformation in settlement, especially with softer soils, where the two failure modes are not so distinct.

3.4.3 Retaining Walls and Lateral Earth Pressures

Although generally not thought of in this way, lateral earth pressures are also related to plastic failure along a slip surface. Figure 20 shows a wedge between a retaining wall and a slip surface behind it. Depending upon the geometry, the well known (and somewhat complicated) lateral earth pressure coefficient equations can be computed from a static analysis of the wedge. In fact, the trial wedge method is commonly used to analyse lateral earth pressures with involved geometries such as broken backfill. A similar approach is used to determine the critical height of vertical cuts in cohesive soils, an important component of trench safety analysis.

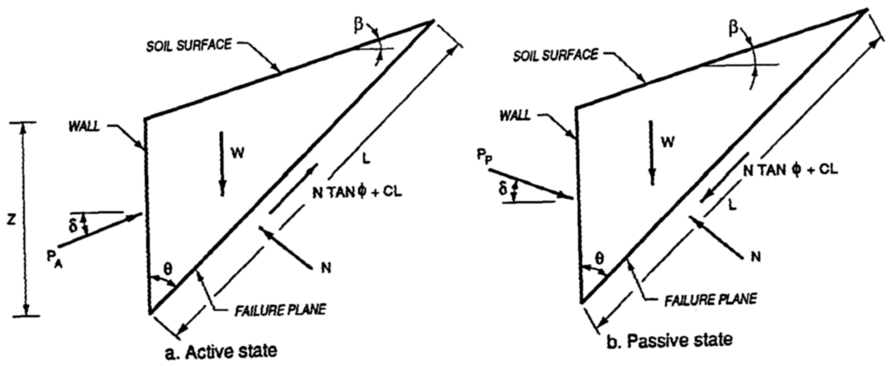


Figure 20: Wedge Analysis of Lateral Earth Pressures (from [EM 1110-2-2504 (1994)])

One assumption that is frequently challenged in retaining wall analysis is the shape of the slip surface. Figure 20 shows a planar surface, which is the assumption behind Rankine and Coulomb earth pressure theories. A log-spiral shaped slip surface–similar to what we saw with shallow foundations–can also be used. The downside is that these slip surfaces frequently defy a simple analytic solution, which means that we either have to use an approximation to determine the needed coefficients or use some kind of a chart or interpolation scheme. Sometime we can have it both ways. AASHTO, for example, recommends the use of Coulomb (planar slip surface) theory for active (wall moving away from soil) pressures, but then turns around and recommends log-spiral theory for passive (wall moving towards soil) pressures [FHWA-NHI-06-088 (2006)].

4 Finite Element Analyses in Geotechnical Engineering: Some Basics

We've referred often to finite element analysis. Many engineers look on the method as a "black box" (or maybe a "grey box" if they have been exposed to it at the undergraduate or graduate level.) This is a brief treatment of some of the concepts behind finite element analysis and how they apply to geotechnical problems.

Finite element analysis, in general, involves several steps [Moaveni (1999)]:

1. Discretise the region to be analysed into finite elements. More mathematically oriented types would use the term "solution domain," but that brings up an important advantage of finite elements: mathematically, each element is a "complete system."

2. Assume a solution that approximates the behaviour of an element. Because of the nature of elements, that approximation doesn't have to be the same for all elements. With care, you can mix and match them. Sometimes the approximation is exact, as is the case with spar and beam elements. There are many approximation schemes, but most solid mechanics applications use a Galerkin method.

3. Develop element equations, from (1) and (2).

4. Assemble the elements mathematically to form a solution of the entire problem.

5. Apply boundary conditions and loads. With solid mechanics we generally think of loads as forces and moments, but they can also be temperatures, etc. Boundary conditions have always been tricky in geotechnical engineering, where we are attempting to simulate the behaviour of semi-infinite masses with very finite computers. There are several ways of dealing with this problem, from special boundary formulations to simply making the system big enough so that the boundary doesn't matter.

6. Solve the system of equations simultaneously. With static linear systems, this can be done in one shot. Geotechnical problems are not linear and frequently not static either, so more effort is required.

7. Extract the information from the solution in tabular and graphic format.

Armed with this information, we can proceed. Consider the finite element model of a pile and soil system shown in Figure 21.

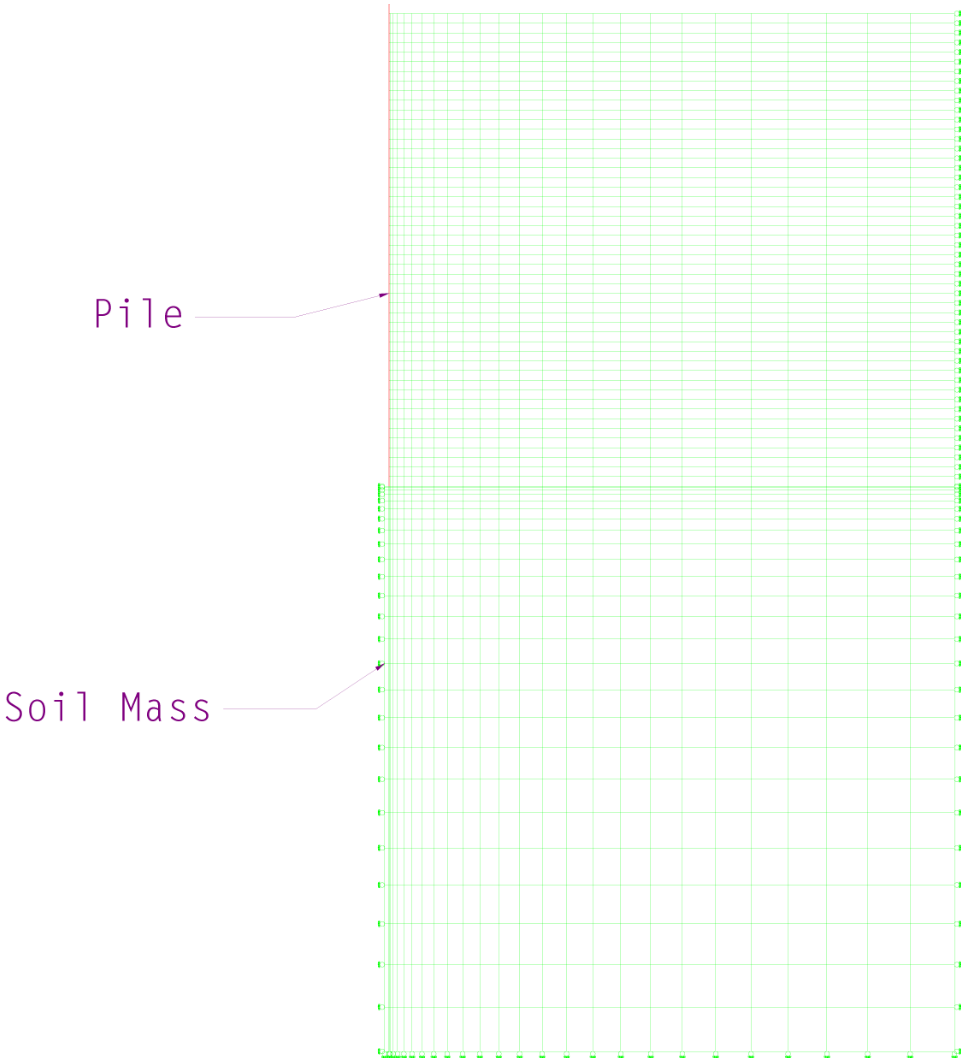


Figure 21: Finite Element Model of Pile and Soil System

The pile located towards the left edge of the model. So, you ask, where's the other half? This is an axisymmetric representation of the system. As we said earlier, to reduce three-dimensional reality to two-dimensional analysis we have to make some reasonable reduction in the system. As shown in Figure 22, in general there are three ways to accomplish this:

1. Plane Stress: the thickness is finite and there is no stress in the z-axis (into the paper or screen) direction. Plane stress assumption is more common with relative thin mechanical members and is used infrequently in geotechnical applications.

2. Plane Strain: the thickness is infinite and there is no strain in the z-axis direction. This is, in reality, just about the "standard" case for two-dimensional geotechnical representations, finite element and otherwise.

Although the thickness is theoretically infinite, in practical application we assume a unit thickness, so we have loads per unit length of wall, continuous foundation, etc.

3. Axisymmetric: the elements wrap themselves around a centre axis, thus they can be analysed in two dimensions. This is very commonly used in deep foundation problems, as is the case in Figure 21.

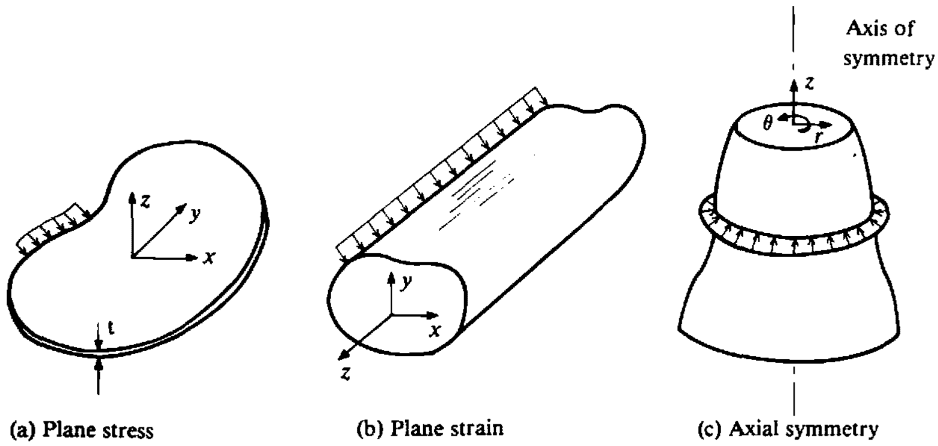


Figure 22: Two-Dimensional Representations (after [Owen and Hinton (1980)])

The lines shown in Figure 21 are divisions between elements. For obvious reasons, the elements shown are quadrilateral elements. The simplest quadrilateral element has four nodes, one at each corner, which both define its geometry and connect the element with its neighbours. A popular element in geotechnical engineering is the eight-node "serendipity" quadrilateral element, with four nodes at the corners and four additional nodes at the midpoints of the element borders. For greater accuracy, we can add another node at the centre of the elements for a nine-node quadrilateral element.

Having constructed this model, now what? As is the case with closed form solutions, we have to determine what we are solving for. For a static model of a solid system, that's easy: we want to sum the forces to zero, as we do in statics. We want not only to do this at the boundaries, but for every node in the system. At the boundaries, we have the reactions, some or all of which are non-zero for a system with forces applied to it. But the interior nodes also have forces from the elasticity of the system. We write this as follows:

$$F_i + F_e = 0 \tag{9}$$

where F_i are the internal forces and F_e are the external forces. (The external forces include gravity, which acts throughout the system.) These forces are not simply scalars but vectors, a series of values for every node and degree of freedom (there are two for each node, in x- and z-directions) in the system.

For a linear system, we reformulate Equation 10 to compute the internal forces as

$$F_i = K_t d \tag{10}$$

The variable d is for the displacements, which are a mixture of x- and z-direction displacements. K_t is the tangential stiffness matrix, the "spring constant" of the system. Substituting we have

$$K_t d + F_e = 0 \tag{11}$$

For a purely linear elastic system, this can be solved in one step. If there is non-linearity, it must be done iteratively. The significance of the K_t is now apparent: with a non-linear situation, the stiffness of the system becomes a moving target, and we search for the actual stiffness at each step of the analysis. In some cases we can get around tinkering with K_t every iteration, but the objective is the same.

So how do we know when non-linearity takes place? The analysis will yield both the deflections at the nodes and the stresses at the elements. Basically, if the principal stresses induce a stress state so that Mohr's Circle crosses the Mohr-Coulomb failure line shown in Figure 9, then the element (or integration point) is assumed to have entered the plastic state, and the stress is limited accordingly. It's the same as with hand calculations, although there some complicating factors:

1. The behaviour in the plastic region can be hardening or softening rather than purely plastic, but this complicates the analysis, especially if the element stresses reduce, in which case we have the phenomenon of Figure 8.

2. Inspection of Figure 9 shows that the principal stress combination that produces failure is not unique; there are an infinite number of them possible in any situation. To choose the "correct" one, in addition to the failure function we use the plastic potential function, which utilises the dilitancy of the soil. If the dilitancy angle $\psi = \phi$, we have what is called an associated flow rule, and K_t remains symmetric and thus easier to analyse. With most soils with any internal friction, $\psi \neq \phi$ and we have a non-associated flow rule, which leads to an asymmetric K_t and thus more computational expense.

For dynamic problems, Equation 11 can be modified to

$$K_t d + F_e = M\ddot{d} \tag{12}$$

where M is the distributed mass and $\ddot{d}$ is the acceleration of the mass at any point. If this looks suspiciously like Newton's Second Law, that's because it is!

Although the concepts are simple, their implementation involves many mathematical and physical implementations that are beyond the scope of this paper. But the existence of these complications should not obscure the basic simplicity of the concept and its relation to more conventional solution techniques.

5 Conclusions

Stresses in soils is an important topic, and we have only scratched the surface here. Hopefully the perspective we have used will make it easier for you to understand these stresses. We have skimmed over some imporant topics such as effective stresses or the details of slip surface failure; more information on both can be found in [NAVFAC DM 7.01 (1982)] and [FHWA-NHI-06-088 (2006)]. And we hope that, if you employ finite element software for geotechnical analysis, you will go into it with a greater understanding for this method, too.

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