APPLICATION OF THE CLOSED FORM SOLUTION FOR THE DAMPED WAVE EQUATION TO PILES[1]

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ABSTRACT

This paper presents the application of the closed form solution for the damped wave equation to piles. The wave equation in numerical solution has been used for many years, generally without even a simple closed form counterpart. In this paper the closed form solution for the damped wave equation will first be stated and related to an actual pile driven into the soil. Following this is a discussion of the boundary conditions: the hammer at the pile top and the soil response at the pile toe. To avoid spectral components in the Fourier series eigenvalues and to preserve orthogonality, a new strain based soil model to simulate radiation dampening from the pile toe is proposed. A solution to this equation which involves the solution of the semi-infinite pile using Laplace transform for the first part of the impact followed by a Fourier series solution for the remainder. Comparison with numerical methods for a sample case is also presented.

INTRODUCTION

Since the early 1930's, wave mechanics have been used to analyse the displacements and stresses in piling during driving. The early work in this field employed closed form solutions; however, it quickly became evident that these were too complex for common use. The advent of the digital computer made numerical methods useful and these became the state of the art for the analysis of piles during driving and other aspects of driven and drilled piles (Warrington, 1996.)

The acceptance of numerical methods has pushed closed form solutions into the background; however, closed form solutions are still useful for a number of applications related to piles. In

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this paper a closed form solution for the damped or Telegrapher's wave equation as applied to piles is proposed. The method used is that proposed by Warrington (1997), where a complete explanation of the background and solution technique for both damped and undamped cases can be found.

STATEMENT OF THE PROBLEM

The basic physical system for a typical hammer-pile-soil system is shown in Figure 1.

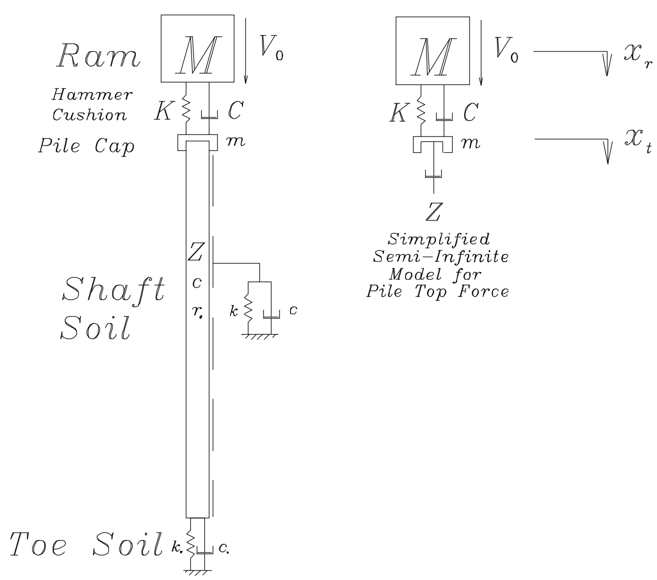

Figure 1 Hammer-Pile-Soil System

Governing Equation

The differential equation for a pile with uniform soil and cross-sectional properties without any non-linearities is given by the equation

$$c^2 u_{xx}(x,t) = u_{tt}(x,t) + 2bu_t(x,t) + au(x,t) \quad \text{(1)}$$

where a = Pile Shaft Elasticity Constant, 1/sec²
b = Pile Shaft Dampening constant, 1/sec
c = Acoustic Speed of Pile Material, m/sec
$u(x,t), u(x,t'), u(x,w)$ = Displacement of Pile Particle, m
t = Time from Zero Point, seconds
x = Distance from Pile Top, m

For longitudinal vibrations, the constant c is the acoustic speed of the material of the bar, given by the equation

$$c = \sqrt{\frac{E}{r}} \quad (2)$$

where E = Pile Young's Modulus of Elasticity, Pa
r = Pile Density, kg/m³

If the geometry ratio is defined as

$$r_g = \frac{A}{P^2} \quad (3)$$

where r_g = Geometry Ratio of Pile
A = Cross-Sectional Area of Pile, m²
P = Pile Surface Perimeter, m

and other substitutions are made for the constants, this equation can be rewritten as

$$c^2 u_{xx}(x,t) = u_{tt}(x,t) + \frac{m}{r\sqrt{Ar_g}} u_t(x,t) + \frac{k}{r\sqrt{Ar_g}} u(x,t) \quad (4)$$

This is the Telegrapher's Equation as applied to piles. Using the soil model of Randolph and Simons (1986) and Corté and Lepert (1986), the soil elasticity and dampening along the shaft can be computed by the equations

$$k = pG_s\sqrt{\frac{r_g}{A}} \quad (5)$$

and

$$m = \sqrt{G_s r_s} \quad (6)$$

where G_s = Soil Shear Modulus of Elasticity, Pa
r_s = Soil Density, kg/m³
k = Soil Shaft Spring or Elastic Constant per Unit Area, N/m³
m = Shaft Soil Dampening Coefficient per Unit Area, N-sec/m³

Boundary Conditions

Pile Top (x = 0)

The simplest way of modelling the pile top in closed form is to use hammer force-time curves derived from semi-infinite pile theory (Deeks and Randolph, 1993.) Unfortunately, this theory is strictly speaking inapplicable to the damped wave equation; however, as a first approximation it is acceptable.

When the hammer force is removed, the condition of the pile top becomes

$$u_x(0,t) = 0 \quad (7)$$

Pile Toe (x = L)

For a visco-elastic pile toe without a discrete mass, the boundary equation is

$$-EAu_x(L,t) = k_t A_t u(L,t) + m_t A_t u_t(L,t) \quad (8)$$

where A_t = Pile Toe Area, m²
k_t = Soil Toe Spring or Elastic Constant per Unit Area, N/m³
m_t = Soil Toe Dampening Constant per Unit Area, N-sec/m³
L = Length of Pile, m

To solve for the constants, Lysmer's Analogue can be employed. This method reduced the response of the half-space to a spring-dampener combination. Using this, the dampening $m_t A_t$ and spring constant $k_t A_t$ in Equation (8) are given as (Lysmer, 1965; Holeyman, 1988)

$$m_t A_t = \frac{3.4 r_t^2 \sqrt{r_s G_s}}{1-n} \quad (9)$$

and

$$k_t A_t = \frac{4 G_s r_t}{1-n} \quad (10)$$

where r_t = Pile Toe Radius, m
n = Poisson's Ratio of Soil

Substituting into Equation (8),

$$-EAu_x(L,t) = \frac{4 G_s r_t}{1-n} u(L,t) + \frac{3.4 r_t^2 \sqrt{r_s G_s}}{1-n} u_t(L,t) \quad (11)$$

The existence of a first time derivative in the boundary condition is a virtual guarantee that difficulties will arise. To address this problem, consider an equivalent solid below the pile toe (Holeyman, 1985, 1988) as shown in Figure 2.

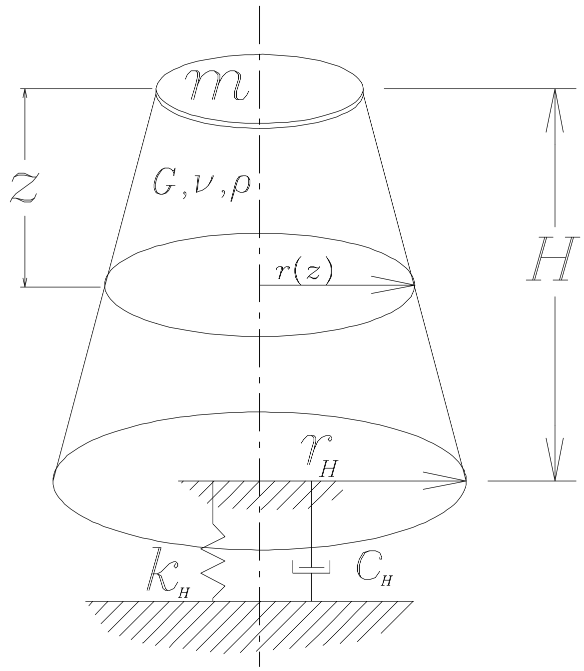

Figure 2 Schematic of Soil Model under Pile Toe (after Holeyman (1988))

The truncated cone is of an indeterminate height H and the radius of the cone is a linear function of the depth z below the pile toe. Holeyman (1988) shows that this radius is given by the equation

$$r(z) = r_t + \frac{1-n}{\sqrt{0.85}} z \qquad (12)$$

where $r(z)$ = Radius of Soil Mass Below Pile Toe, m
z = Distance Below Pile Toe, m

Next consider the acoustic speed of the soil for compressive waves, which is

$$c_s = \sqrt{\frac{E_s}{r_s}} \qquad (13)$$

where c_s = Acoustic Speed of the Soil, m/sec
E_s = Soil Young's Modulus of Elasticity, Pa

which is analogous to Equation (2). Applying semi-infinite pile theory (Warrington, 1997) to a soil column,

$$u_x(L,t) = -u_t(L,t)\frac{1}{c_s} = -u_t(L,t)\frac{1}{\sqrt{\frac{E_s}{r_s}}} \qquad (14)$$

or, rearranging,

$$\sqrt{\frac{E_s}{r_s}}u_x(L,t) = -u_t(L,t) \qquad (15)$$

In this model the Young's modulus is given (Holeyman, 1988) by the equation

$$E_s = \frac{G_s}{0.85(1-n)^2} \qquad (16)$$

where E_s = Soil Young's Modulus of Elasticity, Pa

Substituting this into Equation (15),

$$u_t(L,t) = -\frac{1}{1-n}\sqrt{\frac{G_s}{0.85r_s}}u_x(L,t) \qquad (17)$$

Multiplying both sides by the right hand side of Equation (9), this yields

$$\frac{3.4r_t^2\sqrt{r_sG_s}}{1-n}u_t(L,t) = -\frac{3.4r_t^2\sqrt{r_sG_s}}{1-n}\frac{1}{1-n}\sqrt{\frac{G_s}{0.85r_s}}u_x(L,t) \qquad (18)$$

and reduces to

$$\frac{3.4r_t^2\sqrt{r_sG_s}}{1-n}u_t(L,t) = -\frac{3.69r_t^2G_s}{(1-n)^2}u_x(L,t) \qquad (19)$$

If the pile toe is assumed to be circular (which is the basis for this theory, as was the case with the shaft) and substitute A_t for the area of a circle, and solve Equation (16) for G_s and substitute,

$$\frac{3.4 r_t^2 \sqrt{r_s G_s}}{1-n} u_t(L,t) \cong -E_s A_t u_x(L,t) \quad (20)$$

For use later the approximation sign is treated as an equality. This is an important result; it indicates (but does not necessarily prove) that the soil under the pile toe is in fact a semi-infinite "pile" with special conditions.

Substituting Equation (20) into Equation (11),

$$-EAu_x(L,t) = \frac{4G_s r_t}{1-n} u(L,t) - E_s A_t u_x(L,t) \quad (21)$$

or rearranging,

$$(E_s A_t - EA) u_x(L,t) = \frac{4G_s r_t}{1-n} u(L,t) \quad (22)$$

which is used in the general case for the boundary condition of the toe.

Initial Conditions

Most pile models start with both displacement and velocity at zero, which is expressed as

$$u(x,0) = f(x) = 0 \quad (23)$$

and

$$u_t(x,0) = g(x) = 0 \quad (24)$$

where $f(x)$ = Initial or Momentary Displacement Distribution in Pile, m
$g(x)$ = Initial or Momentary Velocity Distribution in Pile, m/sec

OUTLINE OF THE SOLUTION TECHNIQUE

Basic Solution Method

The basic solution method proposed is as follows:

1. Determine the force-time or displacement-time history of the hammer at the pile top, either using semi-infinite pile theory or actual field data.

2. Using Laplace transforms, solve the wave equation for the semi-infinite pile case. This is the

solution for t < L/c.

3. Compute the displacement and velocity functions as a function of distance at t = L/c. These become the initial conditions for the remainder of the problem.

4. Using the boundary conditions, compute the eigenvalues and eigenfunctions for the Fourier series. The pile top is assumed to be a free end in this case.

5. Using the displacement and velocity functions at t = L/c, compute the Fourier coefficients. This Fourier series is the solution for t $\geq$ L/c.

As stated, this procedure assumes the transition point to be fixed at t = L/c. However, if the impulse force of the hammer system ends before this time, it is most advantageous to make the turnover point at the time when the impulse force becomes zero, or tension begins to develop in the pile top. This is in fact what is done with the solution here.

Assumptions for the Solution

The following assumptions are made:

1. The solution must be reasonably simple; the solution must not require integration or other transformation once it is formulated.

2. The system is a linear system. No plasticity is taken into account in this system.

3. All properties between the boundaries are uniform. These include pile area and material, dampening, soil spring constant.

4. The soil below the pile toe can be modelled as a semi-infinite pile (see above), thus eliminating the first time derivative of the dampening portion.

5. Extensibility considerations of the pile top and toe are not significant. The validity of this assumption is dependent upon how the pile top force is formulated.

6. The force of the hammer is substantially finished before t = L/c. This solution favours long piles relative to the hammer blow duration.

Solution of the Problem

Equation of Motion for the Pile Top

Following Deeks and Randolph (1993) and Warrington (1997), the pile top force using semi-infinite pile theory for an undamped pile has the form

$$u_t(0,t) = g_1\left(g_2 e^{a_1 t} - e^{a_2 t}\left(g_2 \cos(a_3 t) - g_3 \sin(a_3 t)\right)\right) \qquad (25)$$

where a_1, a_2, a_3 = Consolidation Constants for Pile Top Forces
g_1, g_2, g_3 = Consolidation Constants for Pile Top Forces

This cannot be applied directly to this problem, as will be seen.

General Solution for a Semi-Infinite Damped Pile

Beginning with Equation (1), the Laplace transform of this equation with respect to time is

$$c^2 U_{tt}(x,s) = \left(sU(x,s) - u(x,0)\right)s - u_t(x,0) + aU(x,s) + 2b\left(sU(x,s) - u(x,0)\right) \quad (26)$$

where $U, U(x,s)$ = Laplace Transform of Pile Displacement
s, s_n = Laplace Transform Variable

Substituting the initial conditions of Equations (23) and (24), the solution for this differential equation is

$$U(x,s) = C_1 e^{\sqrt{s^2+2bs+a}\frac{x}{c}} + C_2 e^{-\sqrt{s^2+2bs+a}\frac{x}{c}} \quad (27)$$

where $C_1, C_2, C_3 \ldots Cn$ = Constants or Fourier Coefficients

In order to prevent an unbounded condition,

$$C_1 = 0 \quad (28)$$

Substituting this yields

$$U(x,s) = C_2 e^{-\sqrt{s^2+2bs+a}\frac{x}{c}} \quad (29)$$

Consider a generalized forcing function $F_0(t)$ acting on the pile top. The boundary condition for the pile top is given by the equation

$$F_0(t) = -Zcu_x(0,t) \quad (30)$$

where $F_0, F_0(t)$ = Force at Pile Top (x = 0), N
Z = Pile Impedance, N-sec/m

The Laplace transform of this equation is

$$\mathscr{L}\left(F_0(t)\right) = P(s) = -ZcU_x(0,s) \quad (31)$$

where $P(s)$ = Laplace Transform for Pile Top Force

Substituting Equation (29) and then x = 0,

$$P(s) = ZC_2\sqrt{s^2 + 2bs + a} \quad \text{.......... (32)}$$

Solving for C_2, this yields

$$C_2 = \frac{P(s)}{Z\sqrt{s^2 + 2bs + a}} \quad \text{.......... (33)}$$

Substituting this back into Equation (29),

$$U(x,s) = \frac{P(s)}{Z}\frac{e^{-\sqrt{s^2+2bs+a}\,\frac{x}{c}}}{\sqrt{s^2 + 2bs + a}} \quad \text{.......... (34)}$$

Borel's theorem can be used in this case to solve the inverse Laplace transform. First the equation was divided into the expressions

$$\hat{F}(s) = \frac{P(s)}{Z} \quad \text{.......... (35)}$$

and

$$\hat{G}(s) = \frac{e^{-\sqrt{s^2+2bs+a}\,\frac{x}{c}}}{\sqrt{s^2 + 2bs + a}} \quad \text{.......... (36)}$$

where $\hat{F}(s)$ = Laplace Transform of Pile Top Forcing Function
$\hat{G}(s)$ = Laplace Transform of Pile Response Function

The inverse Laplace transforms of these expressions are, respectively,

$$\hat{f}(t) = \frac{F_0(t)}{Z} \quad \text{.......... (37)}$$

for the forcing function and (Oberhettinger and Badii, 1973)

$$\hat{g}(t) = e^{-bt} I_0\left(\sqrt{\left(b^2 - a\right)\left(t^2 - \left(\frac{x}{c}\right)^2\right)}\right), t > \frac{x}{c} \qquad (38)$$

$$\hat{g}(t) = 0, t \le \frac{x}{c} \qquad (39)$$

where $\hat{f}(t)$ = Inverse Laplace Transform of Pile Top Forcing Function
$\hat{g}(t)$ = Inverse Laplace Transform of Pile Response Function

for the response function. The inverse Laplace transform of Equation (34) can be expressed as

$$u(x,t) = \hat{f}(t) * \hat{g}(t) = \frac{1}{Z}\int_{\frac{x}{c}}^{t} e^{-bt} I_0\left(\sqrt{\left(b^2 - a\right)\left(t^2 - \left(\frac{x}{c}\right)^2\right)}\right) F_0(t - t)\,dt, t > \frac{x}{c} \qquad (40)$$

where t = Dummy Variable for Borel's Theorem, sec.

and zero for other times. This is identical to the result of Van Koten et. al (1980) except for changes in the notation. It is similar to the solution of Webster (1960); however, he assumes non-zero initial conditions.

Discussion of the Solution

This solution has a number of important results which need to be understood completely.

First, this equation has no straightforward closed form solution. The most direct method of solving this equation is to substitute a power series or polynomial approximation for the Bessel function and perform termwise integration. How this is performed depends upon the values of the argument of the Bessel function and the desired complexity of the resulting algebra.

Second, for the pile top,

$$u(x,t) = \hat{f}(t) * \hat{g}(t) = \frac{1}{Z}\int_{0}^{t} e^{-bt} I_0\left(\sqrt{\left(b^2 - a\right)\left(t^2\right)}\right) F_0(t - t)\,dt, t > 0 \qquad (41)$$

This result also appears in Zhou and Liang (1996).

Third, any function used as the forcing function results in very difficult integration depending upon what kind of function is used. For the functions derived for pile top force, this can be potentially overwhelming -- especially if one considers that these equations are strictly speaking inapplicable. Combined with the fact that this relationship is different than that of semi-infinite pile theory applied to the undamped pile, i.e.,

$$u(x,t) = f\left(t-\frac{x}{c}\right)H\left(t-\frac{x}{c}\right) \quad (42)$$

where $H(t), H\left(t-\frac{x}{c}\right)$ = Heaviside Step Function

$f(t)$ = Displacement Function at Pile Top, m

the force-time relationships of Equation (25) cannot be directly applied here. These problems are dealt with by substituting a constant force that acts for a time δ after impact. The force is zero afterward. This is expressed as

$$F_0(t) = F_0, t < d, d \leq \frac{L}{c} \quad (43)$$

This is essentially the same forcing function as used by Van Koten et. al. (1980). The difference in this solution is twofold. First, the force-time curve used is matched with the semi-infinite solution by having the two force-time curves have the same impulse and maximum force (the latter to match the pile stresses.) *Second, the time used to begin the Fourier series solution is altered to t = d. rather than t = L/c.* This is as opposed to Van Koten's solution of using a equal negative forcing function after the end of the impulse to simulate a zero pile top forcing function. The method of dividing the solution makes this possible.

Since the Bessel function represents the central difficulty in the analysis of this problem, it was considered first. The square of the argument is first defined as

$$\hat{z} = \left(b^2 - a\right)\left(t^2 - \left(\frac{x}{c}\right)^2\right) \quad (44)$$

where $\hat{z}$ = Bessel Function Argument for Damped Case

The first parentheses has the dimensions of inverse time and the second of time. For simplicity's sake the quantities were rearranged so that both of the parenthetical terms were dimensionless. With judicious rearranging and substitution,

$$\hat{z} = \left(\left(b^2 - a\right)\left(\frac{L}{c}\right)^2\right)\left(\left(\frac{tc}{L}\right)^2 - \left(\frac{x}{L}\right)^2\right) \quad (45)$$

Now the quantity

$$\hat{d} = \left(b^2 - a\right)\left(\frac{L}{c}\right)^2 \qquad (46)$$

where $\hat{d}$ = Pile Shaft Damping and Elasticity Ratio

is defined. Then Equation (45) can be rewritten as

$$\hat{z} = \hat{d}\left(\left(\frac{tc}{L}\right)^2 - \left(\frac{x}{L}\right)^2\right) \qquad (47)$$

To analyse the argument, the remaining parenthetical expression is basically a dimensionless time quantity in "units" of L/c. Thus, the maximum value for this quantity takes place at the pile top (x = 0). For the first "semi-infinite" phase of the analysis, since the maximum time is L/c, the maximum value for this quantity is unity.

For a case defined in this way, the maximum value of $\hat{z}$ is thus completely dependent upon $\hat{d}$. There are three basic cases for this variable, which depend upon a and b since L and c are both positive.

1. $b^2 > a, \hat{d} > 0$. In this case the I_0 Bessel function remains, which is unbounded as the argument increases. This would create difficulties except for the exponential, which approaches zero as I_0 approaches infinity with increasing time.

2. $b^2 = a, \hat{d} = 0$. The Bessel function is valued at unity. This is analogous to the "balanced line" condition which appears in transmission line problems and which simplifies the analysis considerably. Unfortunately this cannot be counted on taking place in piling.

3. $b^2 < a, \hat{d} < 0$. In this case the J_0 Bessel function is used for the negative value of the argument. This results in oscillatory response.

More importantly this variable defines in large part (except for the exponential decay, which is a function of b) the response of the pile to excitation at the top, not only in quantity but in its nature as well. To obtain variables such as this is one of the objects of closed form analysis and thus it is an important result even without a subsequent solution.

Using the notation for the argument developed earlier, the power series representation for the Bessel function is

$$I_0\left(\sqrt{\hat{z}}\right) = \sum_{m=0}^{\infty} \frac{\left(\sqrt{\hat{z}}\right)^{2m}}{2^{2m} m!^2} \qquad (48)$$

This can be simplified to

$$I_0\left(\sqrt{\hat{z}}\right)=\sum_{m=0}^{\infty}\frac{\left(\frac{\hat{z}}{4}\right)^m}{m!^2}=1+\frac{\hat{z}}{4}+\frac{\hat{z}^2}{64}+\frac{\hat{z}^3}{2304}\ldots \quad (49)$$

The series is valid for all values of $\hat{z}$, and furthermore automatically changes the nature of the Bessel function with the changes in sign of the argument.

As is the case with many functions of this type, the function converges everywhere, but how many terms are needed for convergence? This depends on the value of the argument. As $\hat{z}$ increases, the number of terms required for convergence also increases. It is necessary to analyze possible values for the argument to determine the number of terms necessary for convergence.

This analysis was performed by Warrington (1997), who showed that, for many problems involving piling, $\hat{d} > -6$. This would indicate that four terms of the series in Equation (49) would be needed for an ideal approximation. However, the complete argument of the Bessel function is in fact a polynomial; each power of this produces yet a higher value and more involved polynomial. However it is probable that attainment of this low value of $\hat{d}$ is in fact unlikely. Therefore, for simplicity's sake, the first three terms of the series are used.

Practical Statement of the Solution for t<δ

The objective in this analysis is to provide a relatively simple solution for this problem. It is now possible to finalize this solution for the first portion of time.

If the Bessel Function in Equation (40) is changed into the series of Equation (49) for three terms and substitute Equation (43) for the forcing function,

$$u(x,t)=\frac{F_0}{Z}\int_{\frac{x}{c}}^{t}e^{-bt}\left(1+\frac{\left(\left(b^2-a\right)\left(t^2-\left(\frac{x}{c}\right)^2\right)\right)}{4}+\frac{\left(\left(b^2-a\right)\left(t^2-\left(\frac{x}{c}\right)^2\right)\right)^2}{64}\right)dt,\ \frac{x}{c}<t<d \quad (50)$$

where d = Time of Square Wave Simplified Impulse, sec.

Integration (with appropriate substitutions) of this yields

$$u(x,t)=\frac{F_o}{b^5L^4Z}$$

$$\left(\left(\begin{array}{l}-\frac{\hat{d}^2c^4b^4}{64}t^4-\frac{\hat{d}^2c^4b^3}{16}t^3+\left(\frac{\hat{d}^2c^2x^2b^4}{32}-\frac{\hat{d}c^2b^4L^2}{4}-\frac{3\hat{d}^2c^4b^2}{16}\right)t^2\\ \left(-\frac{\hat{d}c^2b^3L^2}{2}-\frac{3\hat{d}^2c^4b}{8}+\frac{\hat{d}^2c^2x^2b^3}{16}\right)t+\frac{\hat{d}^2c^2x^2b^2}{16}+\frac{\hat{d}L^2x^2b^4}{4}-\\ \frac{\hat{d}^2x^4b^4}{64}-b^4L^4-\frac{3\hat{d}^2c^4}{8}-\frac{\hat{d}c^2L^2b^2}{2}\end{array}\right)e^{-bt}\right. \\ \left.+\left(\frac{\hat{d}c^2b^2L^2}{2}+\frac{\hat{d}^2c^2x^2b^2}{8}+\frac{3\hat{d}^2bc^3x}{8}+b^4L^4+\frac{\hat{d}b^3L^2cx}{2}+\frac{3\hat{d}^2c^4}{8}\right)e^{-\frac{bx}{c}}\right) \quad (51)$$

The velocity in the pile is

$$u_t(x,t)=\frac{F_0\left(\hat{d}t^2c^2-\hat{d}x^2+8L^2\right)^2e^{-bt}}{64ZL^4} \quad (52)$$

and the pile stress is

$$s(x,t)=\frac{F_o}{b^5L^4A}$$

$$\left(\begin{array}{l}\left(\left(\frac{\hat{d}^2c^2xb^4}{16}\right)t^2\left(\frac{\hat{d}^2c^2xb^3}{8}\right)t+\frac{\hat{d}^2c^2xb^2}{8}+\frac{\hat{d}L^2xb^4}{2}-\frac{\hat{d}^2x^3b^4}{16}\right)ce^{-bt}\\ +\left(\frac{\hat{d}^2c^2xb^2}{4}+\frac{3\hat{d}^2bc^3}{8}+\frac{\hat{d}b^3L^2c}{2}\right)ce^{-\frac{bx}{c}}\\ -\left(\frac{\hat{d}c^2b^2L^2}{2}+\frac{\hat{d}^2c^2x^2b^2}{8}+\frac{3\hat{d}^2bc^3x}{8}+b^4L^4+\frac{\hat{d}b^3L^2cx}{2}+\frac{3\hat{d}^2c^4}{8}\right)be^{-\frac{bx}{c}}\end{array}\right) \quad (53)$$

where $s, s(x,t)$ = Stress in Pile, Pa

At the turnover time t=δ, the displacement and velocity are

$$u(x,d) = f(x) = \frac{F_0}{b^5 L^4 Z}$$

$$\left(\begin{array}{l}\left(\begin{array}{l} -\frac{\hat{d}^2 c^4 b^4}{64} d^4 - \frac{\hat{d}^2 c^4 b^3}{16} d^3 + \left(\frac{\hat{d}^2 c^2 x^2 b^4}{32} - \frac{\hat{d} c^2 b^4 L^2}{4} - \frac{3\hat{d}^2 c^4 b^2}{16}\right) d^2 \\ \left(-\frac{\hat{d} c^2 b^3 L^2}{2} - \frac{3\hat{d}^2 c^4 b}{8} + \frac{\hat{d}^2 c^2 x^2 b^3}{16}\right) d + \frac{\hat{d}^2 c^2 x^2 b^2}{16} + \frac{\hat{d} L^2 x^2 b^4}{4} - \\ \frac{\hat{d}^2 x^4 b^4}{64} - b^4 L^4 - \frac{3\hat{d}^2 c^4}{8} - \frac{\hat{d} c^2 L^2 b^2}{2} \end{array}\right) e^{-bd} \\ + \left(\frac{\hat{d} c^2 b^2 L^2}{2} + \frac{\hat{d}^2 c^2 x^2 b^2}{8} + \frac{3\hat{d}^2 b c^3}{8} + b^4 L^4 + \frac{\hat{d} b^3 L^2 c x}{2} + \frac{3\hat{d}^2 c^4}{8}\right) e^{-\frac{bx}{c}} \end{array}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots (54)$$

and

$$u_t(x,d) = g(x) = \frac{F_0\left(\hat{d} t^2 c^2 - \hat{d} x^2 + 8L^2\right)^2 e^{-bd}}{64 Z L^4} \quad \ldots\ldots\ldots\ldots\ldots\ldots (55)$$

These last equations are used for the computation of the Fourier coefficients.

Fourier Series Solution for $t > d$

Determination of the Eigenvalues and Eigenfunctions

The time for the Fourier series is defined as

$$t' = t - d \quad \ldots\ldots\ldots\ldots\ldots\ldots (56)$$

where t' = Time from Transition Point t = L/c or t=δ, seconds

and the general solution is assumed to be

$$u(x,t') = e^{\frac{bct' + i l x}{L}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (57)$$

where b, l, l_n = Constants or Eigenvalues

Substituting this into Equation (1) and solving for β yields

$$b = \hat{a} \pm i\hat{w} \quad (58)$$

where

$$\hat{a} = -\frac{bL}{c} \quad (59)$$

and

$$\hat{w} = i\sqrt{\left(\frac{L}{c}\right)^2 \left(b^2 - a\right) - l^2} = \sqrt{l^2 - \hat{d}} \quad (60)$$

This is an important result because it relates the previous results to those in this phase of the analysis.

Substituting these results in to Equation (57),

$$u(x, t') = e^{\frac{act' + ilx \pm iwct'}{L}} \quad (61)$$

and this expands to

$$u(x, t') = e^{-bt'}\left(C_1 \cos\left(\frac{\sqrt{l^2 - \hat{d}}c}{L}t'\right) + C_2 \sin\left(\frac{\sqrt{l^2 - \hat{d}}c}{L}t'\right)\right)\left(C_3 \cos\left(\frac{lx}{L}\right) + C_4 \sin\left(\frac{lx}{L}\right)\right) \quad (62)$$

Applying Equation (7), this expression reduces to

$$u(x, t') = e^{-bt'} \cos\left(\frac{lx}{L}\right)\left(C_1 \cos\left(\frac{\sqrt{l^2 - \hat{d}}c}{L}t'\right) + C_2 \sin\left(\frac{\sqrt{l^2 - \hat{d}}c}{L}t'\right)\right) \quad (63)$$

Turning to the pile toe, first Equation (10) is substituted into (22) and this result is then substituted into the previous equation. Solving for λ,

$$\tan(l) = \frac{\sin(l)}{\cos(l)} = \frac{1}{l}\frac{k_t A_t L}{Zc - E_s A_t} \quad (64)$$

There are three possibilities for this equation.

1) $Zc > E_s A_t$. In this case the right hand side is positive and thus a similar result to Equation (131) is found, namely

$$\tan(l_n) = \frac{\sin(l_n)}{\cos(l_n)} = \frac{1}{l_n}\,\frac{k_t A_t L}{Zc - E_s A_t}\,,$$
$$\left(n - \frac{3}{2}\right)p \le l_n \le \left(n - \frac{1}{2}\right)p\,,\ n = 1,2,3,\ldots\infty,\ Zc > E_s A_t \qquad \text{(65)}$$

2) $Zc < E_s A_t$. Here a unique value for λ is obtained,,

$$\tanh(l) = \frac{\sinh(l)}{\cosh(l)} = \frac{1}{l}\,\frac{k_t A_t L}{E_s A_t - Zc}\,,\ Zc < E_s A_t \qquad \text{(66)}$$

3) $Zc = E_s A_t$. The denominator on the right side is zero. The same result on the left hand side can be obtained if

$$l_n = \left(n - \frac{1}{2}\right)p\,,\ n = 1,2,3,\ldots\infty \qquad \text{(67)}$$

and these are of course very regular eigenvalues.

Generally speaking for piles Case (1) applies and thus it is considered to be the "normative" case, although Case (2) is readily conceivable for closed ended piling. Active consideration to Case (3) is not given.

Assuming Case (1) to be true, the solution for the displacement is

$$u(x, t') = e^{-bt'} \sum_{n=1}^{\infty} \cos\left(\frac{l_n x}{L}\right)\left(C_{1n}\ \cos\left(\frac{\sqrt{l_n^2 - \hat{d}c}}{L}\,t'\right) + C_{2n}\ \sin\left(\frac{\sqrt{l_n^2 - \hat{d}c}}{L}\,t'\right)\right) \qquad \text{(68)}$$

the velocity

$$u_{t'}(x,t') = e^{-bt'} \sum_{n=1}^{\infty} \left(\begin{array}{l} -b\cos\left(\frac{l_n x}{L}\right)\left(C_{1n} \cos\left(\frac{\sqrt{l_n^2 - \hat{d}c}}{L} t'\right) + C_{2n} \sin\left(\frac{\sqrt{l_n^2 - \hat{d}c}}{L} t'\right)\right) \\ + \frac{\sqrt{l_n^2 - \hat{d}c}}{L} \cos\left(\frac{l_n x}{L}\right)\left(-C_{1n} \sin\left(\frac{\sqrt{l_n^2 - \hat{d}c}}{L} t'\right) + C_{2n} \cos\left(\frac{\sqrt{l_n^2 - \hat{d}c}}{L} t'\right)\right) \end{array} \right) \quad \text{..................} (69)$$

and the stress

$$s(x,t') = \frac{Zc}{AL} e^{-bt'} \sum_{n=1}^{\infty} l_n \sin\left(\frac{l_n x}{L}\right)\left(C_{1n} \cos\left(\frac{\sqrt{l_n^2 - \hat{d}c}}{L} t'\right) + C_{2n} \sin\left(\frac{\sqrt{l_n^2 - \hat{d}c}}{L} t'\right)\right) \quad \text{...........} (70)$$

As was the case with the eigenvalues, the form of these expressions depends upon the values of the existing constants. In this case the critical constant is $\hat{w}$. In their present form these expressions are only valid if $\hat{w}$ is real. If $\hat{w}$ is imaginary, the equations for displacement, velocity, and stress respectively are

$$u(x,t') = e^{-bt'} \sum_{n=1}^{\infty} \cos\left(\frac{l_n x}{L}\right)\left(C_{1n} \cosh\left(\frac{\sqrt{\hat{d} - l_n^2 c}}{L} t'\right) + C_{2n} \sinh\left(\frac{\sqrt{\hat{d} - l_n^2 c}}{L} t'\right)\right) \quad \text{..................} (71)$$

$$u_{t'}(x,t') = e^{-bt'} \sum_{n=1}^{\infty} \left(\begin{array}{l} -b\cos\left(\frac{l_n x}{L}\right)\left(C_{1n} \cosh\left(\frac{\sqrt{\hat{d} - l_n^2 c}}{L} t'\right) + C_{2n} \sinh\left(\frac{\sqrt{\hat{d} - l_n^2 c}}{L} t'\right)\right) \\ + \frac{\sqrt{\hat{d} - l_n^2 c}}{L} \cos\left(\frac{l_n x}{L}\right)\left(C_{1n} \sinh\left(\frac{\sqrt{\hat{d} - l_n^2 c}}{L} t'\right) + C_{2n} \cosh\left(\frac{\sqrt{\hat{d} - l_n^2 c}}{L} t'\right)\right) \end{array} \right) \quad \text{...........} (72)$$

$$s(x,t') = \frac{Zc}{AL} e^{-bt'} \sum_{n=1}^{\infty} l_n \sin\left(\frac{l_n x}{L}\right)\left(C_{1n} \cosh\left(\frac{\sqrt{\hat{d} - l_n^2 c}}{L} t'\right) + C_{2n} \sinh\left(\frac{\sqrt{\hat{d} - l_n^2 c}}{L} t'\right)\right) \quad \text{................} (73)$$

It is important to note that these two sets of equations are not mutually exclusive; it is entirely possible that in the progression of eigenvalues, the value of ω may change from imaginary to real. In this case the Fourier series becomes a mixture of hyperbolic and circular functions.

The case of $\hat{w} = 0$ is not considered as it is unlikely that this case would be encountered for reasons stated before.

Computation of the Fourier Coefficients

For either the circular or hyperbolic function series, the initial displacement and velocity can be represented by the Fourier series

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} C_{1n} \cos\left(\frac{l_n x}{L}\right) \quad (74)$$

and

$$u_t(x,0) = g(x) = \sum_{n=1}^{\infty} \left(-bC_{1n} + \frac{\sqrt{l_n^2 - dc}}{L} C_{2n} \right) \cos\left(\frac{l_n x}{L}\right) \quad (75)$$

This is significant because the Fourier coefficients are the same for both the circular and hyperbolic series. A similar technique to the one for the undamped case was used; however, because there are some important differences in the procedure, the derivation is discussed completely. The function f(x) is given by Equation (54). Multiplying both sides of Equation (74) by $\cos(\lambda_m x/L)$, for each term (and coefficient,)

$$C_{1n} \cos\left(\frac{l_n x}{L}\right) \cos\left(\frac{l_m x}{L}\right) = \cos\left(\frac{l_m x}{L}\right) f(x) \quad (76)$$

Integrating both sides,

$$C_{1n} \int_o^L \cos\left(\frac{l_n x}{L}\right) \cos\left(\frac{l_m x}{L}\right) dx = \int_0^{dc} \cos\left(\frac{l_m x}{L}\right) f(x) dx \quad (77)$$

The right hand side is not integrated to L but only to δc. This is because the displacement is zero from this point to the pile toe; thus, this integration is not shown. However, the left hand side is integrated for the full length of the pile.

Orthogonality allows integration only for the case when m=n. Substituting Equation (54) for f(x), performing all integration and solving for the first Fourier coefficient,

$$C_{1n} = \frac{2\,l_n F_0}{Z\left((bL)^2 - (c\,l_n)^2\right)^3 \left(\cos(l_n)\sin(l_n) + l_n\right)}$$

$$\left(\frac{e^{-bd}}{8\,l_n^5} \left(\left(\begin{array}{l} 2\hat{d}^2 l_n^4 b^3 L^3 d^2 c^4 - 3\hat{d}^2 b^5 L^7 - 6\hat{d}^2 c^4 d\,l_n^4 b^2 L^3 \\ - 4L^7 \hat{d}\,l_n^2 b^5 - 15\hat{d}^2 c^4 l_n^4 bL^3 - 12L^3 \hat{d}\,l_n^6 bc^4 \\ - 16L^5 \hat{d}\,l_n^4 b^3 c^2 + \left(\hat{d}cd\,l_n\right)^2 (bL)^5 + \left(\hat{d}d\right)^2 (c\,l_n)^6 bL \\ - 10\left(\hat{d}c\,l_n\right)^2 b^3 L^5 - \left(\hat{d}c\,l_n\right)^2 db^4 L^5 - 8bc^4 L^3 l_n^8 \\ - 5\hat{d}^2 c^6 l_n^6 Ld - 8c^4 L^3 \hat{d}\,l_n^6 b^2 d - 4c^6 L\hat{d}\,l_n^8 d \\ - 16c^2 L^5 l_n^6 b^3 - 4\hat{d}c^2 d\,l_n^4 L^5 b^4 - 8L^7 l_n^4 b^5 \end{array} \right) \sin\left(\frac{l_n dc}{L}\right) + \left(\begin{array}{l} 7\hat{d}^2 c^5 l_n^5 dbL^2 + 10\hat{d}^2 c^3 l_n^3 db^3 L^4 + 4\hat{d}c^5 l_n^7 dbL^2 \\ + 4\hat{d}c\,l_n^3 db^5 L^6 - 8\hat{d}^2 c^5 l_n^5 L^2 + 8\hat{d}c^3 l_n^5 db^3 L^4 \\ - 8\hat{d}c^3 l_n^5 b^2 L^4 - 8\hat{d}c^5 l_n^7 L^2 + \hat{d}^2 c^3 l_n^3 d^2 b^4 L^4 \\ + 2\hat{d}^2 c^5 l_n^5 d^2 b^2 L^2 + \hat{d}^2 c^7 l_n^7 d^2 + 3\hat{d}^2 c\,l_n^1 db^5 L^6 \\ - 8c^5 l_n^9 L^2 - 8c\,l_n^5 b^4 L^6 - 16c^3 l_n^7 b^2 L^4 \end{array} \right) \cos\left(\frac{l_n dc}{L}\right) \right) + cL^2\left(2c^2 l_n^2 b^2 L^2 + b^4 L^4 + \hat{d}^2 c^4 + \hat{d}c^2 b^2 L^2 + \hat{d}c^4 l_n^2 + c^4 l_n^4\right) \right) \quad \ldots\ldots\ldots\ldots (78)$$

Turning to the second Fourier coefficient, this is a little more complicated than the first because it is dependent on the first. The function g(x) is given by Equation (55). Taking Equation (75) and multiplying both sides by $\cos(\lambda_m x/L)$,

$$g(x)\cos\left(\frac{l_m x}{L}\right) = \sum_{n=1}^{\infty} \left(-bC_{1n} + \frac{\sqrt{l_n^2 - dc}}{L}\, C_{2n} \right) \cos\left(\frac{l_n x}{L}\right) \cos\left(\frac{l_m x}{L}\right) \quad \ldots\ldots\ldots\ldots (79)$$

If one term at a time is considered and both sides are integrated, this yields

$$\int_0^{dc} g(x)\cos\left(\frac{l_m x}{L}\right) dx = \left(-bC_{1n} + \frac{\sqrt{l_n^2 - dc}}{L}\, C_{2n} \right) \int_0^L \cos\left(\frac{l_n x}{L}\right) \cos\left(\frac{l_m x}{L}\right) dx \quad \ldots\ldots\ldots\ldots (80)$$

Solving for C_{2n},

$$C_{2n} = \frac{L}{\sqrt{l_n^2 - dc}}\left(\frac{\int_0^{dc} g(x)\cos\left(\frac{l_m x}{L}\right)dx}{\int_0^{L}\cos\left(\frac{l_n x}{L}\right)\cos\left(\frac{l_m x}{L}\right)dx} + bC_{1n}\right) \quad (81)$$

Substituting Equation (55) for g(x) and integrating,

$$C_{2n} = \frac{1}{4cL\, l_n^4 Z\left(\cos(l_n)\sin(l_n) + l_n\right)\sqrt{l_n^2 - \hat{d}}}\left(\begin{array}{l} F_0 e^{-bd}\left(\begin{array}{l}\left(3\hat{d}^2 + 8\,l_n^4 + 4\hat{d}\,l_n^2 - \left(\frac{\hat{d}cd\,l_n}{L}\right)^2\right)L^2\sin\left(\frac{l_n dc}{L}\right) \\ -\left(3\hat{d} + 4\,l_n^2\right)\hat{d}\,l_n L dc\cos\left(\frac{l_n dc}{L}\right)\end{array}\right) \\ +4bC_{1n}L^2\,l_n^4 Z\left(\cos(l_n)\sin(l_n) + l_n\right)\end{array}\right) \quad (82)$$

Knowing these coefficients, and transforming the time reference to the original one using Equation (56), the Fourier series with circular functions for the displacement and stress are given by the equations

$$u(x,t) = e^{-bt'}\sum_{n=1}^{\infty}\cos\left(\frac{l_n x}{L}\right)\left(C_{1n}\cos\left(\frac{\sqrt{l_n^2 - \hat{d}}\,c}{L}(t-d)\right) + C_{2n}\sin\left(\frac{\sqrt{l_n^2 - \hat{d}}\,c}{L}(t-d)\right)\right) \quad (83)$$

$$s(x,t) = \frac{Zc}{AL}e^{-bt'}\sum_{n=1}^{\infty} l_n\sin\left(\frac{l_n x}{L}\right)\left(C_{1n}\cos\left(\frac{\sqrt{l_n^2 - \hat{d}}\,c}{L}(t-d)\right) + C_{2n}\sin\left(\frac{\sqrt{l_n^2 - \hat{d}}\,c}{L}(t-d)\right)\right) \quad (84)$$

For the hyperbolic functions, the displacement and stress are

$$u(x,t) = e^{-bt'}\sum_{n=1}^{\infty}\cos\left(\frac{l_n x}{L}\right)\left(C_{1n}\cosh\left(\frac{\sqrt{\hat{d} - l_n^2}\,c}{L}(t-d)\right) + C_{2n}\sinh\left(\frac{\sqrt{\hat{d} - l_n^2}\,c}{L}(t-d)\right)\right) \quad (85)$$

$$s(x,t)=\frac{Zc}{AL}e^{-bt'}\sum_{n=1}^{\infty} l_n \sin\left(\frac{l_n x}{L}\right)\left(C_{1n}\cosh\left(\frac{\sqrt{\hat{d}-l_n^2}c}{L}(t-d)\right)+C_{2n}\sinh\left(\frac{\sqrt{\hat{d}-l_n^2}c}{L}(t-d)\right)\right)\quad(86)$$

This is the solution for the damped case using a uniform intensity impulse function of force F_0 and time duration from impact δ. These of course are only valid after time δ.

COMPARISON OF RESULTS WITH NUMERICAL METHODS

Given the large number of possible cases that exist for hammer/pile/soil combinations, the possibility for comparison of the closed form solutions described above are literally endless. The central purpose of this part of the paper is to illustrate the possible application of these methods and to compare them with existing numerical methods, both to verify the basic soundness of the closed form solution and to further explore the relationship between numerical methods and the closed form solution.

Computer Implementations of the Calculations

It is evident from both the solutions proposed here and a review of those who have gone on before that any viable use of any solution of the wave equation for piles involves computer solutions of some kind. Following are the methods which were used in this analysis.

Closed form Solution using Maple V Release 3

Advances in computer software made it possible to consider closed form solutions that would have been impossible or impractical in the past. For the purposes of this paper this means Maple V Release 3, which is a general purpose mathematical software package capable of both symbolic solution and numerical computation. A detailed description of this software is given in Abell and Braselton (1994a, 1994b). Much of the derivations given earlier, although possible by hand, were in fact done with Maple V, especially as they relate to integration, algebra and complex analysis. Although Maple's capabilities with Laplace transforms were used, limitations with inverse transforms and other areas required occasional "intervention" in the calculation sequence.

With the implementation of these solutions, Maple V was used in a different way. Although it is possible in principle to use the same routines to make numerical computations as were used with derivation, both limitations in both software and hardware and the need for a relatively efficient code suggested the division of the code into a program most suited for symbolic manipulation and one for numerical computation.

Direct Stiffness Solution using ANSYS-ED 5.0-56

One interesting concept that has not been widely pursued either by researchers or practitioners has been the use of general purpose finite element codes for stress wave analysis of piles. For both undamped and damped cases the closed form solution was compared with results from the

ANSYS general purpose computer program. The pile top force can be simulated either by applying a force-time relationship or simulating the drop of a mass onto the hammer cushion. Although the educational version is limited as to the number of nodes and elements, by finite element standards this is a relatively simple problem, so this limitation does not pose any problem here, because here the soil is modelled using visco-elastic elements and not an axisymmetric solid around the pile.

Finite Difference Solution using WEAP87

From both an historical and a practical standpoint, the most important comparison is with the finite difference techniques that have been the industry standard since the days of Smith (1960). For this purpose the WEAP87 program was used. This is similar to the WEAP86 program as described by Goble and Rausche (1986). This program has a relatively undemanding personal computer implementation and many options for input and output. These are necessary in this case as the entry of soil parameters that are similar to those used in the closed form solution require some care because their theoretical basis is different. This program, however, can only be used to compare the damped case, not because it does not analyze undamped piles but because the undamped case uses a hammer system without a cap, which is not permitted by this program.

Solution Implementation using the Example Case

Statement of the Problem

The basic problem under consideration is the driving of a 1000 mm diameter steel pipe pile, 50 m long, with a wall thickness of 40 mm. The pile is driven open ended into a medium dense sand. The hammer used has a ram mass of 15 metric tons; it has an equivalent stroke of 1.5 m and a mechanical efficiency of 80%. The cushion block has a stiffness of 2.45 GN/m and has no damping (this is to avoid comparing the static hysteresis cushion concept of WEAP87 with the viscous material damping concept of the closed form solution and ANSYS.) This example was analysed for $0 < t < 4L/c$ for displacement-time and stress-time histories at the pile top ($L = 0$ m), pile middle ($L = 25$ m) and pile toe ($L = 50$m).

Values for the variables of the solution are shown in Table 1. These are either given variables or computed using the appropriate equations given earlier.

Table 1 Variables for Example Case

Variable Designation	Nomenclature	Value of Variable in Example Case
a	Pile Shaft Elasticity Constant	36858.97436 1/sec²
b	Pile Shaft Dampening constant	221.8649536 1/sec
c	Acoustic Speed of Pile Material	5188.745215 m/sec
c'	Cushion Dampening/Hammer Impedance Ratio	0
d	Pile Inside Diameter	920 mm
d'	Pipe Pile Diameter Ratio	0.92
$\hat{d}$	Pile Shaft Damping and Elasticity Ratio	1.148186304
k	Soil Shaft Spring or Elastic Constant per Unit Area	11.04 MPa
k_t	Soil Toe Spring or Elastic Constant per Unit Area	11.08 MN/m³
$\hat{m}$	Mass of Driving Accessory for Pile Hammer	3000 kg
m'	Pile Cap/Ram Mass Ratio	0.2
r_g	Geometry Ratio of Pile	0.0122
r_t	Pile Toe Radius	500 mm
A	Cross-Sectional Area of Pile	0.12064 m²
A_t	Pile Toe Area	0.12064 m²
C	Cushion Dampening Coefficient	0 N-sec/m
D	Pile Outside Diameter	1000 mm
E	Pile Young's Modulus of Elasticity	210 GPa
E_s	Soil Young's Modulus of Elasticity	27.6 MPa
G_s	Soil Shear Modulus of Elasticity, Pa	11.04 MPa
K	Cushion Material Spring Constant	2.45 GN/m
L	Length of Pile	50 m
L/c	Time Length for Wave Transmission from Top to Toe	9.636 msec
M	Mass of Pile Hammer Ram	15,000 kg
P	Pile Surface Perimeter	3142 mm
V_0	Initial Velocity of Pile Hammer Ram	4.85 m/sec
Z	Pile Impedance	4.882 MN-sec/m
Z_h	Pile Hammer Impedance	6.062 MN-sec/m
Z'	Pile-Hammer Impedance Ratio	0.8054
a_0	Pile Top Consolidation Variable	250.8985 1/sec
a_1	Consolidation Constant for Pile Top Forces	-835.176
a_2	Consolidation Constant for Pile Top Forces	-396.154
a_3	Consolidation Constant for Pile Top Forces	401.678
b_0	Pile Top Consolidation Variable	1.26279
d	Time of Square Wave Simplified Impulse	4.673 msec
m	Shaft Soil Dampening Coefficient per Unit Area	132.906 kN.-sec/m³
m_t	Soil Toe Dampening Constant per Unit Area	191.785 kN.-sec/m³
n	Poisson's Ratio of Soil	0.25
r	Pile Density	7800 kg/m³
r_s	Soil Density	1600 kg/m³
	Time Step for Newmark's method using Maple V	46.4 µsec
	Default Time Step for ANSYS	12.05 µsec
	Pile Shaft Surface Area	157.08 m²
	Total Shaft Spring Constant	1.74 GN/m
	Total Shaft Dampening Constant	20.88 MN-sec/m
	Total Toe Spring Constant	29.44 MN/m

Total Toe Dampening Constant	23.137 kN-sec/m
Assumed Quake for WEAP87	½”
Total Pile Capacity for WEAP87	5052.2 kips
Percentage of Capacity at Shaft for WEAP87	98%
Smith Shaft Dampening Constant for WEAP87	0.288 sec/ft
Smith Toe Dampening Constant for WEAP87	0.009412 sec/ft

Computation of Pile Top Force

The closed form solution is dependent upon the function of the pile top force, as is the Newmark method using Maple V and most of the ANSYS runs.

The velocity-time relationship for the undamped semi-infinite pile with a ram, damped cushion and pile cap as shown in Figure 1 is given in Equation (25). Multiplying this by the impedance and substituting the example variables gives the force-time relationship (again in Newtons) of

$$F_0(t) = 54.61\times10^6\left(e^{-835.176t} - e^{-396.154t}\cos(401.6678t)\right) + 59.69\times10^6 e^{-396.154t}\sin(401.6678t) \quad \text{.............} (87)$$

However, this cannot be directly applied to the damped case because it is only valid for piles without dampening. The solution to this problem is first to note that the maximum force is 15.566 MN at a time of 3.2 msec; this becomes the constant force for the assumed duration of the square wave pulse. That duration was computed by multiplying the hammer mass and its impact velocity to compute the hammer’s total impulse, than dividing this by the maximum force. Because there is no dampening in the cushion material and (as it turns out) not a great deal of rebound from the pile top, this approximation can be made with little error. The impulse time calculates to 9.636 msec.

Presentation and Discussion of the Results

The parameters and special aspects of all of the solution types having been detailed, it is possible to proceed to the presentation and discussion of the results.

Because of the nature of the results, graphical comparison is the most expedient method to view these results. They are compared in two ways: a) between differing places on the pile for a single method, and b) between methods for given points on the pile.

Displacements

Figure 3 shows the displacement-time histories for the damped case by comparing the three pile locations using the same method for each graph, and Figure 4 shows these histories by comparing the methods with each other at each pile location.

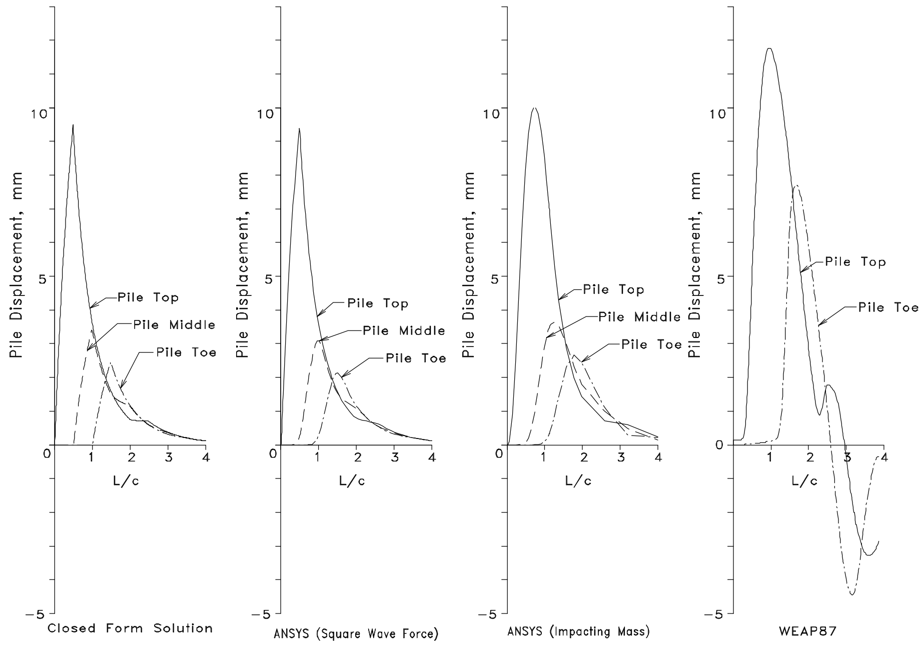

Figure 3 Comparison of Pile Locations, Displacements

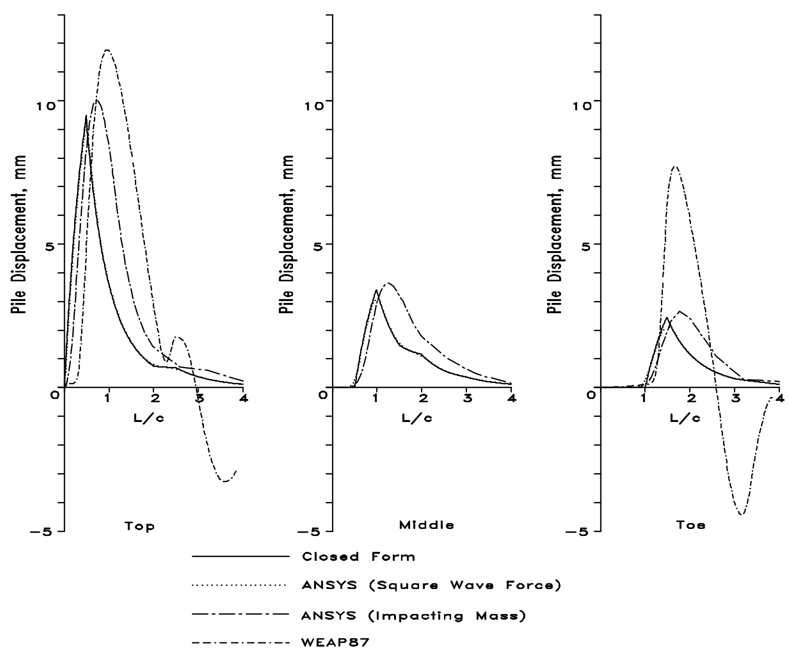

Figure 4 Comparison of Methods, Displacements

The time delay characteristics of the displacements are typical of problems such as this, i.e., the pile top motion starts at t = 0, the pile middle stars at t = L/2c, and the pile toe at t = L/c. The dissipative effects of the shaft soil are very evident, as the displacements diminish with distance from the pile top. The closed form solution and the ANSYS solutions show a similar dissipation pattern, even with different pile top loading. There is not a great deal of pile toe rebound; only the "knee" in the displacement curves around $t = 2L/c$ is evidence of this. This is because most of the soil "resistance" here is along the shaft. Given the length of the pile and the fact that it is being driven open ended, this is to be expected.

With the closed form solutions, it should be noted that the transition from Laplace transform solution to Fourier series takes place around $t = L/2c$; therefore, almost all of the results for the pile middle and toe are derived from the Fourier series. This indicates the complicated nature of these series and why a large number of terms is necessary to obtain a reasonable solution.

Except for "rounding" at the corners, the ANSYS results for the square wave pile top force are virtually indistinguishable from the closed form solution. This is a major result; in addition to confirming both solutions (since they are obtained using very different methods,) it shows that the difficulties with finite element methods can be overcome depending upon how the solution is set up. This last point is true with virtually any finite element solution.

The ANSYS results with the impacting mass is similar to the square wave solution but shows that the use of the undamped semi-infinite pile solution to determine a substitute force-time curve has its limitations. Differences in the timing of the peak displacement were expected because of the nature of the approximation. The slightly higher displacements, however, indicate that, in order to accurately determine the peak force and displacement, a solution of the damped semi-infinite pile at the pile top would be needed.

The WEAP87 results, however, are very different from the other methods. Both the pile top and especially the pile toe have much higher displacements than the other methods and the "knee" is much more pronounced than the other methods as well. Since this program, its predecessors and its successors are important in the actual analysis of wave propagation in piles, some reasonable explanation is necessary.

The most important things to keep in mind about WEAP87 is that it has been developed a) largely without the benefit of closed form solution for comparison on a theoretical basis and b) with the aim of correlation with field results. This latter point includes the very important consideration that virtually all actual pile driving problems involve plastic displacement of the soil; without it there can be no penetration of the pile. This fact is underscored by observing that the displacements for the other solutions approach zero for all pile points as time advances. In this case WEAP87 is asked to analyse a pile driving problem completely devoid of plastic deformation, something it was not really intended to do. Moreover, although every attempt was made to input the parameters of the problem into WEAP87 to have the same meaning as they did with the other problems, the necessary inclusion of empirical factors into the program (the source code was not available for this study) may make an exact comparison impossible and thus alter the results.

One option considered at the start was the use of a finite difference program for which there is available source code. This was rejected because a) most of these programs are at least twenty years old, and thus may not be very relevant to programs currently in use, and b) would have had the result of a new theoretical method to the problem, a role which ANSYS is well suited for.

Stresses

Figure 5 shows the stress-time histories by comparing the three pile locations using the same method for each graph, and Figure 6 shows these histories by comparing the methods with each other at each pile location.

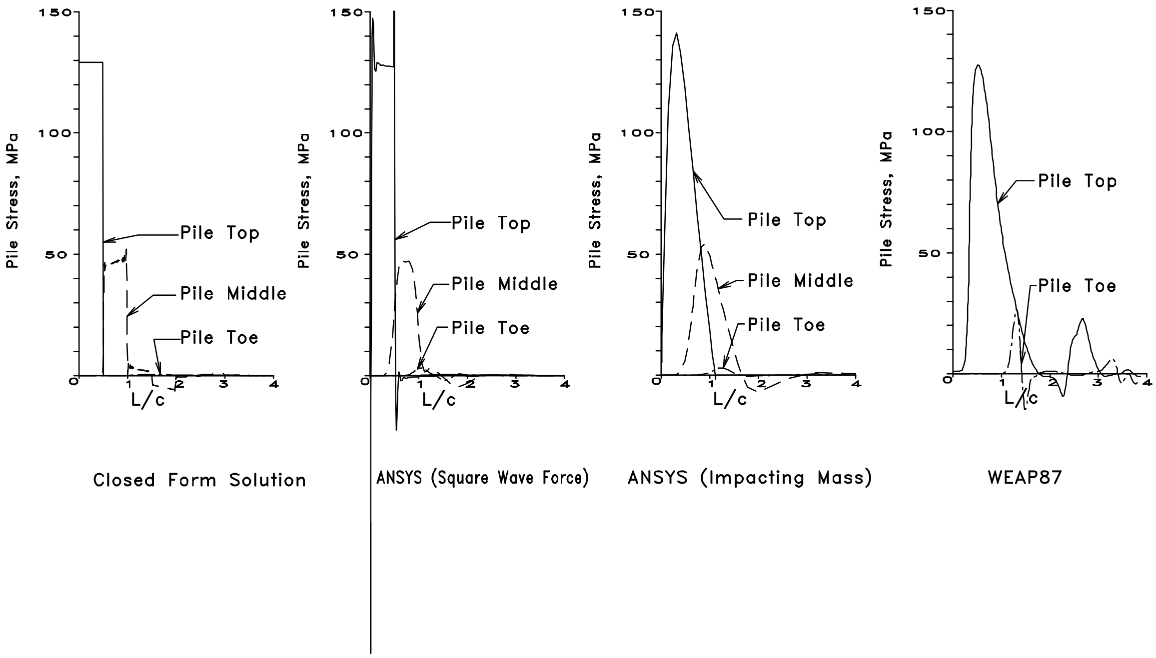

Figure 5 Comparison of Pile Locations, Stresses

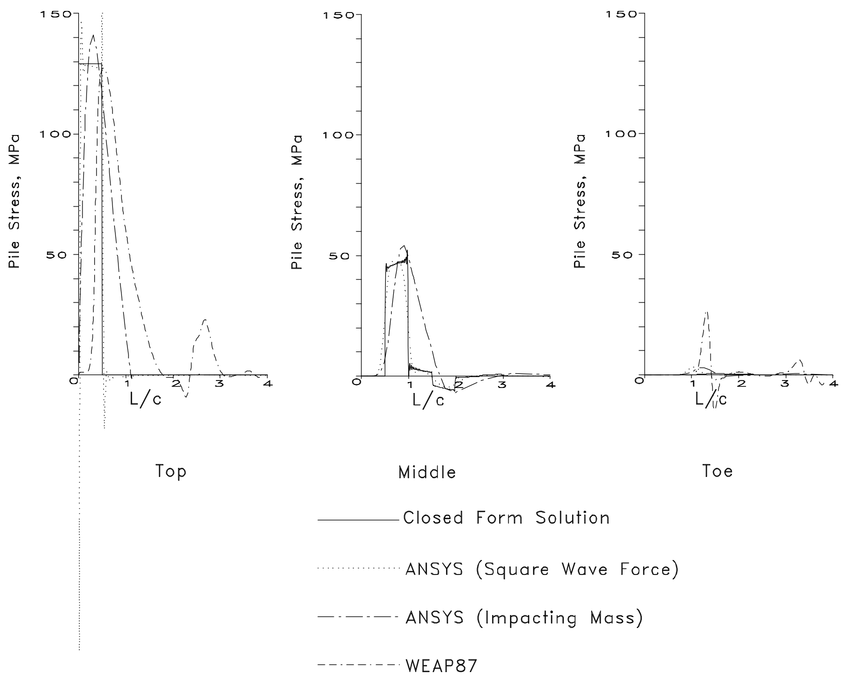

Figure 6 Comparison of Methods, Stresses

The comparison between the closed form solution and the ANSYS run with the square wave load is not as precise as with the displacements. This is probably due to a discretization problem. However, the peak results correlate very well. ANSYS also experienced problems with the step type of loading at the corners, as did the closed form solution.

The situation with the stresses from the ANSYS run with the impacting mass is very similar to that with the displacements.

The stress results from WEAP87 compares more closely with the other solutions than the displacements, although there are still oscillations at the toe.

CONCLUSIONS

1. Within its stated limitations, the closed form solution presented is a viable tool for the analysis of stress wave phenomena in piles on a theoretical basis. The method generally both correlates with and illuminates the results of numerical analysis, provided that the numerical method used can be properly compared with the theoretical basis of the closed form solution.

2. The substitution of a strain related pile toe dynamic resistance (as opposed to a velocity related one) by assuming the existence of a semi-infinite soil column under the toe shows initial success but requires further study, both theoretical and experimental, as the entire subject of pile shaft and toe response is still not adequately quantified.

3. The limitations on the Bessel function argument induced by the series substitution and the reduction of the force to a square wave pulse indicate that, although the damped solution is an adequate first approximation, the ultimate solution of this problem is either a numerical one or the numerical integration of a closed form solution.

4. The results of the ANSYS modelling shows that this program is capable of the basic modelling of wave propagation in piles.

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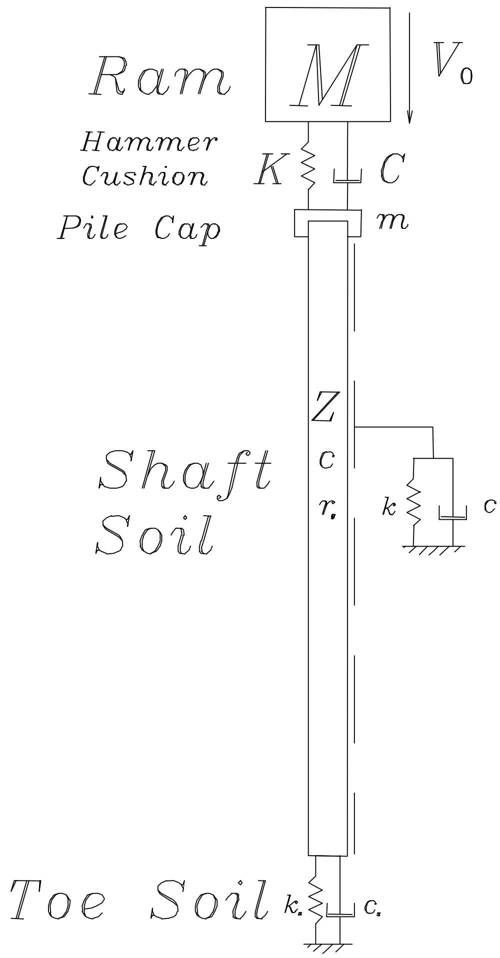

Now that x=L…

For those of us involved in the one-dimensional wave equation, this means that you have reached the end! We trust that the information presented in the article concerning the wave equation or other technical matters has been useful to you. We should now like to take the time to make some other observations.

It is our conviction that the beauty of our world and universe, especially as it is expressed mathematically but certainly in other ways, speaks of its formation by an intelligent Creator. This is underscored by the unity that appears both in mathematics and in the physical laws which mathematics are used in to quantify and qualify. As scientists and engineers we depend upon this unity to both make sense out of what we observe and to make progress both in our knowledge and in our application of that knowledge to practical problems.

But as we turn away from the reverie of beautiful formulations, we see a world that is marred by human failing. This manifests itself in many forms that we are reminded of daily. The longer we live on this earth the more those failings come home to inflict pain upon us, no matter how hard we try to escape them.

It was not God's intent to leave us with this pain alone in his creation but to offer us a way by which we finite beings be united into his perfect infinity, something which is both definable and beyond definition. In infinity past he was with his Son Jesus Christ and Jesus came to live amongst us, share our situation and ultimately face torture and execution by those who were threatened by his message.

But this was not the end, for Jesus being God rose from the dead and offers us both a way out of our present condition in this life and eternal life with God, not by simply following a set of rules but by having God himself live in us and both empowering and leading us in a better way. If we commit ourselves to Jesus then for us $L=\infty$, which means that we have life forever.

All of these things are described in the book called the *Bible*; but in the meanwhile you can learn more at the website

http://www.geocities.com/penlay

or by emailing us at uttc2uxx@ geocities.com. We look forward to hearing from you.