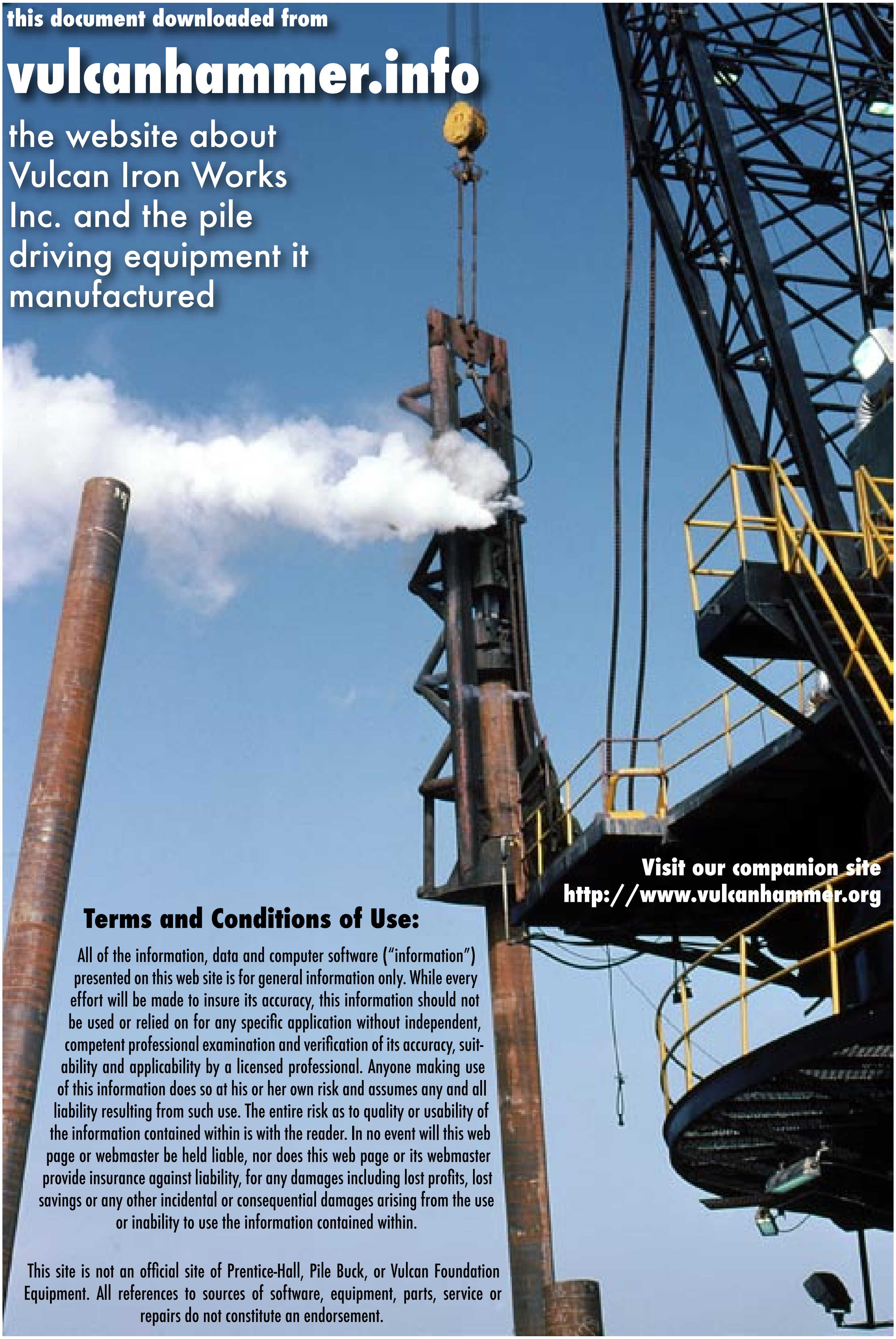

Deflections of Pile Toe Plates on Elastic Foundations

Don C. Warrington
The Wave Equation Page for Piling[1]

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Abstract

This paper is an analysis of pile toe plates that are assumed to interact with elastic foundations. A solution to the deflection and moment equations is derived and discovered to be in fact made up of Bessel functions with complex arguments. A solution based on the analysis of the series that make up the Bessel functions is performed. The solution is presented in the form of charts based on dimensionless parameters. A sample case is analysed and discussed.

Statement of the Problem

Consider the case of the pipe pile toe shown in Figure 1, with outside diameter D. The wall thickness of the pile is considered small relative to the diameter. The pile is closed ended with a flat plate with thickness h. The plate is considered thin enough relative to its diameter to be modelled by thin plate considerations. It is welded to the pipe in such a way that the toe plate is considered to be rigidly clamped at the edge. The plate is assumed to be made of metal with ν=0.3.

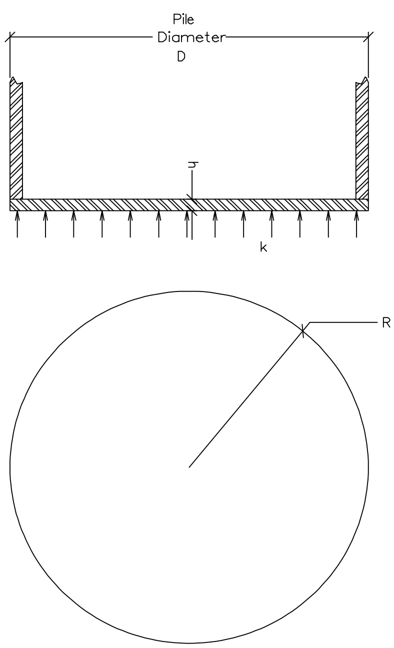


Figure 1
Basic Configuration of Pile Toe

The toe plate interfaces with a Winkler type soil of linear subgrade modulus k. This k is considered to be uniform over the entire pile toe.

The toe is loaded with load P which is applied from the pile. The load is uniform around the circumference of the pipe. Alternately a settlement s can be applied and the load computed from the settlement. This represents the load not transferred to the pile shaft.

[1] http://www.geocities.com/CapeCanaveral/Hangar/2955. All of the information, data and computer software ("information") presented on this web site is for *general information only*. While every effort will be made to insure its accuracy, this information should not be used or relied on for any specific application without independent, competent professional examination and verification of its accuracy, suitability and applicability by a licensed professional. Anyone making use of this information does so at his or her own risk and assumes any and all liability resulting from such use. The entire risk as to quality or usability of the information contained within is with the reader. In no event will this web page or webmaster be held liable, nor does this web page or webmaster provide insurance against liability, for any damages including lost profits, lost savings or any other incidental or consequential damages arising from the use or inability to use the information contained within.

Using Maple V Release 3 mathematical software, the deflection of the plate, the moments and the stresses in the plate are to be computed. Actual test cases will be based on reasonable geotechnical considerations.

Solution of the Plate Equation

We first define the radius of the plate by

R = D/2 .... (1)

where R = plate radius, m
D = plate diameter, m

To generalize the results, we make the principal variables dimensionless. We do this by making the radius R the reference dimension of the system; all other length dimensions are thus in "units" of R, or dimensionless. Thus,

$$r = \frac{r}{R} \qquad (2)$$

and

$$w = \frac{v}{R} \qquad (3)$$

and also

$$d = \frac{s}{R} \qquad (4)$$

where r = dimensionless radius
w = dimensionless lateral deflection
δ = dimensionless settlement of pile at outer edge of plate
ρ = distance from the centre of the plate to a given point on the plate, m
ϖ = deflection of the plate at a given point, m
s = pile settlement at rim, m

Now that we have made the principal length quantities dimensionless, we can generalize the results so they may be applied to a wide variety of cases. According to Vinson (1974), for an axially symmetrical plate with symmetrical loading, the plate deflection equation (with some rearranging) can be given by the equation

$$\nabla^4 w = R^3 \nabla^4 v = \frac{1}{r}\frac{d}{dr}\left\{ r\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)\right]\right\} \qquad (5)$$

Usually, this is equated to the function of the lateral loading; however, in the case of an elastic foundation, the lateral loading is a function of the stiffness of the foundation and the deflection of the plate. Again from Vinson (1974), for plates with no other form of lateral loading, this means that

$$\nabla^4 w = \frac{1}{r}\frac{d}{dr}\left\{ r\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)\right]\right\} = -\frac{kwR^4}{D} \qquad (6)$$

where k = linear foundation modulus, N/m³.

D is of course

$$D = \frac{Eh^3}{12(1-n^2)} \quad (7)$$

where D = plate stiffness variable, J
E = Young's Modulus of the plate material, Pa
ν = Poisson's Ratio for plate material.

Expanding Equation (6), we have

$$rw' - r^2 w'' + 2r^3 w''' + r^4 (w'''' + b^4) = 0 \quad (8)$$

where

$$b = \sqrt[4]{\frac{kR^4}{D}} \quad (9)$$

The solution of (8) (Murphy, 1960) is

$$w = C_1 J_0(zr) + C_2 Y_0(zr) + C_3 J_0(izr) + C_4 Y_0(izr) \quad (10)$$

where J_0 and Y_0 are Bessel functions and

$$z = be^{\frac{i(2n+1)p}{4}}, n = 0,1,2 or 3 \quad (11)$$

The coefficients C_2 and C_4 can be immediately set to zero as Y_0 is unbounded at r=0. This leaves the equation for w as

$$w = C_1 J_0(zr) + C_3 J_0(izr) \quad (12)$$

and the first derivative of w as

$$\frac{dw}{dr} = -zC_1 J_1(zr) - izC_3 J_1(izr) \quad (12)$$

We use Equations (11) and (12) for solving for C_1 and C_3 because the boundary conditions for the problem are

$$w\big|_{r=1} = d \quad (14)$$

and

$$\left.\frac{dw}{dr}\right|_{r=1} = 0 \quad (15)$$

With Equation (14), the given value for the edge deflection is set rather than a force because the actual force for any deflection will vary depending upon the uneven deflection of the plate. In effect we are solving this problem in reverse, i.e., giving a deflection and computing a force, but this simplifies the solution, as we will see. Equation (15) reflects the fact that the outside radius (r=1) is fixed.

Although the occurrence of Bessel functions in such a problem is not unexpected, the appearance of complex arguments of Bessel functions complicates matters. To solve this problem, it is best to start with the series definition of the Bessel function J_0, which is

$$J_0(zr) = \sum_{m=1}^{\infty} \frac{(-1)^m (zr)^{2m}}{2^{2m} m!^2} \quad \text{(16a)}$$

$$J_0(izr) = \sum_{m=1}^{\infty} \frac{(-1)^m (izr)^{2m}}{2^{2m} m!^2} = I_0(zr) \quad \text{(16b)}$$

To manipulate the series effectively, Maple V was employed. Because of the fact that series were used and the number of terms actually used (30), the actual manipulation of the equations is not fully reproduced but summarized here. The steps are as follows:

1. Because any expansion of (16) with ß unknown would result in an infinite power series in both beta and r, a value of ß is assumed. The range of betas used is discussed in the next section.
2. The series is expanded for both of the Bessel functions in Equation (12). This results in two series with both real and imaginary terms. These are substituted into Equation (12).
3. Equation (12) with the series terms is termwise differentiated to obtain w'.
4. The value of r=1 is substituted into both of these equations generated by steps (2) and (3).
5. The expression for w is equated to the initial deflection of the edge δ and the expression for w' is equated to zero.
6. For each case (w and w') values for C_1 are determined. These become complex numbers which are functions of C_3. Fractional arithmetic with integer denominators and numerators are maintained as long as possible in order to insure accuracy in calculation.
7. The two values of C_1 are subtracted from each other and set to zero (they should be the same). C_3 is then solved for.
8. The calculated value of C_3 is substituted into one of the values of C_1 and C_1 is solved for.
9. The values for C_1 and C_3 are substituted into the series version of Equation (12). A series results which is a) a linear function of delta (see Equation (25)) and b) a real function of r (all imaginary terms drop out) results.

It is noteworthy that in selected cases this result is substituted into Equation (8) to check whether or not it is a solution of the equation. In each case the result shows only a residual error in the highest terms of the series, indicating series truncation error.

Once a valid expression for w is determined, it is converted into floating point form in Maple V for computational purposes.

Computation of Moments and Shears

Once a valid expression of w is determined for a given beta, the moments and shears are given. As was the case with w and r, these are first determined dimensionlessly. The dimensionless values for the moments and shears are (Vinson, 1974)

$$M_r = w'' + \frac{nw'}{r} \quad \text{.......(17)}$$

$$M_q = \frac{w'}{r} + nw'' \quad \text{.......(18)}$$

$$Q_r = (\nabla^2 w)' \quad \text{.......(19)}$$

Again the series form of Equation (12) is substituted for each quantity. Because the case is axisymmetric, any derivative of theta is zero; thus, in this case

$$V_r\big|_{r=1} = Q_r\big|_{r=1} \quad \text{.......(20)}$$

and thus the total force exerted by the pile on the plate (and thus the soil) is given by the expression

$$V_{Rtot} = 2pV_r\big|_{r=1} \quad \text{.......(21)}$$

As is the case with w, Equation (21) is linear function of delta (see Equations (26) and (27)).

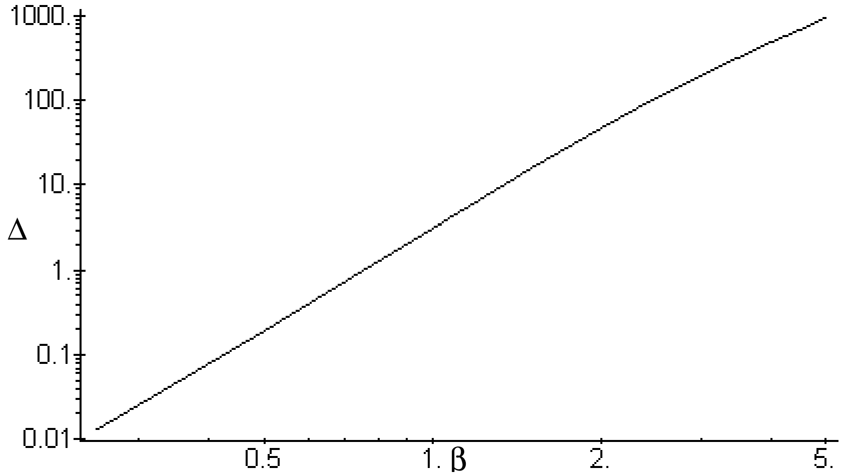
The total shear load at the radius R is important because, in the case where the pile toe load is known rather than the settlement, it is possible to estimate the settlement required for a particular load if the "delta ratio" for various values of ß is known. Figure 2 shows the relationship between these two values.

If the toe load and beta is known, the value of the dimensionless settlement can be computed by the relationship

$$d = \frac{V_{Rtot}}{\Delta} \quad \text{.......(21)}$$

where Δ = "delta ratio" as shown in Figure 2.

Figure 2
Plot of Delta Ratio for Various Values of Beta



Estimation of Possible Values of Beta

Because of the nature of the infinite series used to solve the equations, it is necessary to solve the equation for certain given values of ß and to construct a series of solutions for this set of values.

According to Das (1984), the value of the soil subgrade modulus k can be approximated by the equation

$$k = \frac{E_s}{2R(1-n_s^2)} \quad \text{.....} (22)$$

where E_s = Young's modulus of soil, Pa
ν_s = soil Poisson's Ratio

Substituting this and Equation (7) into Equation (9), we have

$$b = \sqrt[4]{\frac{6E_s(1-n_p^2)}{E_p(1-n_s^2)}\left(\frac{R}{h}\right)^3} \quad \text{.....} (23)$$

where E_p, ν_p are Young's modulus and the Poisson's ratio for the pile material.

Although the values of soil Poisson's Ratio vary considerably (from 0.15 to 0.50), most gravitate around the values for steel, the usual material for a bottom plate (ν=0.3). Since we are not determining here an exact value but attempting to determine a range, we will assume that the Poisson's ratios for pile and soil are the same and that they cancel each other in (23). Also, it should be noted that Equation (22) is not the only value permissible for k; it can be estimated by other methods as well. Our purpose here is to establish a range of plausible values of beta.

With these things in mind, we can simplify (23) to

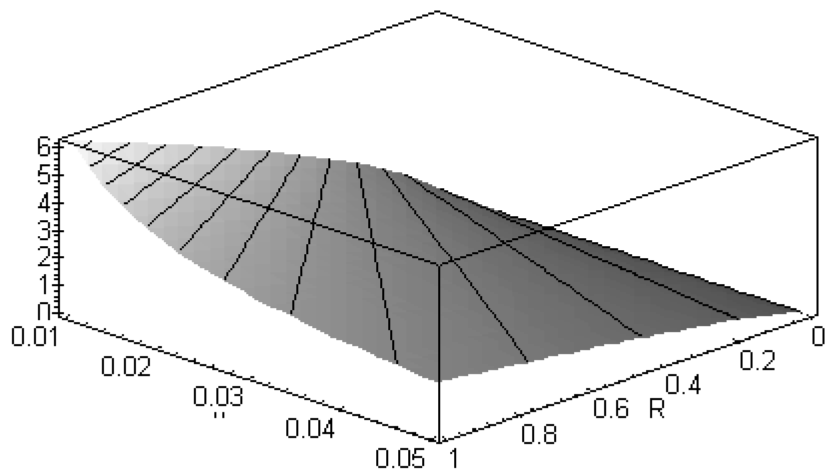
$$b = \sqrt[4]{\frac{6E_s}{E_p}\left(\frac{R}{h}\right)^3} \quad \text{......} (24)$$

The Young's modulus for the pile material can be taken to be E_p = 200 GPa. Although as usual the Young's modulus for the soil varies widely, a value of E_s = 50 MPa is assumed. This is higher that most soils and will produce candidate values of beta which are also high.

To generate a plot which will enable us to estimate candidate values of beta, we varied R from 0 to 1 meter and h from 10 to 50 mm, which is a reasonable range of possible bottom plates used in piling. All of these values are inserted into Equation (24) and a plot of estimated beta values is shown in Figure 3.

Figure 3

Plot of Estimated Beta Values for Various R and h



Based on this result, it was decided to compute w, the moments and the delta ratio from the following values of beta:

1/4	1	5/2	4
1/2	3/2	3	9/2
3/4	2	7/2	5

Although some of the values indicated exceed five, it is unlikely, for example, that a 10mm thick bottom plate be used with a 2 m diameter (1 m radius) pile. The data generated from the various values of beta is given and explained in the next section.

Presentation and Use of the Results

To calculate desired deflections, moments and stresses from the data, it is necessary to realize that for each beta chosen two charts were generated: 1) a plot of dimensionless plate deflection w vs. dimensionless plate radius r, and 2) a plot of dimensionless plate moments M_r and M_θ vs. dimensionless plate radius r. All of the other desired quantities are computed from those given in the chart.

The procedure to use the charts is as follows:

1. Determine the values of k, R, E_p, h and ν_p using a consistent set of units. Determine the settlement from zero load s as well.
2. Compute the value of beta using Equation (9).
3. Compute the value of δ using Equation (4).
4. Select the chart(s) desired from this value. Two charts and linear interpolation can be used if necessary, since all charts are drawn on a consistent, dimensionless, basis.
5. Select a value or values of r (Equation 2) where results are desired. Maximum deflection and moments generally occur at the centre of the plate (r=0), except in the higher values of beta (stiffer soils).
6. Read the values of w, M_r and M_θ from the charts. For the moment charts, the values of M_r *are the solid lines* while the values of M_q *are the broken lines*.
7. Perform additional chart readings and linear interpolation if necessary.
8. Determine the actual deflection from the equation (keep in mind that this is relative to the zero point before settlement is applied, not the edge of the plate!)

$$v = wRd \qquad (25)$$

If desired, the deflection of the plate relative to the edge can be computed by subtracting the result of (25) from the settlement s.

9. Determine the actual moments using the equations

$$m_r = \frac{DM_r d}{R} \qquad (26)$$

$$m_q = \frac{DM_q d}{R} \qquad (27)$$

where m_r, m_θ = actual plate radial and tangential moments, N

10. Determine the plate bending stresses using the equations

$$s_r = \frac{12 z m_r}{h^3} \qquad (28)$$

$$s_q = \frac{12 z m_q}{h^3} \qquad (29)$$

where z = plate thickness coordinate relative to the plate centre, m

Obviously the maximum stress (compressive or tensile) takes place at the plate surface, or at z = h/2. Once these stresses are computed they can be analysed using a suitable failure criterion.

Example Problem

Consider the case of a closed ended pipe pile 1500mm in diameter with a bottom plate 40mm thick. Both pile and plate are made of steel (E_p = 200 GPa, ν_p = 0.3.) The soil surrounding the pile toe is a medium dense sand (E_s = 25 MPa, ν_s = 0.3.) The toe is subject to an elastic settlement of 1.25mm. Determine the deflection, moments and stresses at the centre of the plate.

First, we must compute ß. We first compute k from Equation (22),

$$k = \frac{(25000000)}{(2)(.75)(1-.3^2)} = 18.32\,MPa$$

Then we compute D from Equation (7),

$$D = \frac{(200000000000)(.04)^3}{(12)(1-.3^2)} = 1.172\,MJ$$

Then we compute ß from Equation (9),

$$b = \sqrt[4]{\frac{(18.32)(.75)^4}{1.172}} = 1.49$$

Since this is very close to 3/2, we can use the charts for ß = 3/2.

We need also to compute δ from Equation (4),

$$d = \frac{(0.00125)}{0.75} = 0.00167$$

These charts show that, for r=0, w = 0.925 and $M_r = M_\theta$ = -0.39. We can then compute the actual quantities. First we compute ϖ, given by Equation (25),

$$v = (0.925)(.75)(.00167) = 1.16mm$$

The two moments are equal; they are by Equations (26) and (27),

$$m_r = m_q = \frac{(1172000)(-0.39)(.00167)}{.75} = -1017.8N$$

The stresses are computed by Equations (28) and (29),

$$s_r = s_q = \frac{(12)\left(\frac{.04}{2}\right)(-1017.8)}{(0.04)^3} = -3.81MPa$$

Although the sign of this solution is negative, there are both tensile and compressive stresses at all points on the plate, depending upon the size. Also, as we said before the value of the deflection is taken from the zero position of the plate before settlement is applied. If the deflection of the centre of the plate relative to the edge is desired, it can be obtained by simply subtracting the deflection from the elastic settlement, or in this case,

$$w_{diff} = 1.25 - 1.16 = 0.09mm$$

Comments on the Solution

The test case showed a situation where the stresses and deflection of the plate was little affected by the elastic foundation loading. It indicates that a thinner plate may be possible in this particular application. There are two cautions to this and any other attempt to reduce the plate thickness by using this method.

The first is that the plate stresses and loading increase exponentially with an increase of the subgrade modulus (see Figure 2.) Underestimation of this value (and thus of ß) can have serious consequences once the pile is installed.

The second is that only static loading is considered here. It is reasonable to assume that dynamic loading during impact or vibratory driving will be considerably greater than static loading; moreover, the stiffness of the soil will vary with a time dependent loading. On the other hand, most dynamic analyses consider a bottom plate to be rigid, only considering the stiffness and damping of the soil to be significant during driving. For values of "dynamic" ß which are sufficiently high, this may not be the case, and may alter the dynamic stiffness of the bottom plate and thus the pile toe.

Conclusion

The result of this study has produced a method for computing the plate deflection, bending and bending stress for pile toe plates that is relatively simple to use, considering the complexity of the underlying mathematics. The results are also applicable to any circular plate on an elastic foundation subject to edge loading only and clamped at the edges, although the selection of trial values of ß may vary for different situations.

Acknowledgements

The author would like to thank Dr. Edwin P. Foster of the University of Tennessee at Chattanooga for his review of the article, and Chris Smoot at *Pile Buck* for his many years of support and his work in the dissemination of information within the industry.

Finally the author would like to give thanks to God for his blessings, remembering that, as Paul wrote, "For I am confident of this very thing, that He who began a good work in you will perfect it until the day of Christ Jesus." (Phil 1:6)

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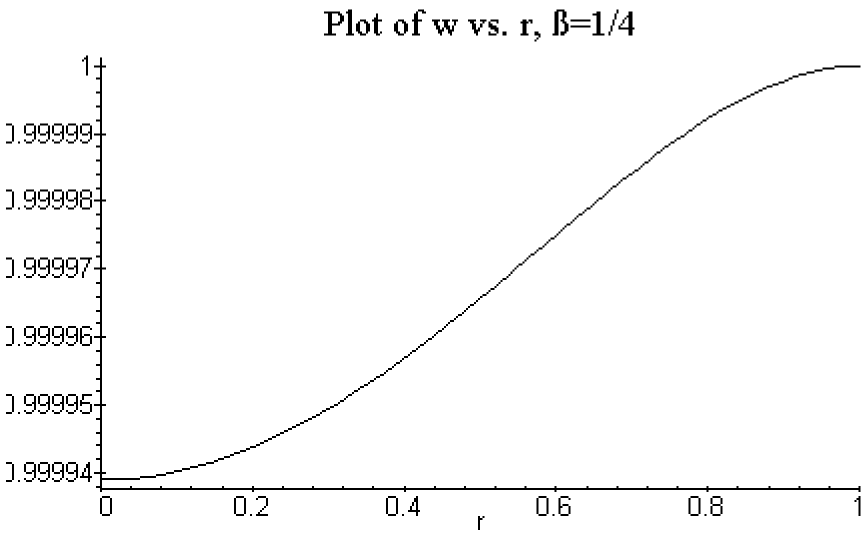
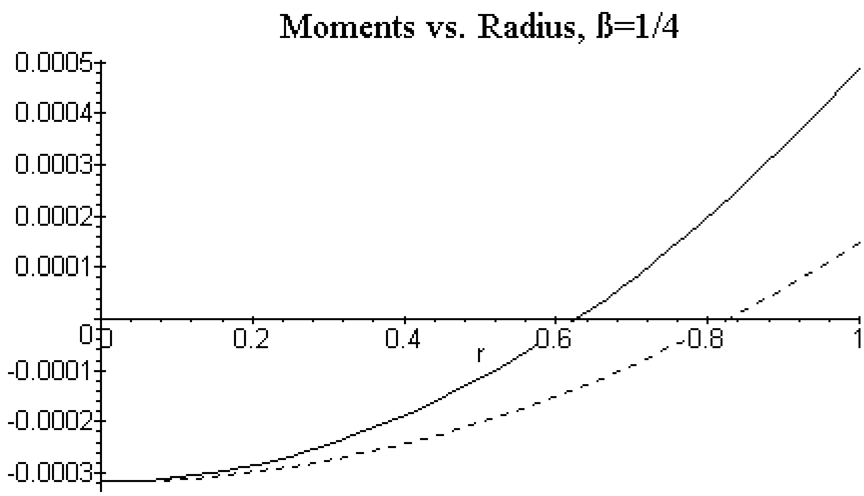
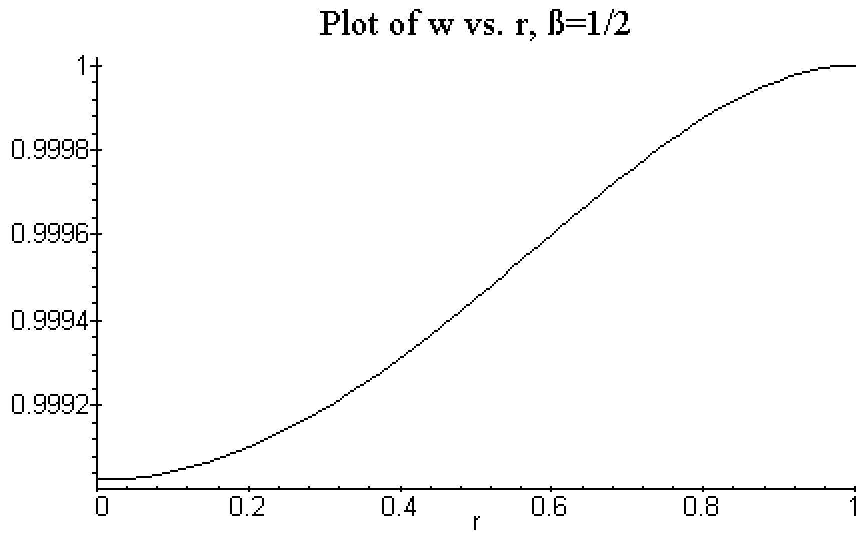
Charts for Deflection and Moment

Deflections and Moments for ß=1/4






Plot of w vs. r, ß=1/2
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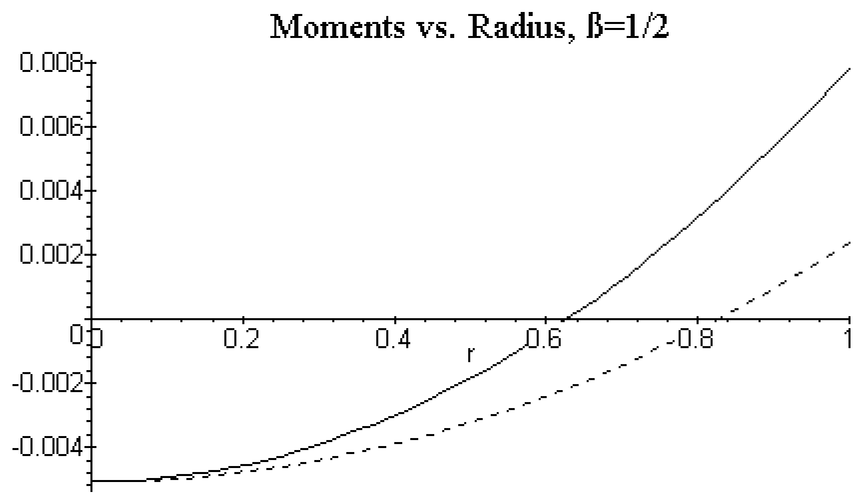
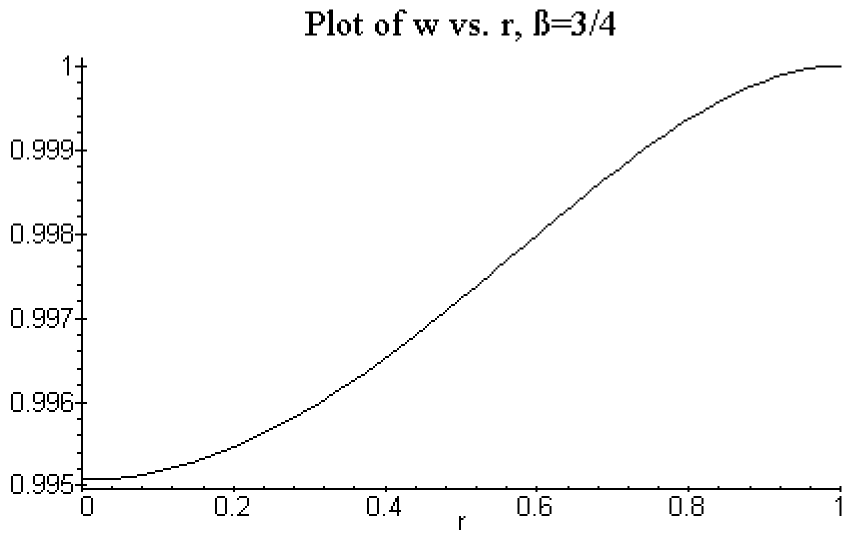
0.9996
0.9994
0.9992
0
0.2
0.4
r
0.6
0.8
1


Moments vs. Radius, ß=1/2
0.008
0.006
0.004
0.002
0
-0.002
-0.004
0
0.2
0.4
r
0.6
0.8
1


Plot of w vs. r, ß=3/4
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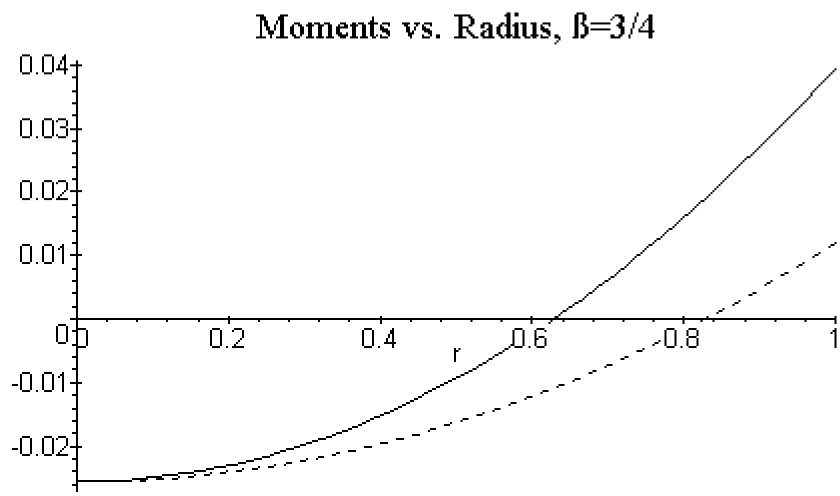
0.995
0
0.2
0.4
0.6
0.8
1
r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0
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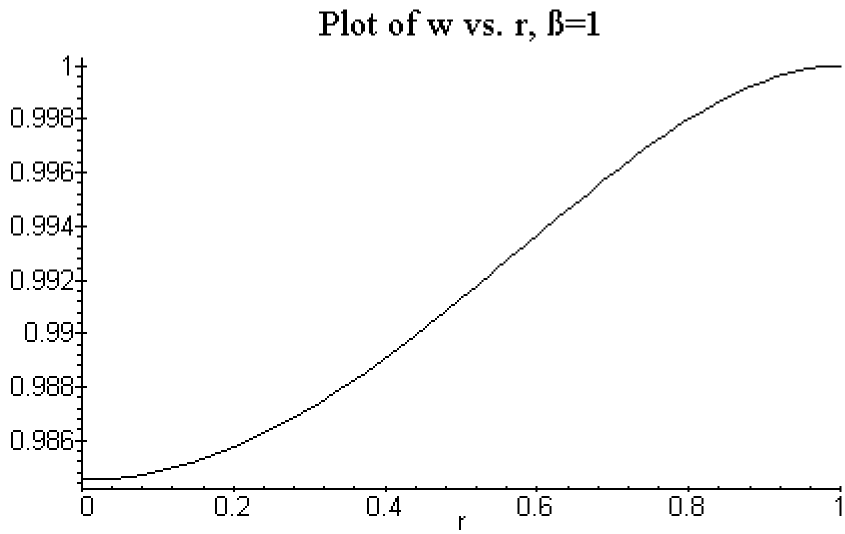
0
0.2
0.4
0.6
0.8
1
r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0.99
0.988
0.986
0
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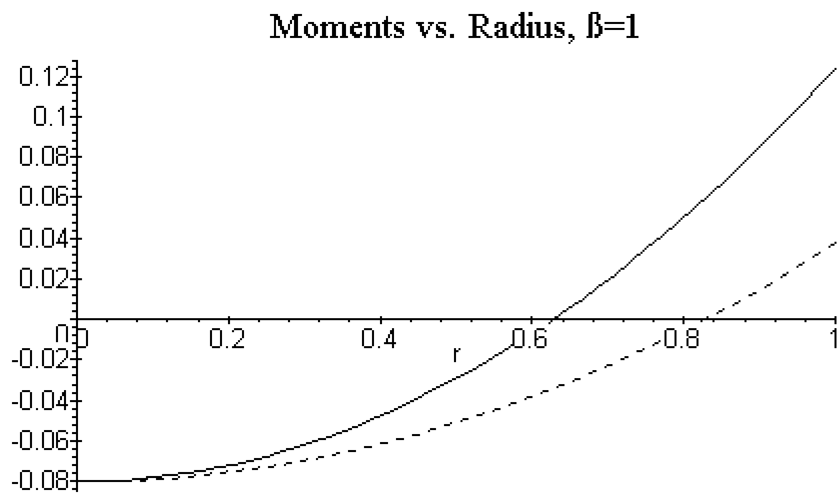
0.4
0.6
0.8
1
r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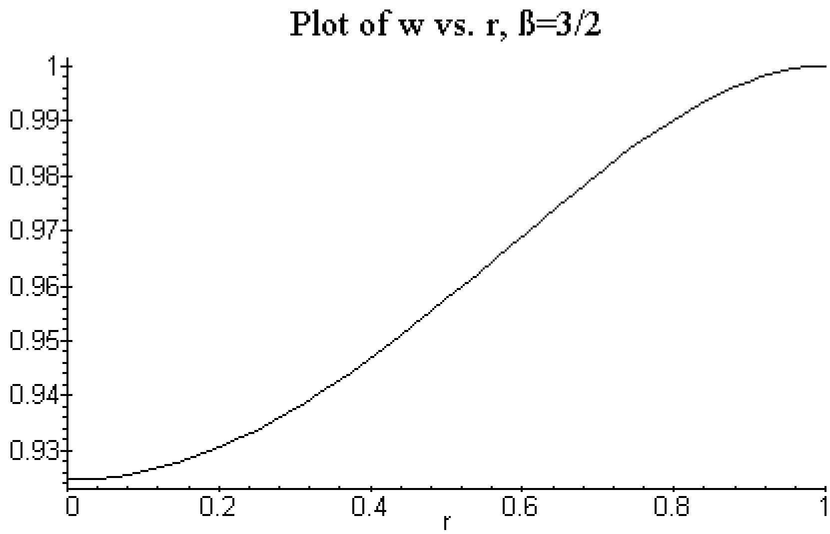
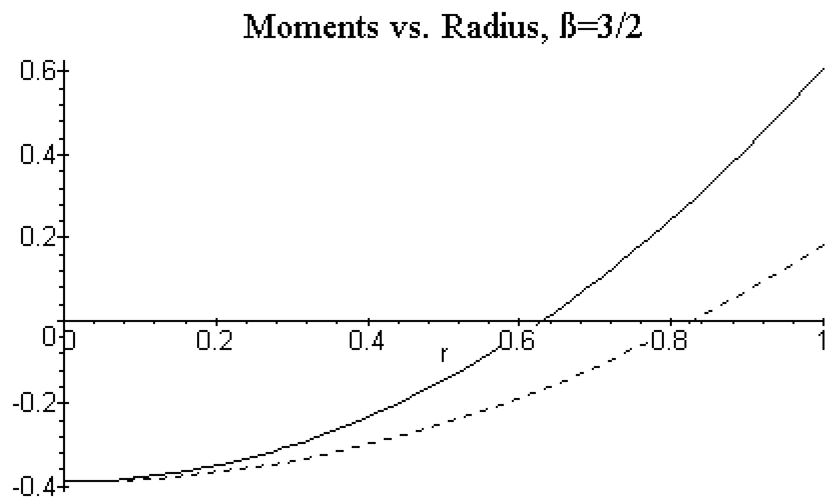
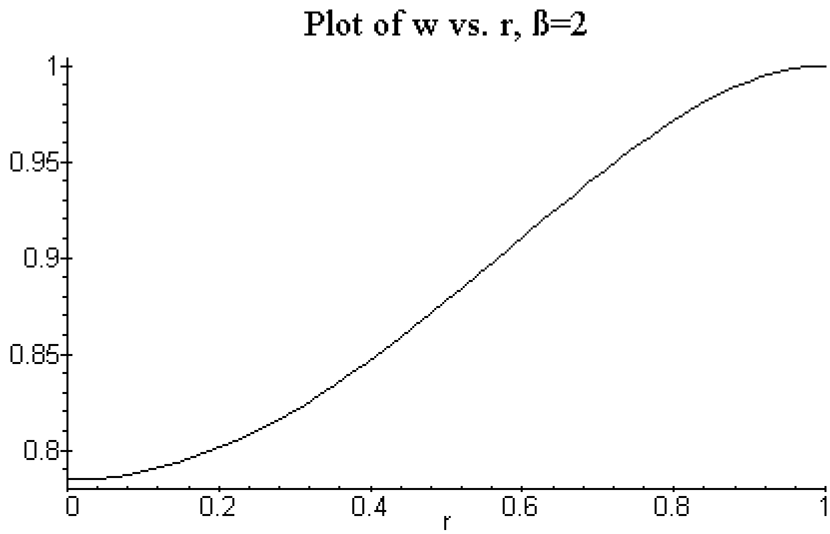
0.1
0.08
0.06
0.04
0.02
0
-0.02
-0.04
-0.06
-0.08
0.2
0.4
0.6
0.8
1
r

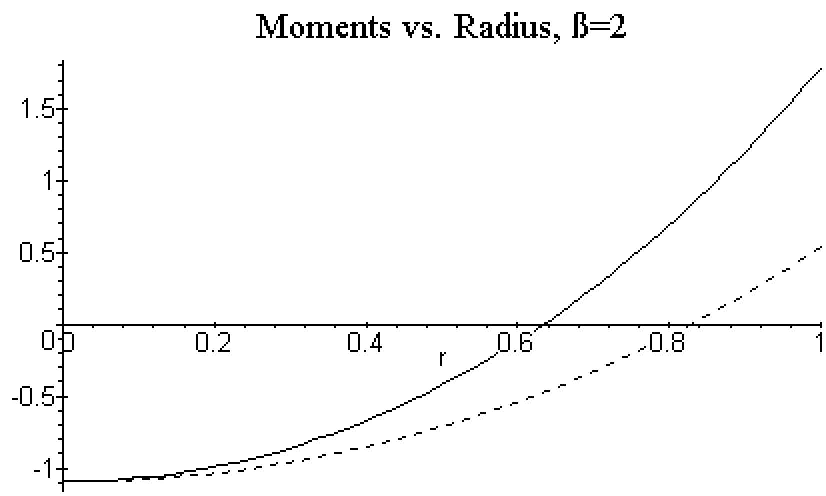
Deflections and Moments for ß=3/2

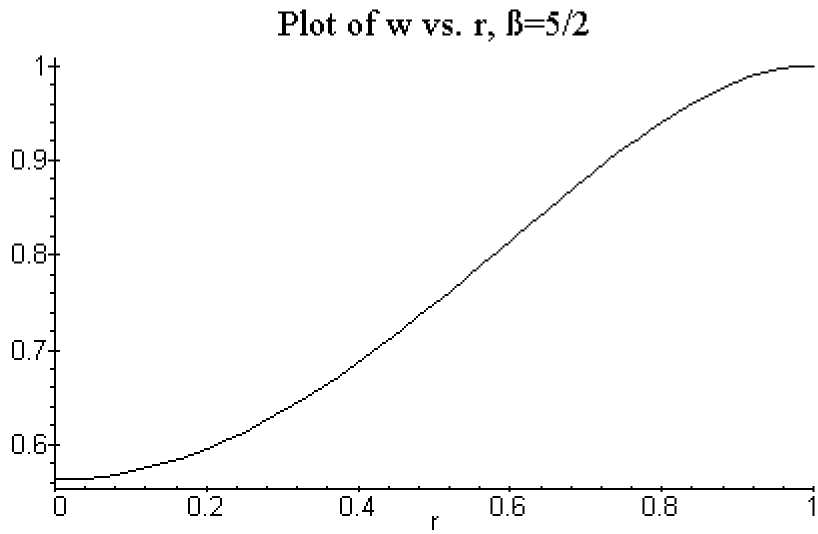




Deflections and Moments for ß=2






Plot of w vs. r, ß=5/2
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0
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0.4
r
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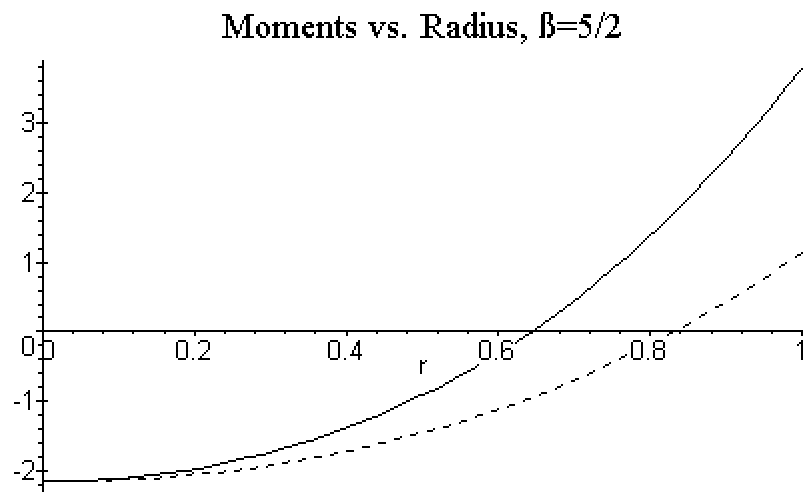
0.8
1


Moments vs. Radius, ß=5/2
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-1
-2
0
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0.4
r
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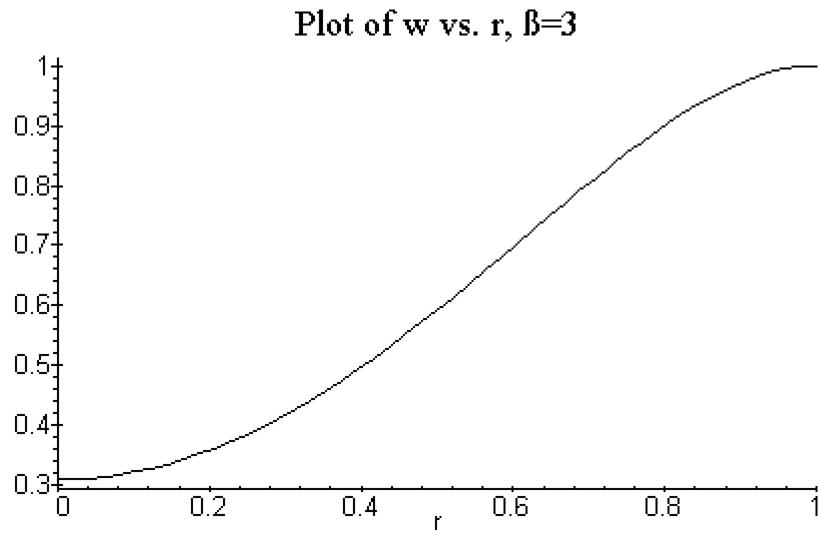
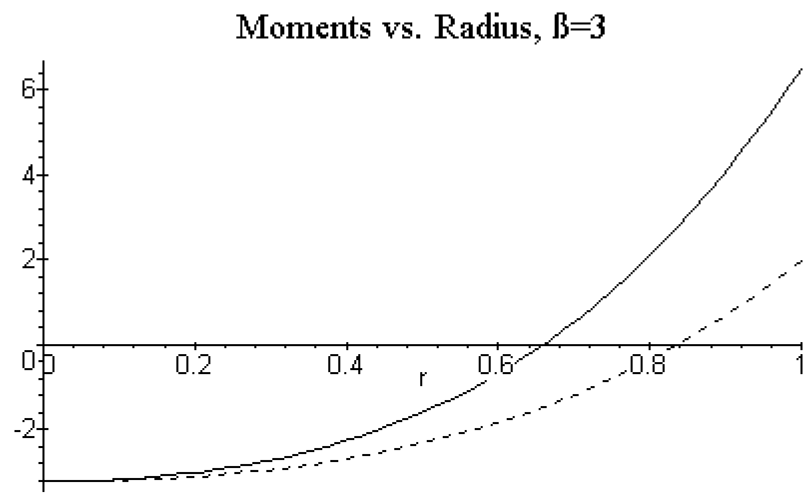
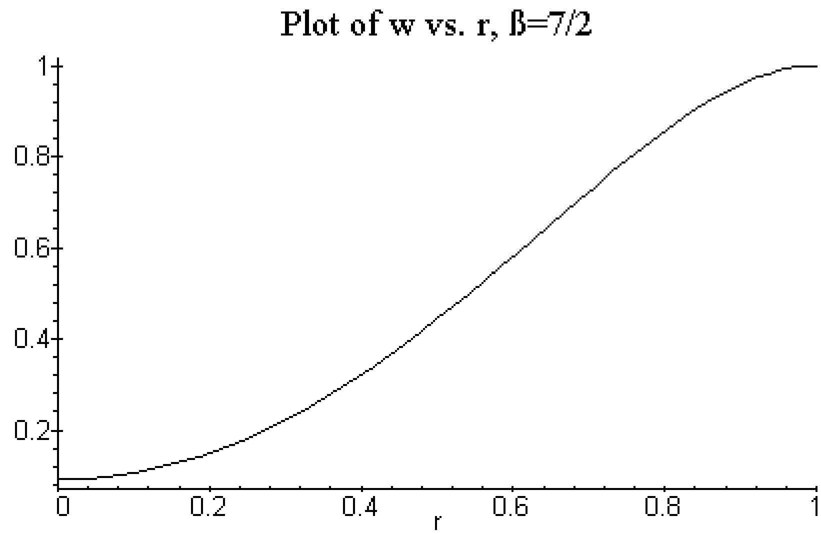
0.8
1

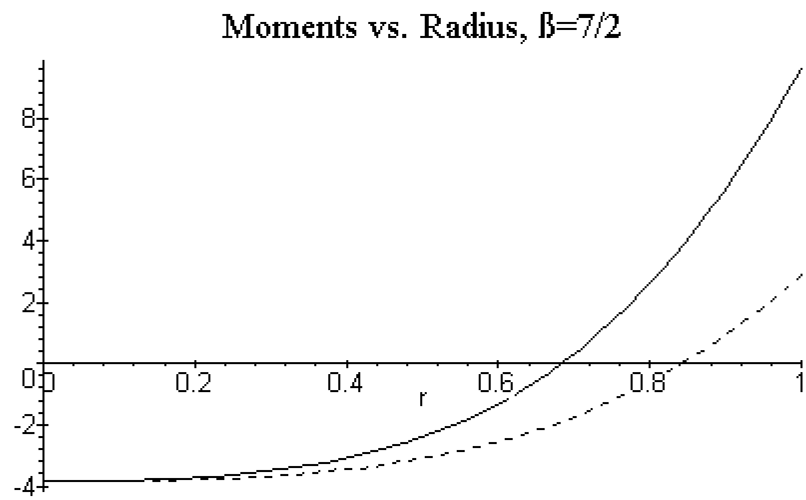
Deflections and Moments for ß=3



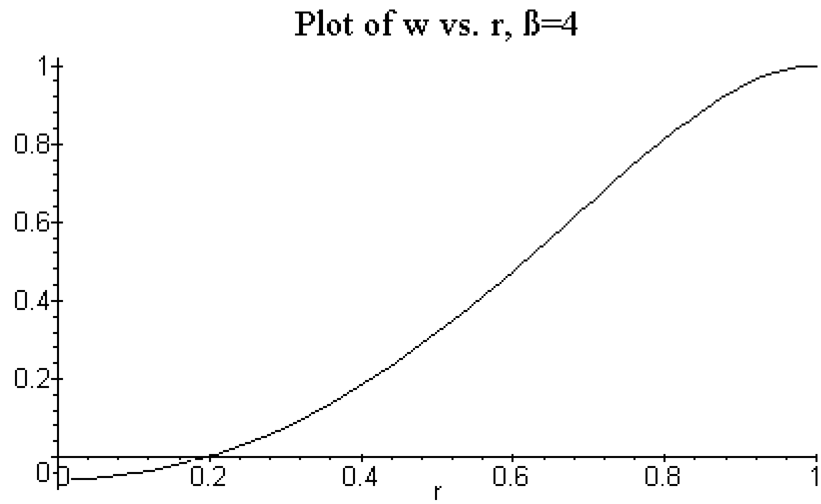


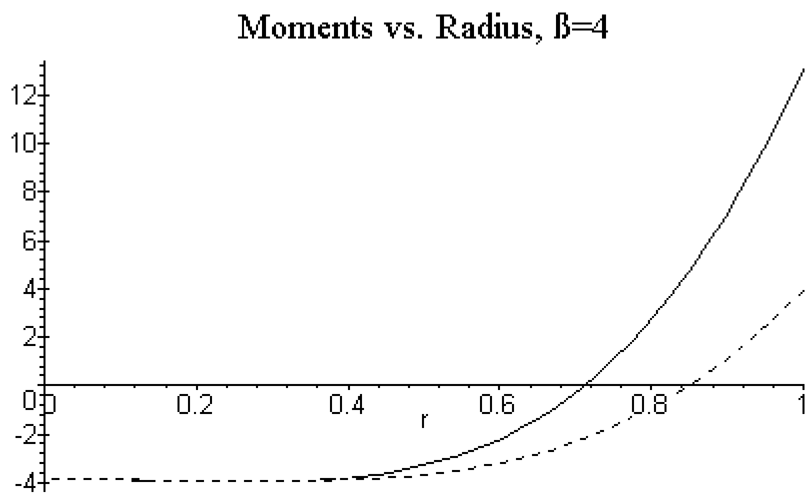
Deflections and Moments for ß=7/2

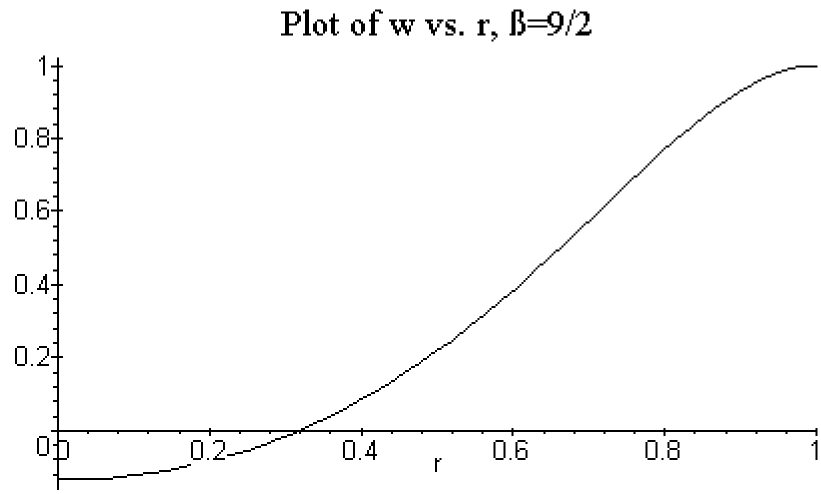


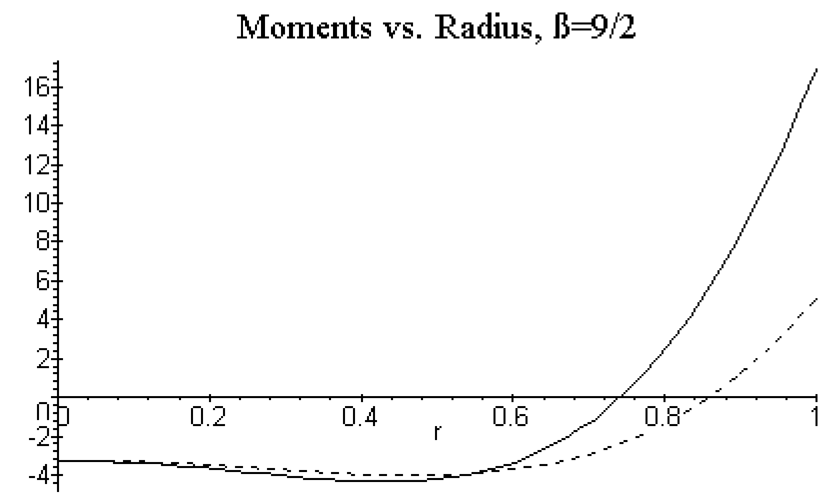


Design of Pile Toe Plate on an Elastic Foundation
Deflections and Moments for ß=4

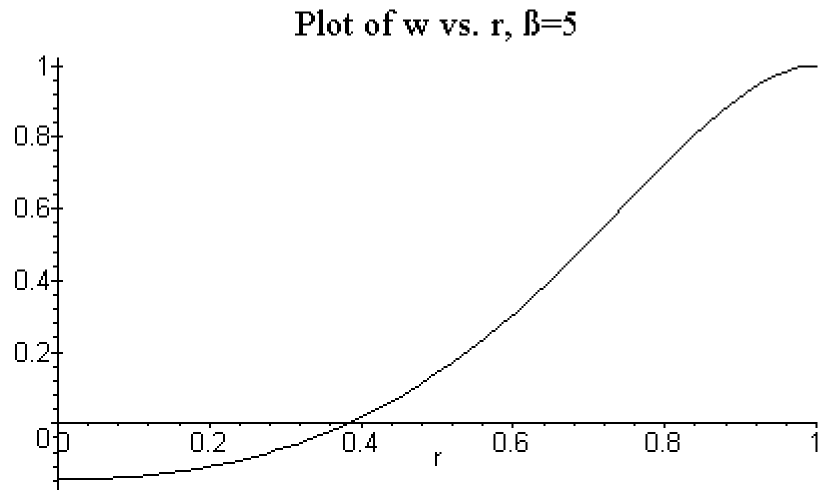


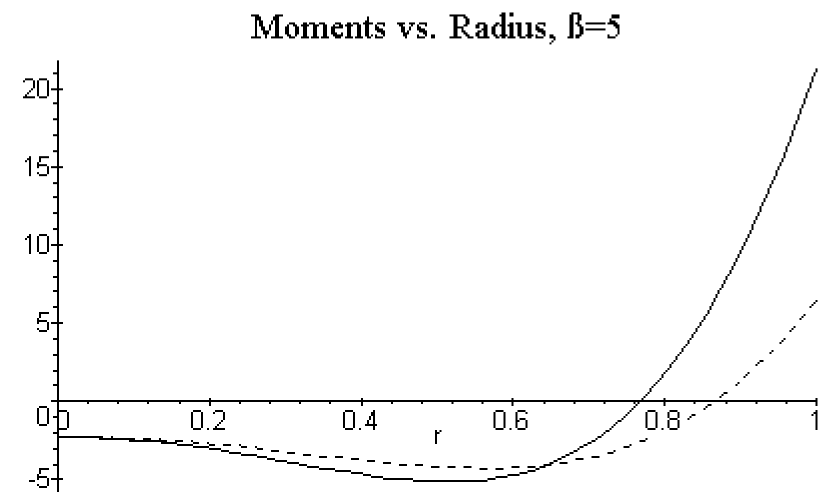


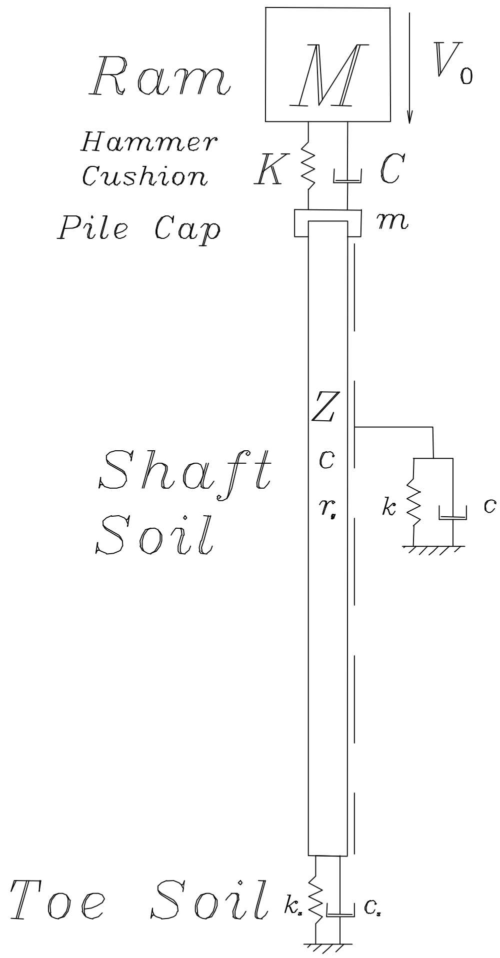

Plot of w vs. r, ß=9/2
1
0.8
0.6
0.4
0.2
0
0.2
0.4
0.6
0.8
1
r


Moments vs. Radius, ß=9/2
16
14
12
10
8
6
4
2
0
-2
-4
0.2
0.4
0.6
0.8
1
r

Deflections and Moments for ß=5







Now that x=L…

For those of us involved in the one-dimensional wave equation, this means that you have reached the end! We trust that the information presented in the article concerning the wave equation or other technical matters has been useful to you. We should now like to take the time to make some other observations.

It is our conviction that the beauty of our world and universe, especially as it is expressed mathematically but certainly in other ways, speaks of its formation by an intelligent Creator. This is underscored by the unity that appears both in mathematics and in the physical laws which mathematics are used in to quantify and qualify. As scientists and engineers we depend upon this unity to both make sense out of what we observe and to make progress both in our knowledge and in our application of that knowledge to practical problems.

But as we turn away from the reverie of beautiful formulations, we see a world that is marred by human failing. This manifests itself in many forms that we are reminded of daily. The longer we live on this earth the more those failings come home to inflict pain upon us, no matter how hard we try to escape them.

It was not God's intent to leave us with this pain alone in his creation but to offer us a way by which we finite beings be united into his perfect infinity, something which is both definable and beyond definition. In infinity past he was with his Son Jesus Christ and Jesus came to live amongst us, share our situation and ultimately face torture and execution by those who were threatened by his message.

But this was not the end, for Jesus being God rose from the dead and offers us both a way out of our present condition in this life and eternal life with God, not by simply following a set of rules but by having God himself live in us and both empowering and leading us in a better way. If we commit ourselves to Jesus then for us $L = \infty$, which means that we have life forever.

All of these things are described in the book called the *Bible*; but in the meanwhile you can learn more at the website

http://www.geocities.com/penlay

or by emailing us at uttc2uxx@ geocities.com. We look forward to hearing from you.